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Exact self-similar solutions for axisymmetric wakes

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ABSTRACT

This Note presents an analytical solution of two-dimensional axisymmetric wakes valid in the development region, before the ultimate equilibrium state. Based on the boundary layer equations in polar coordinates, assuming a small velocity defect, the problem reduces to a linear diffusive equation and can be expressed as an eigenvalue problem. Then a complete set of eigenfunctions is analytically obtained, which are damped and evolve self-similarly in space. The first mode corresponds to the Schlichting's solution, in agreement with the downstream asymptotic behavior.

1. Introduction

This Note investigates the generic problem of the flow past fixed axisymmetric bodies such as spheres, disks or bullet-shaped bodies. This old problem was first considered by Stokes [1] in 1851 for the case of the flow past a sphere, in the limit of negligible inertia. Later Whitehead [2] and Oseen [3] attempted to improve upon this solution for small, but not negligible, Reynolds numbers. Higher approximation of the flow past a circular sphere than those represented by the solutions of Stokes and Oseen has been presented by Proudman and Pearson [4].

In the present article the intermediate far field is considered. Leaving aside the near field, that is potentially highly non-parallel since it can host counter-rotating vortices, the intermediate and long term wake is a near parallel flow. The velocity defect in the intermediate area can be adequately described by the Prandtl's boundary layer equations, linearized around the uniform flow field. In the next section a method is presented to build a complete set of self-similar solutions based on the invariance property of the equation under dilatation symmetry. Thus self-similarity implies that the flow has reached a kind of asymptotic state.

The search for self-similar solutions is old and fruitful. The ideas of self-similarity in fluid flows appear to have been first applied by Blasius in 1908 for the laminar boundary layer. The turbulent wakes are as well as known to develop self-similarly sufficiently far downstream from obstacles that generate them (Townsend [5]). Recently new self-similar solutions have been presented, for vortex flows, in the paper by Satijn et al. [6]. These authors present some two-dimensional unsteady vortex models, which are based on similarity solutions of the linear diffusion equation, modeling diffusing monopoles, dipoles, and tripolar vortices. Self-similarity occurs when the velocity profile can be brought into congruence by a simple scale factor. As consequence, the dynamical equations are reduced by one geometrical variable in their functional. In the present case, the original equations being axisymmetric, they are reduced to ordinary differential equations. Moreover, in the case presented here, this differential equation can be expressed as a Sturm–Liouville problem, it then becomes possible to provide an analytical solution, as it was demonstrated by Kloosterziel [7].

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2. Governing equations

We consider an axially symmetric wake, evolving in a uniform flow U_0, such as occurs downstream of an axially symmetric body placed in a stream parallel to its axis. The velocity field, supposed to be stationary, may be represented as a deviation superposed to the uniform flow: $[U_0 + u(x,r); v(x,r)]$, where u, v are respectively the streamwise and the radial velocity and x, r the streamwise and the radial coordinates. By considering the slow streamwise variation of the velocity field, we apply the Prandtl's approximation. Then the governing equations are reduced to the boundary layer equations in polar coordinates:

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial (rv)}{\partial r} = 0 \tag{1}$$

$$(U_0 + u)\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{2}$$

This nonlinear equation can be transformed into the classical Falkner–Skan equation, see for example Barker and Wilks [8] and the references therein. In the following we are interested by finding an analytical solution. Thus the velocity deficit u, is assumed small as compared to U_0 sufficiently far downstream, consequently the problem is reduced to a linear diffusion-like equation for the streamwise velocity:

$$U_0\frac{\partial u}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{3}$$

associated to the radial boundary conditions plus an inflow condition:

$$\frac{\partial u}{\partial r}(x, r = 0) = 0, \qquad \lim u(x, r \to \infty) = 0 \quad \text{and} \quad u(x = 0, r) = u_0(r) \tag{4}$$

From the physical point of view, the radial direction is characterized by diffusive transport while the longitudinal direction is characterized by advection. Thus the solutions are sought invariant under dilatation symmetry. This transformation assigns to each variable or function the transformed form defined such that: $\bar{x} = \lambda^a x$; $\bar{r} = \lambda^b r$; $\bar{u} = \lambda^c u$. The procedure consists in two stages:

- Invariance condition for the function u:

  $$\bar{u}(x,r) = u(\bar{x}, \bar{r}) \quad \to \quad \lambda^c u = u\left(\lambda^a x, \lambda^b r\right)$$

  After differentiation with respect to λ and posing λ equal to one, a first-order partial differential equation is obtained,

  $$cu(x,r) = ax\frac{\partial u}{\partial x} + br\frac{\partial u}{\partial r}$$

  which is solved using the method of characteristics:

  $$u(x,r) = x^{c/a} F\left(\frac{r^a}{x^b}\right)$$

- Invariance condition for the equation:

  $$\lambda^{c-a} U_0 \bar{u}_{\bar{x}} - \frac{\nu}{\bar{r}}\lambda^{c-2b}\bar{u}_{\bar{r}} - \nu\lambda^{c-2b}\bar{u}_{\bar{r}\bar{r}} = A(\lambda)\left(U_0 u_x - \frac{\nu}{r}u_r - \nu u_{rr}\right)$$

which gives the relation $c - a = c - 2b \Rightarrow a = 2b$. This relation is representative of the core size's diffusive nature. The axisymmetric wake is a thin free flow. The wake slowly becomes thicker according to a diffusive scaling law, i.e. $\sqrt{x}$.

