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A generalized notion of compliance

Une notion généralisée de compliance

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ABSTRACT

It is a known fact in structural optimization that for structures subject to prescribed non-zero displacements the work done by the loads is not a good measure of compliance, neither is the stored elastic energy. We briefly discuss a possible alternative measure of compliance, valid for general boundary conditions. We also present the adjoint states (necessary for the computation of the structural derivative) for the three functionals under consideration.

RÉSUMÉ

Il est un fait établi dans la littérature sur l'optimisation de structures que, pour mésurer la compliance de structures dont une partie de la frontière a un déplacement prescrit non nul, le travail effectué par les charges n'est pas un critére correcte; l'energie élastique emmagasiné dans le corps n'en est non plus. Cette notte propose un critére alternative pour mésurer la compliance d'une structure pour des conditions de frontière très générales. Les états adjoints (nécéssaires pour le calcul de la dérivée structurale) sont presentés pour les trois fonctionnelles considerées.

Version française abrégée

En optimisation de structures, il est fréquent de chercher des structures raides (qui résistent sans trop se déformer). Rémarquons qu'il s'agit d'optimiser la structure par rapport a un paramètre structural qui peut être la forme du corps, ou le tenseur élastique du matériau, entre autres. On a l'habitude d'achever cet objectif en minimisant le travail effectué par les forces appliquées (2) (cette quantité est parfois appelée « compliance », un synonime de « souplesse », opposé à la « raideur »). Dans le cas òu les conditions de Dirichlet sont nulles, c'est-à-dire pour des structures modelées par l'équation (1), cette approche est justifiée : comme f et g sont des charges données, leur travail será petit si le déplacement u est petit. Pour des structures modelées par l'équation (1), il est facile à vérifier que le travail (2) est égal à deux fois l'énergie élastique emmagasinée (3).

Le cas opposé est celui òu toutes les charges sont nulles, la déformation élastique étant due à un déplacement prescrit non nul ; ce cas est décrit par l'équation (5). Une telle structure mérite être appelée « raide » si l'éffort nécessaire pour imposer le déplacement prescrit $\bar{u}$ est grand. En conséquence, on peut chercher des structures raides en *maximisant* le

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travail (6), ce qui va (en apparence au moins) dans la direction opposée par rapport au premier cas. Une fois de plus, il est facile à vérifier que le travail (6) est égal à deux fois l'énergie élastique emmagasinée (3).

Nous proposons la quantité (8) comme une mésure de la souplesse d'une structure sujette à des conditions de bord générales, c'est-à-dire pour des structures modelées par l'équation (7). Rémarquons que, si on se situe dans le prémier cas (1), cette quantité est égale au travail effectué par les charges (2), tandis que pour le second cas (5) la quantité (8) est égale à moins le travail (6). Rémarquons aussi que l'état adjoint associé au calcul de la dérivée de (8) (par rapport à n'importe quel paramètre structural) est nul.

1. Introduction

It is common in structural optimization to look for stiff structures (that is, structures which do not deform much): beams that do not bend much, bridges that can stand heavy loads, and so on (see, for instance, [1]). This is usually achieved by minimizing the work done by the applied loads (this quantity is sometimes called "compliance", as opposed to "stiffness"). If all Dirichlet boundary conditions are homogeneous, then the work of the applied loads is equal to the elastic energy stored in the body. However, it has been observed that, if non-zero Dirichlet boundary conditions are considered, the work of the applied loads is no longer a good measure of the compliance of the structure, neither is the stored elastic energy (see [2, Section 3.1] and [3]). The goal of the present Note is to propose a quantity called "generalized compliance" which describes correctly the behaviour of a structure under any boundary conditions.

The *design domain* is an open set $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$; E will stand for the *elasticity tensor*, u designates the state variable, that is, the *displacement*, and $e(u)$ is the symmetric part of the jacobian matrix of u. The boundary $\partial\Omega$ of the design domain is partitioned into disjoint parts Γ_N (where some surface loads may be applied) and Γ_D (where some displacements are imposed); ν denotes the outward unit normal to $\partial\Omega$.

2. Zero Dirichlet boundary conditions

We start by considering the following elliptic problem describing the equilibrium of a linearly elastic structure:

$$\begin{cases} -\operatorname{div} E\, e(u) = f, & \text{in } \Omega \\ u = 0, & \text{in } \Gamma_D \\ E\, e(u)\nu = g, & \text{in } \Gamma_N \end{cases} \tag{1}$$

In the previous formulation, only zero Dirichlet boundary conditions are allowed (or no Dirichlet boundary conditions appear at all, when $\Gamma_D = \varnothing$).

