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# On the post-buckling of elastic beams on gradient foundation

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ABSTRACT

The post-buckling of an axially loaded elastic beam resting on linearly elastic medium is investigated in this paper from a geometrically exact analysis. It is known that the elastic foundation increases the bifurcation limit, but it may have a destabilizing effect on the post-buckling behavior associated to imperfection sensitivity. This unstable nature of the post-buckling behavior may lead to drastic softening phenomena, as already investigated for plasticity or Continuum Damage Mechanics media. It is suggested in this paper to study the influence of gradient terms in the interaction foundation model on the post-buckling behavior of this structural system. The gradient elasticity foundation model of Pasternak is used and introduced by variational arguments in a geometrically exact framework. A general nonlinear fourth-order differential equation is obtained, and numerically solved with a nonlinear boundary value solver. The post-buckling behavior is analyzed from an asymptotic method. The gradient elasticity constitutive law significantly affects the post-localization process.



## 1. Introduction

The beam model on elastic foundation is of particularly interest for civil engineers, first because it can be considered as a very generic model of soil–structure interaction problems (see Hetenyi [1] or Selvadurai [2]), but also for its capability to cover both stable and unstable post-buckling behavior of elastic structures. The simplest representation of a continuous elastic foundation has been provided by Winkler who assumed the base consisting of closely spaced, independent linear elastic springs (Winkler [3]). The Winkler model can be classified as a local elastic relationship between the reaction and the beam deflection. It has been already shown that imperfection sensitivity can be observed in the post-buckling of such structural systems resting on Winkler foundation (see for instance El Naschie [4]), as also appearing in shell problems (see Koiter [5] for the discussion of the effect of imperfection on stability of elastic structures). The aim of this Note is to investigate again the post-buckling behavior of such an academic problem with an asymptotic and a numerical approach. We would like to show that the unstable nature of the post-buckling behavior may lead to drastic localization softening phenomena, as already investigated for plasticity or Continuum Damage Mechanics media (see for instance Challamel et al. [6,7], Challamel [8]). It is suggested in this paper to add some nonlocality in the soil–structure interaction law, to affect the post-buckling behavior of this structural system. The gradient elasticity foundation model of Pasternak is used and introduced by variational arguments in a geometrically exact framework. Pasternak [9] generalizes the Winkler model and assumes existence of shear interactions between the spring elements. The Pasternak model involves an additional term which is proportional to the second-order derivative of displacements, as in gradient theory of elasticity. Henceforth, di Paola et al. [10] suggested the classification of Pasternak model as gradient elastic model of foundation (see also Challamel et al. [6,7]). Note that the post-buckling behavior of elastic beam on Pasternak foundation was already studied by El Naschie [4], without specific mention on the need of nonlocality in the softening post-localization response. This problem has been

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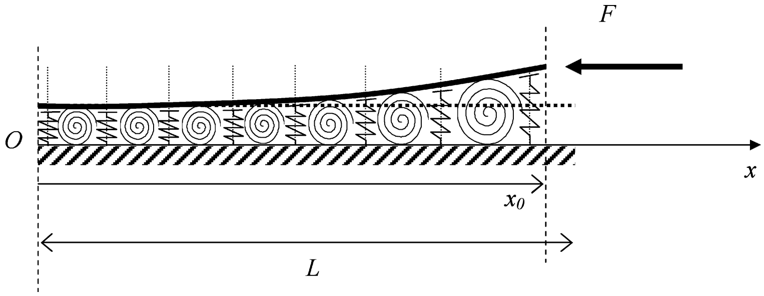


**Fig. 1.** Elastic beam on Pasternak foundation.

tackled again by Nguyen [11] in a more systematic numerical study. This study is also related to the instability studied by Damil and Potier-Ferry [12] even if they treated the nonlinear Winkler reaction (see also Amazigo et al. [13]) whereas in the present article the reaction is restricted to the linear format, the only nonlinearity being of geometrical nature.

In fact, the present Note deals with the post-buckling behavior of gradient or nonlocal structures, even if the nonlocality is located at the soil–beam interaction constitutive law, and the beam model is simply the "local" Euler–Bernoulli one. The post-buckling of gradient-based elasticity beam has been studied by Lazopoulos [14]. More recently, Wang et al. [15] investigated the post-buckling behavior of Eringen's based nonlocal elastic beam. Reddy [16] classified the nonlinear formulations of generic structural beam models with an Eringen's based nonlocal elastic constitutive law. The need of nonlocality in soil–structure interaction problem has been inspired us from the recent papers of Tortesillas et al. [17] or Hunt et al. [18] who studied the strong softening process in granular materials with some simple structural models. The localization process has been correlated to some geometrical and physical parameters and some numerical sensitivity studies have been performed. We believe that the present model could be understood as the continuous analogy with the discrete model considered by Hunt et al. [18]. The localization process with this analogy has definitively to be firmly controlled by some scale parameters related to the nonlocal nature of the interaction constitutive law.