By injecting this solution into Eq. (3), the single admissible value for the parameter b is $b = 1$. With the notation $m = c/a$, the self-similar problem takes the form of an ordinary differential equation:

$$\eta F'' + (1 + \eta)F' - mF = 0 \tag{5}$$

$$F(0) = 1; \qquad F(\infty) = 0 \tag{6}$$

with $\eta = U_0 r^2/(4\nu x)$. The coefficient $U_0/4\nu$ was included to render the problem dimensionless. Note that the centreline boundary condition $\partial u/\partial r = 0 = x^m 2U_0 r/(4\nu x)F'(\eta)$ is satisfied if the function F' is continuous, or equivalently the function F remains finite, on the axis. The equation being linear, the centreline value has been arbitrary chosen as unity.

Eq. (5) represents an eigenvalue problem. To conclude regarding completeness of the eigenfunctions system, the equation can be reduced to the Sturm–Liouville one with the transformation: $F(\eta) = \tilde{F}(\eta)e^{-\eta}$ and $\lambda = -1 - n$, leading to the new ODE:

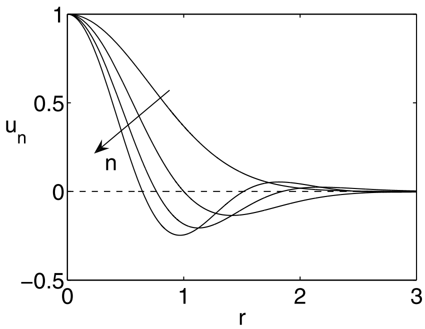

Fig. 1. First four solutions for the streamwise velocity.

$$\left(\eta e^{-\eta}\tilde{F}'\right)' + \lambda e^{-\eta}\tilde{F} = 0$$

which is the Laguerre's differential equation. The solutions, or eigenfunctions, are referred to as Laguerre polynomials:

$$\tilde{F}_n(\eta) = \frac{1}{n!}e^{\eta}\frac{\mathrm{d}^n}{\mathrm{d}\eta^n}\left(\eta^n e^{-\eta}\right) \quad \text{for } n = 0, \ldots, \infty$$

associated to the eigenvalues

$$\lambda_n = n$$

The centreline velocity is decaying like $1/x^{n+1}$. The relevance of the Laguerre functions for the asymptotic behavior of the diffusion equation, on cylindrical semi-infinite domains, was previously shown by Kloosterziel [7]. Laguerre polynomials form a complete orthogonal set with respect to the weighting function $\exp(-\eta)$:

$$\langle \tilde{F}_m, \tilde{F}_n \rangle = \int_0^\infty \tilde{F}_m(\eta)\tilde{F}_n(\eta)e^{-\eta}\,\mathrm{d}\eta = \delta_{mn} \tag{7}$$

3. Solution

Even if each mode is a solution of Eq. (3), a general velocity field u can be expressed as a linear combination of functions u_n:

$$u(x,r) = \sum_{n=0}^{\infty} A_n u_n(x,r) \tag{8}$$

with

$$u_n(x,r) = x^{-(1+n)}F_n(\eta) = \frac{x^{-(1+n)}}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}\eta^n}\left(\eta^n e^{-\eta}\right)$$

These functions take the form of Laguerre–Gaussian distributions in radial direction. The coefficients A_n are given by using the orthogonality property (7). Thus the global solution can be constructed, for example, from an experimental inflow condition. Note that the streamwise coordinate can be translated along x in order to introduce a fictitious origin x_0. Alternatively, the solution can be also constructed from the knowledge of the centreline velocity. In the latter case, the coefficients A_n have to be computed in a least square sense.

The first four solutions u_n are expressed as follows:

$$u_0 = (x/U_0)^{-1}e^{-\eta}$$
$$u_1 = (x/U_0)^{-2}(1-\eta)e^{-\eta}$$
$$u_2 = (x/U_0)^{-3}\left(\eta^2/2 - 2\eta + 1\right)e^{-\eta}$$
$$u_3 = (x/U_0)^{-4}\left(-\eta^3/6 + 3/2\eta^2 - 3\eta + 1\right)e^{-\eta}$$

and are plotted in Fig. 1.