We are interested in optimizing for stiffness a structure defined by the domain Ω and by the elasticity tensor $E \in L^\infty(\Omega; \mathbb{R}^{d^4})$. Note that high stiffness is equivalent to low compliance. It does not matter whether we are talking about shape optimization (when the shape of Ω varies), free material optimization (when the elastic tensor E is the optimization parameter), or any other kind of optimization.

The most common way to evaluate the compliance of an elastic structure is through the work done by the applied loads

$$\mathcal{W} = \int_\Omega f \cdot u \,\mathrm{d}x + \int_{\Gamma_N} g \cdot u \,\mathrm{d}s \tag{2}$$

It is not difficult to see that the smaller the work done by the applied loads, the stiffer the structure. This is so because f and g are fixed (they are data of the problem); thus, the work done by the loads f and g is small if the displacement u is small, that is, if the structure does not deform much.

It is a simple exercise to check that $\mathcal{W} = 2\mathcal{E}$, where $\mathcal{E}$ is the elastic energy stored in the body.

$$\mathcal{E} = \frac{1}{2}\int_\Omega E\, e(u) : e(u) \,\mathrm{d}x \tag{3}$$

It suffices to take u as test function in the variational formulation of (1):

$$\begin{cases} \text{find } u \in V \text{ such that} \\ \displaystyle\int_\Omega E\, e(u) : e(v) \,\mathrm{d}x = \int_\Omega f \cdot v \,\mathrm{d}x + \int_{\Gamma_N} g \cdot v \,\mathrm{d}s, \quad \forall v \in V \end{cases} \tag{4}$$

where V is the space of functions in $H^1(\Omega)$ whose trace on Γ_D is zero.

3. Structures subject to a prescribed displacement

The extreme opposite case is when no loads are applied at all, the deformation being caused by non-zero Dirichlet conditions:

$$\begin{cases} -\operatorname{div} E\,e(u) = 0, & \text{in } \Omega \\ u = \bar{u}, & \text{in } \Gamma_D \\ E\,e(u)\nu = 0, & \text{in } \Gamma_N \end{cases} \tag{5}$$

The description of a stiff structure as one which "does not deform much" is no longer valid; the displacement is prescribed on Γ_D. Such a structure should be called stiff if the effort it takes to impose the displacement $\bar{u}$ on Γ_D is large. That is, the work done by the force $g = E\,e(u)\nu$ should be large. Note that g cannot be called "applied load" now, since it is not a datum of problem (5).

So, a structure described by problem (5) should be called "stiff" if the work

$$\int_{\Gamma_D} E\,e(u)\nu \cdot \bar{u}\,\mathrm{d}s \tag{6}$$

is *large*, which is very different from the situation described in Section 2, where for a structure to be stiff the work done by the applied loads had to be *small*.

Again, it is easy to check that the quantity (6) is equal to twice the elastic energy $\mathcal{E}$ stored in the body. It suffices to integrate by parts in (3) and use the boundary conditions in (5).

In this case then, one should *maximize* the stored elastic energy in order to obtain a stiff structure, which is the opposite of the situation described in Section 2. This phenomenon has already been observed in the literature; see [4, Remark 11], [3] and [5, Remark 9].

4. Loads and prescribed displacement

Consider now the general case, when loads are applied (f in Ω, g on Γ_N) and a non-zero displacement $\bar{u}$ is prescribed on Γ_D:

$$\begin{cases} -\operatorname{div} E\,e(u) = f, & \text{in } \Omega \\ u = \bar{u}, & \text{in } \Gamma_D \\ E\,e(u)\nu = g & \text{in } \Gamma_N \end{cases} \tag{7}$$

How can one define the notion of stiff structure (or its reverse, the compliance) in this context? Should one minimize or maximize the stored elastic energy $\mathcal{E}$?

Claim. *We propose the difference*

$$\mathcal{C} = \int_{\Omega} f \cdot u\,\mathrm{d}x + \int_{\Gamma_N} g \cdot u\,\mathrm{d}s - \frac{1}{2}\int_{\Omega} E\,e(u) : e(u)\,\mathrm{d}x \tag{8}$$

as a measure of compliance for structures subject to general boundary conditions, described by (7).