## 2. Elastic beam on Pasternak foundation – Variational method

The principle of virtual work is written in the geometrically exact framework, for the axially-loaded inextensible Euler–Bernoulli beam on Pasternak elastic support as:

$$\delta U = \int_0^L EI\theta'\delta\theta'\,\mathrm{d}s - F\int_0^L \sin\theta\delta\theta\,\mathrm{d}s + \int_0^{x_0} kw\delta w + G\theta\delta\theta\,\mathrm{d}x = 0 \quad \text{with}$$
$$\theta = \arcsin w' \quad \text{and} \quad x_0 = \int_0^L \cos\theta\,\mathrm{d}s \tag{1}$$

$s \in [0; L]$ is the arc abscissa. The prime denotes the derivative with respect to $s$, i.e. $\theta' = \mathrm{d}\theta/\mathrm{d}s$ is the curvature. $k$ is the stiffness of the Winkler foundation, and $G$, the rotational stiffness of the shear layer (Pasternak foundation). In the problem considered in the paper, it is assumed that during deformation the foundation springs shift freely in the $x$-direction so as to remain vertical and exert only vertical forces for the Winkler foundation (see Fig. 1). The springs are assumed to be fixed at the foundation support, and there is a sliding connection between the beam and the springs. It is also worth mentioning that the model of El Naschie [4] for the Pasternak foundation is not strictly equivalent to the one that is presented in this paper. Nevertheless, a similar Winkler model has been investigated by Panayotounakos [19] for the post-buckling of elastic column on Winkler foundation, from direct equilibrium methods. However, the variational expression of this problem has not been studied before to the author's knowledge, including the Pasternak foundation. The additional shear layer of the Pasternak foundation is modeled in a similar way. The Pasternak foundation is a gradient elasticity foundation with an internal length scale calculated from $a = \sqrt{G/k}$. Introducing the geometrical relationship $\cos\theta = \mathrm{d}x/\mathrm{d}s$ in Eq. (1) leads to the equivalent variational equality:

$$\delta U = \int_0^L EI\theta'\delta\theta'\,\mathrm{d}s - F\int_0^L \sin\theta\delta\theta\,\mathrm{d}s + \int_0^L kw\cos\theta\delta w + G\theta\cos\theta\delta\theta\,\mathrm{d}s = 0 \tag{2}$$

We note that the work $W$ associated with the present Winkler foundation is different from the following functional:

$$W \neq W^* = \int_0^{x_0} \frac{1}{2}kw^2\,\mathrm{d}x \quad \text{with}\, \delta W^* = \int_0^L kw\cos\theta\delta w - k\frac{w^2}{2}\sin\theta\delta\theta\,\mathrm{d}s \tag{3}$$

The model considered in the paper, based on Eq. (2) cannot be derived from a potential, and typically belongs to a non-conservative elastic problem. However, the linearized problem is conservative and one could consider that the problem is therefore only nonconservative in the large. The static post-buckling path is studied in the following, restricted to possible divergence instabilities.

An integration by parts of Eq. (2) gives the variational equality:

$$\int_0^L \left(-EI\theta'' + ka^2\theta\cos\theta - F\sin\theta\right)\delta\theta + kw\cos\theta\delta w\,\mathrm{d}s + \left[EI\theta'\delta\theta\right]_0^L = 0 \tag{4}$$

This variational equality can be expressed with variational deflection quantities based on $\delta w' = \cos\theta\delta\theta$:

$$\int_0^L \left(\frac{-EI\theta'' + ka^2\theta\cos\theta - F\sin\theta}{\cos\theta}\right)\delta w' + kw\cos\theta\delta w\,\mathrm{d}s + \left[EI\theta'\delta\theta\right]_0^L = 0 \tag{5}$$

An other integration by parts finally gives the variational result:

$$\int_0^L \left[-\left(\frac{-EI\theta'' + ka^2\theta\cos\theta - F\sin\theta}{\cos\theta}\right)' + kw\cos\theta\right]\delta w\,\mathrm{d}s + \left[\left(\frac{-EI\theta'' + ka^2\theta\cos\theta - F\sin\theta}{\cos\theta}\right)\delta w\right]_0^L + \left[EI\theta'\delta\theta\right]_0^L = 0 \tag{6}$$

## 3. Nonlinear fourth-order differential equation

A nonlinear differential equation is obtained from application of the variational principle equation (6):

$$-\frac{1}{\cos\theta}\left(\frac{-EI\theta'' + ka^2\theta\cos\theta - F\sin\theta}{\cos\theta}\right)' + kw = 0 \tag{7}$$

Eq. (7) can also be developed as:

$$EI\frac{\theta''' + \theta'\theta''\tan\theta}{\cos^2\theta} - \frac{ka^2\theta'}{\cos\theta} + \frac{F\theta'}{\cos^3\theta} + kw = 0 \tag{8}$$

This differential equation has already been obtained by Panayotounakos [19] for the Winkler model ($a = 0$). Differentiating Eq. (7) and using $w' = \sin\theta$ leads to a nonlinear fourth-order differential equation in $\theta$:

$$\left[\frac{1}{\cos\theta}\left(\frac{EI\theta'' - ka^2\theta\cos\theta + F\sin\theta}{\cos\theta}\right)'\right]' + k\sin\theta = 0 \tag{9}$$

This nonlinear differential equation is also written as:

$$EI\theta^{(4)} + EI\theta''^2\tan\theta + 3EI\theta'\theta'''\tan\theta + EI\theta'^2\theta''\left(1 + 3\tan^2\theta\right) - ka^2\left(\theta''\cos\theta + \theta'^2\sin\theta\right) + F\frac{\theta'' + 3\theta'^2\tan\theta}{\cos\theta} + k\sin\theta\cos^2\theta = 0 \tag{10}$$