The first mode is characterized by a centreline velocity inversely proportional to the downstream distance. It corresponds to the classical Gaussian solution found by Schlichting [9]. Being the least damped mode, it describes the asymptotic behavior of the general solution.

The physical constraint imposed by the conservation of momentum requires that the drag coefficient D, evaluated from the momentum of the wake, must be independent of the distance from the origin x. This leads to the relation:

$$D = 2\pi\rho U_0 \int_{r=0}^{+\infty} ur\,\mathrm{d}r = \text{const} \tag{9}$$

From this relation, Schlichting [9] has deduced, in an elegant way, the classical solution, presented in many handbooks. Thus the self-similar solution, given in Eq. (8), must satisfy the condition:

$$D = 2\pi\rho U_0 \sum_n A_n \int_{r=0}^{+\infty} u_n r\,\mathrm{d}r = 4\pi\rho\nu \sum_n A_n x^{-n} \int_{\eta=0}^{+\infty} F_n\,\mathrm{d}\eta = \text{const}$$

It is seen that, for $n \neq 0$, the integral vanishes:

$$\int_{\eta=0}^{+\infty} F_n\,\mathrm{d}\eta = \frac{1}{n!}\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}\eta^{n-1}}\left(\eta^n e^{-\eta}\right)\right]_0^{+\infty} = 0 \quad \text{for } n = 1, 2, \ldots$$

Thus the relation (9) is always satisfied, the drag coefficient D being determined by the first mode, i.e. the Schlichting's solution.

It is now possible to express the radial velocity with the continuity equation. The radial velocity takes the form:

$$v_n = \frac{r}{2x}\left(\frac{U_0}{x}\right)^{n+1} G_n(\eta)$$

with G_n solution of the ordinary differential equation:

$$G_n + \eta G_n' = (n+1)F_n + \eta F_n'$$

For example, the first solution is: $v_0 = U_0 r/(2x^2)e^{-\eta}$. From the point of view of specifying conditions at the lateral boundaries, the analytical solutions, presented herein, may be useful to test several radial boundary conditions with respect to their ability to conserve the mass and the momentum and to handle low and high entrainment flows.

4. Discussion and conclusion

The development of axisymmetric laminar wake can be divided into three zones beyond the body, the immediate downstream area where the flow is potentially reversed, an intermediate zone where the flow velocity profile is self-similar at different distances and finally the fully developed flow field farther away, where the centreline velocity decreases as $1/x$, as predicted by the Schlichting's similarity solution [9]. A general expression of the axisymmetric wakes has been presented which takes into account the effects of the streamwise diffusion and entrainment. Nonetheless it must be noted that the method is based on the linearization of the equations. Despite this hypothesis, the new solution should constitute an improvement of the classical Schlichting's solution in the intermediate-far field.

That general solution is a linear combination of self-similar eigenfunctions (see Eq. (8)). In that sense, the global solution is not itself self-similar since each constituting self-similar mode decreases more or less rapidly. That implies that the shape of the velocity profiles changes with axial distance. It is a consequence of the fact that, in the intermediate zone, the flow is still not an equilibrium state. In fact the equilibrium state corresponds to the asymptotic solution, which depends only on the net force on the fluid and not on the details of how the force was applied. The contribution of the other self-similar modes is a memory effect of the near body velocity profile.

That property bears some similarities with the turbulent wake flows for which it has long been assumed that the spreading rate of the wake in the self-similar regime is independent of the details of the body, being dependent only on the total drag or momentum deficit. George [10] presented a critical analysis of the self-similarity argument in the context of certain apparent discrepancies of self-similar solutions with experimental results on jets and wakes. In fact even flows which appears to scale in similarity variables can be dependent on their initial conditions. That distinction between self-similar state and equilibrium state have lead Bevilaqua and Lykoudis [11] to introduce the concept of hierarchy of self-preservation for turbulent flows. The self-preservation is of order one when the mean velocity profiles are self-preserving; of order two when in addition the Reynolds stresses are self-preserving and so on through the higher moments. An equilibrium state, or fully developed flow, as defined by Townsend [5] would then be the asymptotic state that occurs when all moments are self-preserving.

To conclude, a perspective concerns the possible extension of this method to algebraic growth instabilities. Algebraic transient growth is believed to play an important role in the subcritical transition of laminar flows. By following the methodology used here to find general self-similar solutions of the linearized Navier–Stokes equations, it would be interesting to identify the linear perturbations algebraically unstable (i.e. with positive eigenvalue m, assuming that perturbations behave like t^m). This approach was successfully pursued by Luchini [12] for the boundary layer developing over a flat plane. The limitation is that the equations, in order to admit self-similar solutions, are necessarily simplified by neglecting some terms and then they can be physically uninteresting.

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