The above quantity will be called *generalized compliance*. It is clear that $\mathcal{C}$, $\mathcal{W}/2$ and $\mathcal{E}$ are equal for structures governed by (1). On the other hand, for structures governed by (5), $\mathcal{C}$ and $\mathcal{E}$ are equal in modulus and have opposite signs. So, for any boundary conditions, one should minimize $\mathcal{C}$ in order to obtain a stiff structure.

It should be noted that an independent work (see [6]) points to the same conclusion, although in a discretized setting.[1]

5. Computation of sensitivities

In Section 6, a comparison will be made (by means of numerical tests) between the objective functionals $\mathcal{W}$, $\mathcal{E}$ and $\mathcal{C}$ in terms of how adequately they measure the performance of a structure governed by (7). Since a gradient algorithm will be used, the sensitivity of the objective functional with respect to some structural parameter(s) — which, for mere convenience, will be assumed to be the material tensor itself — has to be determined.

Let us begin by writing the variational formulation of (7):

[1] The results of [6] were brought to the knowledge of the authors at a late stage, after the publication of the preprint [7].

$$\begin{cases} \text{find } u \in \mathcal{U} \text{ such that} \\ \displaystyle\int_\Omega E\, e(u) : e(v)\, \mathrm{d}x = \int_\Omega f \cdot v\, \mathrm{d}x + \int_{\Gamma_N} g \cdot v\, \mathrm{d}s, \quad \forall v \in V \end{cases} \tag{9}$$

where $\mathcal{U}$ is the affine space of functions in $H^1(\Omega)$ whose trace on Γ_D is equal to $\bar{u}$ and V is the space of functions in $H^1(\Omega)$ whose trace on Γ_D is zero. Because f and g are data of problem (7), hence they do not depend on E, the derivatives of $\mathcal{W}$, $\mathcal{E}$ and $\mathcal{C}$ at E in a direction δE are given, respectively, by

$$\delta\mathcal{W} = \int_\Omega f \cdot \delta u\, \mathrm{d}x + \int_{\Gamma_N} g \cdot \delta u\, \mathrm{d}s \tag{10}$$

$$\delta\mathcal{E} = \frac{1}{2}\int_\Omega \delta E\, e(u) : e(u)\, \mathrm{d}x + \int_\Omega E\, e(\delta u) : e(u)\, \mathrm{d}x \tag{11}$$

and

$$\delta\mathcal{C} = \int_\Omega f \cdot \delta u\, \mathrm{d}x + \int_{\Gamma_N} g \cdot \delta u\, \mathrm{d}s - \frac{1}{2}\int_\Omega \delta E\, e(u) : e(u)\, \mathrm{d}x - \int_\Omega E\, e(\delta u) : e(u)\, \mathrm{d}x \tag{12}$$

where by δu we mean the derivative of u at E in the direction δE; note that, because $\bar{u}$ is fixed, it vanishes upon differentiation, thus yielding $\delta u = 0$ on Γ_D. Of course, to obtain a final formula in terms of δE only, it is essential to eliminate δu from these expressions since, being u the solution of (9), it depends in a highly implicit manner on E. Amazingly, in the case of the generalized compliance $\mathcal{C}$, δu disappears altogether from (12): multiplication of the state equation in (7) by δu and integration by parts gives

$$\int_\Omega E\, e(u) : e(\delta u)\, \mathrm{d}x = \int_\Omega f \cdot \delta u\, \mathrm{d}x + \int_{\Gamma_N} g \cdot \delta u\, \mathrm{d}s$$

therefore, (12) evaluates simply to

$$\delta\mathcal{C} = -\frac{1}{2}\int_\Omega \delta E\, e(u) : e(u)\, \mathrm{d}x \tag{13}$$

However the same reasoning does not prove to be effective either for the work $\mathcal{W}$ of the applied loads or for the stored energy $\mathcal{E}$; with those objective functionals, the *adjoint method* must be used.

First observe that by differentiating (9), with respect to E and in a direction δE, one concludes that the variation δu of u is the solution of the variational problem

$$\begin{cases} \text{find } \delta u \in V \text{ such that} \\ \displaystyle\int_\Omega E\, e(\delta u) : e(v)\, \mathrm{d}x = -\int_\Omega \delta E\, e(u) : e(v)\, \mathrm{d}x, \quad \forall v \in V \end{cases} \tag{14}$$

This problem alone will not suffice to get rid of δu in the derivatives (10) and (11); to complete that task, an auxiliary problem has to be introduced — the so-called *adjoint problem*. The adjoint method is a well documented procedure, whose technical aspects will be kept aside for simplicity's sake; it can be found, for instance, in [8, Section 4.3], [9, Chapter 5], or [10, Theorem 4.1].