This fourth-order differential equation has also already been obtained by Panayotounakos [19] for the Winkler model ($a = 0$). The natural and essential boundary conditions are given in Eq. (6) as:

$$\left[EI\theta'\delta\theta\right]_0^L = 0 \quad \text{and} \quad \left[\left(\frac{-EI\theta'' + ka^2\theta\cos\theta - F\sin\theta}{\cos\theta}\right)\delta w\right]_0^L = 0 \tag{11}$$

We note that the deflection $w$ can be expressed in term of $\theta$ variable thanks to Eq. (8). The dimensionless parameters are chosen for the dimensionless formulation of the buckling problem:

$$\beta = \frac{FL^2}{EI}; \qquad a^* = \frac{a}{L}; \qquad k^* = \frac{kL^4}{EI}; \qquad w^* = \frac{w}{L} \quad \text{and} \quad s^* = \frac{s}{L} \tag{12}$$

The derivative is now expressed with respect to the dimensionless spatial variable $s^*$, leading to the dimensionless nonlinear fourth-order differential equation:

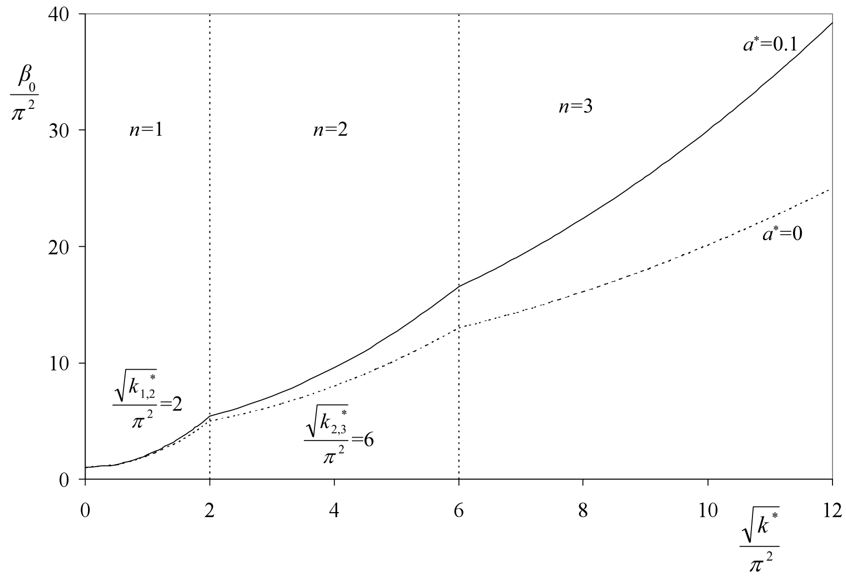


**Fig. 2.** Buckling load with respect to the foundation modulus; Influence of the shear layer; $\frac{\sqrt{k^*_{n,n+1}}}{\pi^2} = n(n+1)$.

$$\theta^{(4)} = -\theta''^2 \tan\theta - 3\theta'\theta''' \tan\theta - \theta'^2\theta''\left(1 + 3\tan^2\theta\right) + k^* a^{*2}\left(\theta'' \cos\theta + \theta'^2 \sin\theta\right) - \beta \frac{\theta'' + 3\theta'^2 \tan\theta}{\cos\theta} - k^* \sin\theta \cos^2\theta \tag{13}$$

with the dimensionless boundary conditions:

$$\left[\theta'\delta\theta\right]_0^1 = 0, \qquad \left[\left(\frac{-\theta'' + k^* a^{*2}\theta \cos\theta - \beta \sin\theta}{\cos\theta}\right)\delta w^*\right]_0^1 = 0 \quad \text{and}$$
$$w^* = -\frac{1}{k^*}\frac{\theta''' + \theta'\theta'' \tan\theta}{\cos^2\theta} + \frac{a^{*2}\theta'}{\cos\theta} - \frac{\beta}{k^*}\frac{\theta'}{\cos^3\theta} \tag{14}$$

We note that the Pasternak contribution (gradient foundation model) does not affect the order of boundary conditions. For a hinged–hinged beam, as considered for instance by Challamel et al. [6,7] for the linearized buckling on Reissner foundation (comprising the Pasternak model), the four boundary conditions are written as:

$$\theta'(0) = 0, \qquad \theta'(1) = 0, \qquad \theta'''(0) = 0 \quad \text{and} \quad \theta'''(1) = 0 \tag{15}$$

## 4. Buckling behavior and asymptotic post-buckling behavior

The buckling load of the hinged–hinged problem can be determined from the linearization of Eq. (13):

$$\theta^{(4)} = k^* a^{*2}\theta'' - \beta\theta'' - k^*\theta \tag{16}$$

with the boundary conditions (15). The solution can be sought in the following format:

$$\theta\left(s^*\right) = \theta_0 \cos\left(n\pi s^*\right) \tag{17}$$

The buckling load of the hinged–hinged beam is calculated as:

$$\beta_0 = \min_n \left\{ (n\pi)^2 + k^* \frac{1 + a^{*2}(n\pi)^2}{(n\pi)^2} \right\} \tag{18}$$

It is easy to check that this formulae leads to the usual buckling load of Winkler model for $a = 0$ (see for instance Timoshenko and Gere [20], Alfutov [21], Bažant and Cedolin [22], Wang et al. [23] for the Winkler problem). The number of half-waves $n$ of the fundamental buckling mode depends on the dimensionless reaction modulus $k^*$, as shown in Figs. 2 and 3.

In fact, one can compute the characteristic foundation modulus at the boundary of each adjacent buckling mode from the nonlinear equation:

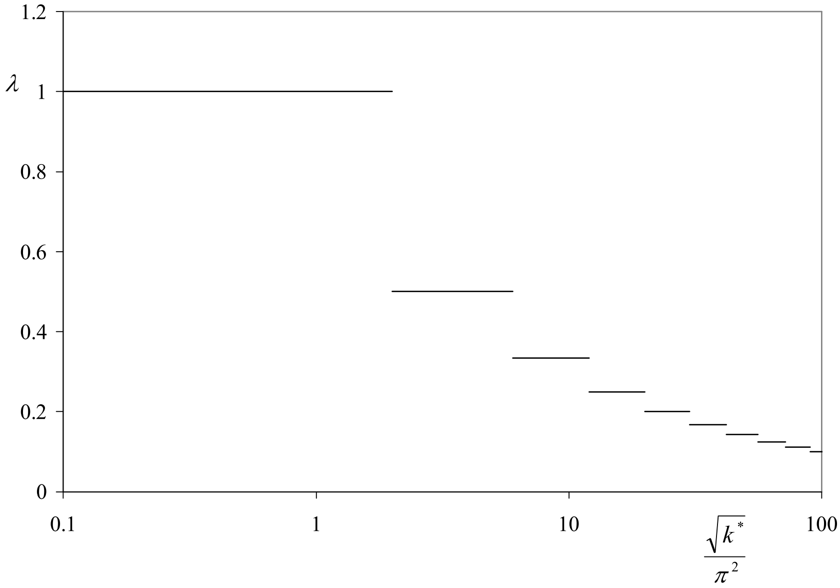


**Fig. 3.** Evolution of the wave length versus the dimensionless foundation modulus for the Pasternak foundation; $\lambda = 1/n$.

$$(n\pi)^2 + k^* \frac{1 + a^{*2}(n\pi)^2}{(n\pi)^2} = \left[(n+1)\pi\right]^2 + k^* \frac{1 + a^{*2}[(n+1)\pi]^2}{[(n+1)\pi]^2} \tag{19}$$

leading to the analytical evaluation of the characteristic foundation modulus

$$\frac{\sqrt{k^*_{n,n+1}}}{\pi^2} = n(n+1) \tag{20}$$

Note that this characteristic foundation modulus $k^*_{n,n+1}$ does not depend on the rotational stiffness of the shear layer $G$. Therefore, the number $n$ of half-wave for the fundamental buckling mode is an increasing function of the foundation modulus $k^*$, given by the relationship:

$$n = E\left[-\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\frac{\sqrt{k^*}}{\pi^2}}\right] + 1 = E\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\frac{\sqrt{k^*}}{\pi^2}}\right] \tag{21}$$

The number $n$ of half-wave for the fundamental buckling mode is asymptotically obtained for a rigid support, as:

$$k^* \to \infty \quad \Rightarrow \quad n \to \frac{\sqrt[4]{k^*}}{\pi} = \sqrt[4]{\frac{k}{EI}}\frac{L}{\pi} \tag{22}$$

A consequence is that for rigid support, the wavelength $\lambda$ is infinitely small:

$$\lim_{k^* \to \infty} n \to \infty \quad \text{or} \quad \lim_{k^* \to \infty} \lambda \to 0 \quad \text{with } \lambda = 1/n \tag{23}$$

The post-buckling behavior of the elastic column on Pasternak foundation is investigated using an asymptotic expansion.

$$\begin{cases} \beta = \beta_0 + \varepsilon\beta_1 + \varepsilon^2\beta_2 + \varepsilon^3\beta_3 + \cdots \\ \theta = \vartheta_0 + \varepsilon\vartheta_1 + \varepsilon^2\vartheta_2 + \varepsilon^3\vartheta_3 + \cdots \end{cases} \quad \text{and} \quad \varepsilon = \theta(0) = \alpha \tag{24}$$

where $\varepsilon$ is a small parameter related to the amplitude of the post-buckling behavior. The methodology is the same as the one presented by Thompson and Hunt [24] for the purely *elastica* problem (see also Koiter [5]). The fundamental path is characterized by no prebuckling deformation $\vartheta_0 = 0$. Furthermore, for symmetrical reasons, it can be shown that some terms are vanishing in the asymptotic expansion:

$$\text{for } k \geqslant 1, \quad \vartheta_{2k}(\bar{s}) = 0 \quad \text{and} \quad \beta_{2k-1} = 0 \tag{25}$$

leading to the third-order asymptotic expansion:

$$\begin{cases} \beta = \beta_0 + \varepsilon^2\beta_2 + \cdots \\ \theta = \varepsilon\vartheta_1 + \varepsilon^3\vartheta_3 + \cdots \end{cases} \quad \text{and} \quad \varepsilon = \theta(0) = \alpha \tag{26}$$

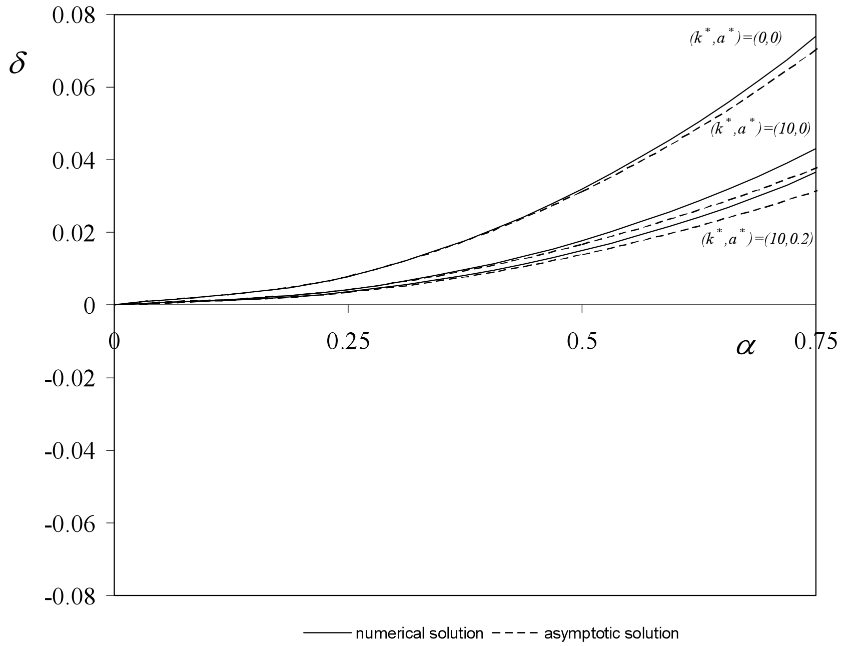


**Fig. 4.** Evolution of the rotation versus the dimensionless buckling load; $\delta = \frac{\beta}{\beta_0} - 1$ versus $\alpha = \theta(0)$. Comparison of the numerical solution with the asymptotic solution; $k^* = 10$.

Inserting this asymptotic expansion in the nonlinear differential equation (13) and considering each power of the small parameter $\varepsilon$ leads to the following system of two differential equations:

$$\begin{cases} \vartheta_1^{(4)} + \beta_0 \vartheta_1'' - k^* a^{*2} \vartheta_1'' + k^* \vartheta_1 = 0 \\ \vartheta_3^{(4)} + \beta_0 \vartheta_3'' - k^* a^{*2} \vartheta_3'' + k^* \vartheta_3 \\ \quad = -\vartheta_1 \vartheta_1''^2 - 3\vartheta_1 \vartheta_1' \vartheta_1''' - \vartheta_1'^2 \vartheta_1'' + k^* a^{*2} \left( -\frac{\vartheta_1^2}{2} \vartheta_1'' + \vartheta_1'^2 \vartheta_1 \right) \\ \qquad - \left( 3\beta_0 \vartheta_1'^2 \vartheta_1 + \beta_0 \frac{\vartheta_1^2}{2} \vartheta_1'' + \beta_2 \vartheta_1'' \right) + \frac{7}{6} k^* \vartheta_1^3 \end{cases} \tag{27}$$

associated with the boundary conditions for $i \in \{1..3\}$:

$$\theta_i'(0) = 0, \qquad \theta_i'(1) = 0, \qquad \theta_i'''(0) = 0 \quad \text{and} \quad \theta_i'''(1) = 0 \tag{28}$$

with the normalization procedure (see also Thompson and Hunt [24]):

$$\vartheta_1(0) = 1 \quad \text{and} \quad \vartheta_i(0) = 0 \quad \text{for } i \geqslant 2 \tag{29}$$

The first differential equation gives the linearized buckling mode:

$$\vartheta_1(s^*) = \cos(n\pi s^*) \quad \text{with } \beta_0 = \min_n \left\{ (n\pi)^2 + k^* \frac{1 + a^{*2}(n\pi)^2}{(n\pi)^2} \right\} \tag{30}$$

Inserting this first-order buckling mode in the second differential equation of (27) leads to:

$$\vartheta_3^{(4)} + \beta_0 \vartheta_3'' - k^* a^{*2} \vartheta_3'' + k^* \vartheta_3 = A \cos(n\pi s^*) + B \cos(3n\pi s^*) \quad \text{with}$$
$$\begin{cases} A = \frac{(n\pi)^4}{4} + \frac{7}{8} k^* + \frac{5}{8} (n\pi)^2 k^* a^{*2} - \frac{3}{8} \beta_0 (n\pi)^2 + \beta_2 (n\pi)^2 \\ B = -5 \frac{(n\pi)^4}{4} + \frac{7}{24} k^* - \frac{1}{8} (n\pi)^2 k^* a^{*2} + \frac{7}{8} \beta_0 (n\pi)^2 \end{cases} \tag{31}$$

Introducing the boundary conditions for the solution of this linear differential equation necessarily shows that $A$ is vanishing, associated with a second-order buckling load factor $\beta_2$ equal to:

$$\beta_2 = \frac{(n\pi)^2}{8} - \frac{k^*}{2(n\pi)^2} - \frac{k^* a^{*2}}{4} \tag{32}$$

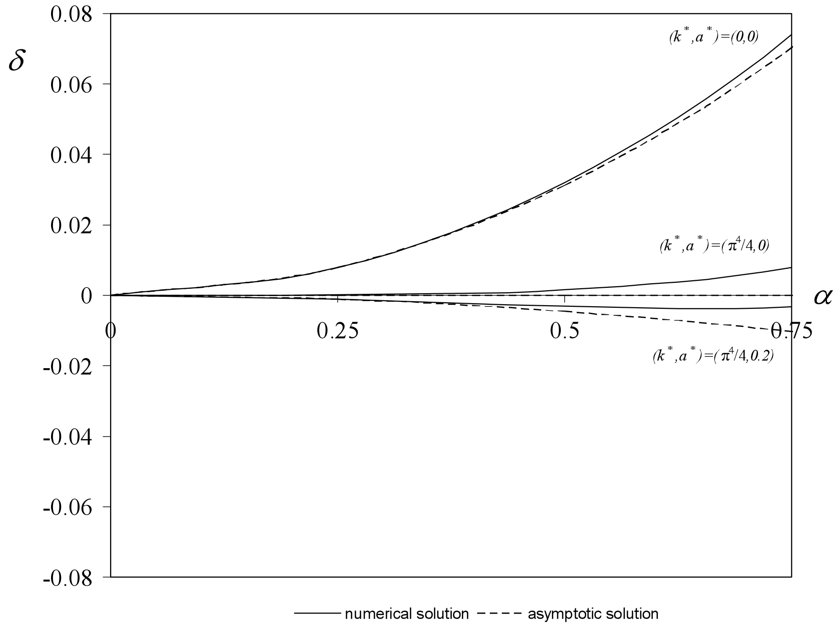


**Fig. 5.** Evolution of the rotation versus the dimensionless buckling load; $\delta = \frac{\beta}{\beta_0} - 1$ versus $\alpha = \theta(0)$. Comparison of the numerical solution with the asymptotic solution; $k^* = \frac{\pi^4}{4}$.

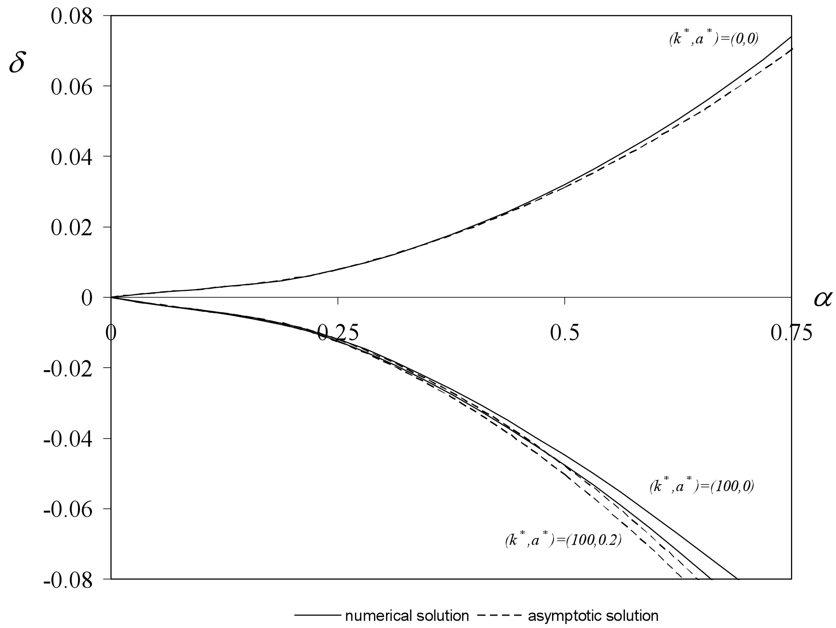


**Fig. 6.** Evolution of the rotation versus the dimensionless buckling load; $\delta = \frac{\beta}{\beta_0} - 1$ versus $\alpha = \theta(0)$. Comparison of the numerical solution with the asymptotic solution; $k^* = 100$.

The post-buckling path of the elastic column on Pasternak foundation is written as:

$$\delta = \frac{\beta}{\beta_0} - 1 = \frac{\beta_2}{\beta_0}\alpha^2 + \cdots \tag{33}$$

As shown by El Naschie [4], $\beta_2$ is a measure of the initial curvature of the post-buckling path. The softening behavior of the post-buckling path leads to instability according to the second variation energy criterion (see also Bažant and Cedolin [22], or Challamel [25]) for conservative systems. For the nonconservative problem studied in this paper, only divergence instabilities are studied. We see from Eq. (32) that $\beta_2$ is positive for the *elastica* problem ($k^* = 0$), leading to a stable post-

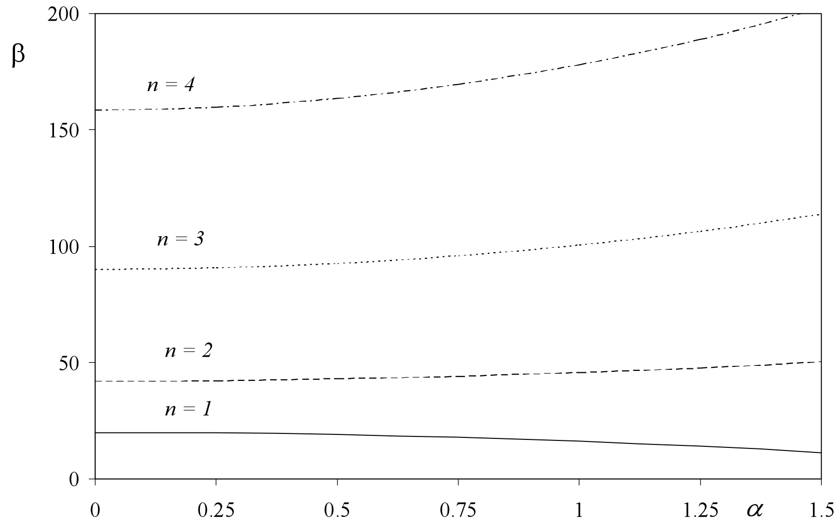


**Fig. 7.** Evolution of the rotation versus the dimensionless buckling load for the asymptotic solution; $k^* = 100$; $a^* = 0$.

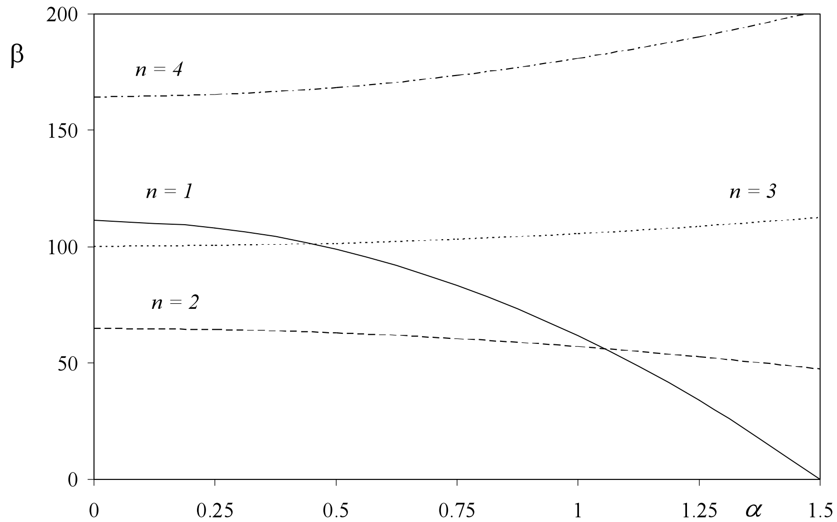


**Fig. 8.** Evolution of the rotation versus the dimensionless buckling load for the asymptotic solution; $k^* = 1000$; $a^* = 0$. Possible mode exchange.

buckling behavior. However, there is a characteristic foundation modulus $k^*_{cr}$ associated with $\beta_2 = 0$ and with a negative initial curvature and consequently an unstable initial post-buckling behavior:

$$k^*_{cr} = \frac{(n\pi)^4}{4}\frac{1}{1+\frac{a^{*2}}{2}(n\pi)^2} \tag{34}$$

Furthermore, it can be shown that:

$$\forall n \geqslant 2, \quad k^*_{cr} = \frac{(n\pi)^4}{4}\frac{1}{1+\frac{a^{*2}}{2}(n\pi)^2} \leqslant \frac{(n\pi)^4}{4} \leqslant k^*_{n-1,n} = (n-1)^2 n^2 \pi^4 \tag{35}$$

This means that the initial curvature $\beta_2$ is negative for all fundamental buckling mode with $n$ half-waves and $n$ greater than 2. In other words, the transition from the stable to the unstable fundamental buckling path is obtained for:

$$k^*_{cr,n=1} = \frac{\pi^4}{4}\frac{1}{1+\frac{a^{*2}}{2}\pi^2} \tag{36}$$

A fundamental stable post-buckling path is obtained for $k^* \leqslant k^*_{cr,n=1}$, whereas a fundamental unstable post-buckling path (softening path) is observed for larger foundation modulus, i.e. $k^* \geqslant k^*_{cr,n=1}$.

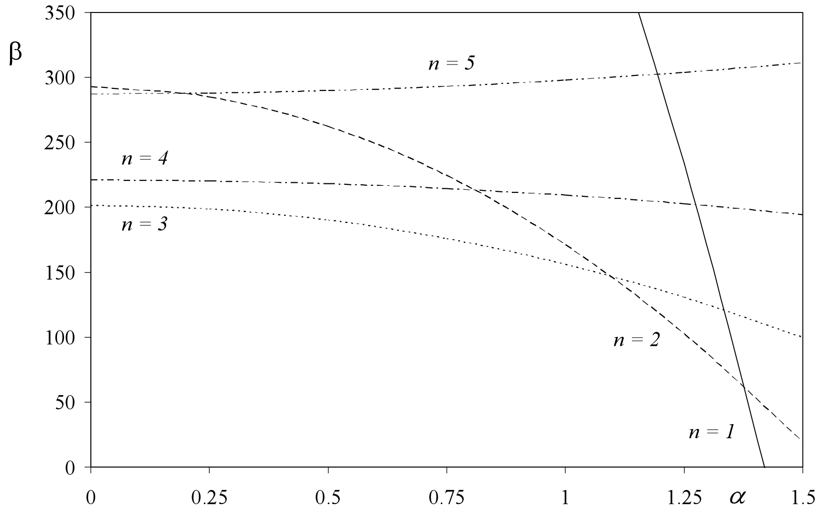


**Fig. 9.** Evolution of the rotation versus the dimensionless buckling load for the asymptotic solution; $k^* = 10\,000$; $a^* = 0$. Possible mode exchange.

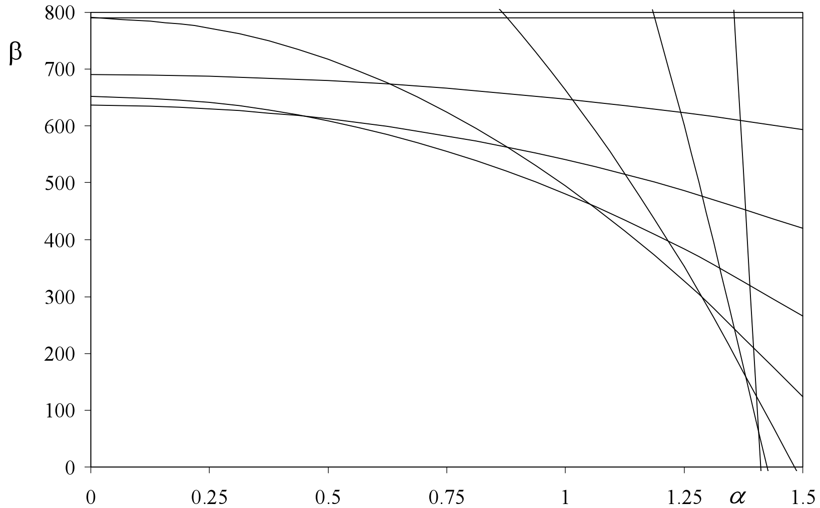


**Fig. 10.** Evolution of the rotation versus the dimensionless buckling load for the asymptotic solution; $k^* = 100\,000$; $a^* = 0$. Possible mode exchange.

## 5. Numerical results

The validity of this asymptotic expansion with respect to the exact numerical solution is given in Figs. 4, 5 and 6. The computation of the nonlinear boundary value problem is based on the MATLAB program bvp4c, a finite difference code that implements the three-stage Lobatto IIIa formula. This is a collocation formula and the collocation polynomial provides a C1-continuous solution that is fourth-order accurate uniformly in the constant interval domain. A stable post-buckling path is observed for sufficiently small foundation modulus with $k^* = 10$ in Fig. 4. The transition between the two regimes is shown in Fig. 5 for $k^* = \pi^4/4$, i.e. the critical foundation modulus for the Winkler foundation. The unstable softening regime is clearly recognized in Fig. 6. Furthermore, the asymptotic solution given by Eq. (33) appears to be valid even for large values of the rotation angle $\alpha$. For instance, in the case $k^* = 100$ and $a^* = 0.2$, the difference between the asymptotic solution and the numerical solution is smaller than 5.7% for $\alpha$ smaller than 0.5 rad. Figs. 7–10 show the possible buckling mode exchange for sufficiently stiff foundation modulus. This is clearly illustrated in Fig. 11 where the fundamental buckling mode with two half-waves is converted into a fundamental buckling mode with only one half-wave.

## 6. Concluding remarks

Some physical arguments have been detailed for the mathematical identification of the nonlocal constitutive law associated with the reaction soil medium. However, the development of some other soil–beam interaction law with vertical and axial coupling, and included in a conservative framework (the interaction law derives from a potential), needs to be further investigated.

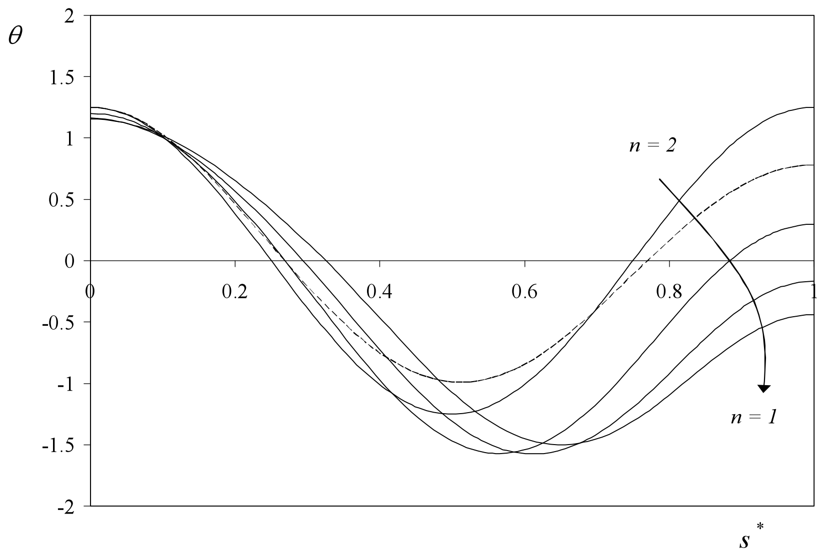


**Fig. 11.** Buckling mode transition from $n = 2$ to $n = 1$; $k^* = 1000$; $a^* = 0$.

The post-buckling of a simply-supported axially loaded elastic beam resting on linearly elastic nonlocal (or gradient) medium has been analytically and numerically studied in this paper. A nonlinear fourth-order differential equation similar to the equation of Panayotounakos [19] has been found in case of Winkler-type interaction law, and has been generalized to take into account some gradient terms.

The structural model considered in this paper is very generic, including possible softening phenomena, mode exchange, and imperfection sensitivity phenomena. The initial post-buckling path is characterized by periodic modes. This initial periodic buckling mode goes through secondary bifurcations leading to possible localization state (see recently Hunt et al. [18]). The gradient terms are expected to play a crucial role in the localization process.

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