When the work $\mathcal{W}$ is considered, the *adjoint state* p is defined as the solution of

$$\begin{cases} \text{find } p \in V \text{ such that} \\ \displaystyle\int_\Omega E\, e(p) : e(v)\, \mathrm{d}x = \int_\Omega f \cdot v\, \mathrm{d}x + \int_{\Gamma_N} g \cdot v\, \mathrm{d}s, \quad \forall v \in V \end{cases} \tag{15}$$

which is nothing more than the variational formulation of

$$\begin{cases} -\operatorname{div} E\, e(p) = f, & \text{in } \Omega \\ p = 0, & \text{in } \Gamma_D \\ E\, e(p)\, \nu = g, & \text{in } \Gamma_N \end{cases} \tag{16}$$

Since δu and p both belong to V, we conclude, first by (15) and then by (14), that

$$\int_\Omega f \cdot \delta u\, \mathrm{d}x + \int_{\Gamma_N} g \cdot \delta u\, \mathrm{d}s = \int_\Omega E\, e(p) : e(\delta u)\, \mathrm{d}x = -\int_\Omega \delta E\, e(u) : e(p)\, \mathrm{d}x$$

that is, the derivative (10) writes as

$$\delta\mathcal{W} = -\int_\Omega \delta E\, e(u) : e(p)\, \mathrm{d}x \tag{17}$$

When the stored energy $\mathcal{E}$ is to be differentiated, it gives rise to the same adjoint state p, solution of (16). Note that, since u satisfies (9), the variational formulation (15) is equivalent to

$$\begin{cases} \text{find } p \in V \text{ such that} \\ \displaystyle\int_\Omega E\, e(p) : e(v)\, \mathrm{d}x = \int_\Omega E\, e(u) : e(v)\, \mathrm{d}x, \quad \forall v \in V \end{cases} \tag{18}$$

Similarly to the previous case, using (18) and then (14), plus the fact that δu and p are in the space V of test functions, it follows that

$$\int_\Omega E\, e(\delta u) : e(u)\, \mathrm{d}x = \int_\Omega E\, e(p) : e(\delta u)\, \mathrm{d}x = -\int_\Omega \delta E\, e(u) : e(p)\, \mathrm{d}x \tag{19}$$

Thus, the derivative (11) can be reduced to

$$\delta\mathcal{E} = \int_\Omega \delta E \left\{ \frac{1}{2} e(u) : e(u) - e(u) : e(p) \right\} \mathrm{d}x \tag{20}$$

Remark 1. Regarding the derivative of $\mathcal{C}$, with u solution of (7), the adjoint method can be likewise followed, yielding in this case a null adjoint state $p = 0$.

Remark 2. The only difference between problems (16) and (7) is the space where the solutions are to be found (V in one case, W in the other); in other words, the only difference lays in the Dirichlet condition. For structures with no prescribed displacements (considered in Section 2), the two problems are identical, which means that the minimization of either $\mathcal{W}$ or $\mathcal{E}$ is a self-adjoint problem.

Remark 3. It may seem strange that two different quantities, $\mathcal{E}$ and $\mathcal{W}$, give rise to the same adjoint state p, defined by problem (16). This is especially queer for structures with no prescribed displacements (considered in Section 2) where $\mathcal{W} = 2\mathcal{E}$. Recall, however, that

$$\int_\Omega f \cdot u\, \mathrm{d}x + \int_{\Gamma_N} g \cdot u\, \mathrm{d}s = \int_\Omega E\, e(u) : e(u)\, \mathrm{d}x \tag{21}$$

only when u satisfies problem (1). Otherwise, if u is some arbitrary function in V, formula (21) has no reason to hold.

6. Numerical tests

The numerical simulations in [6, Section 4.2] demonstrate that minimizing the work of the applied loads (for a structure subject to applied loads only) is equivalent to maximizing the stored elastic energy (for a structure subject to an equivalent imposed displacement); they also show clearly that both of those formulations can be recovered from the generalized compliance formulation. The main concern in the present section is to see what happens when mixed non-homogeneous boundary conditions are present, and how each of the three measures $\mathcal{W}$, $\mathcal{E}$ and $\mathcal{C}$ describe the structure's behaviour in that case.

In order to make a comparison in the previous terms, we consider the problem depicted in Fig. 1: a rectangular design domain $\Omega = [0, 6] \times [0, 3]$, clamped at the lower corners $A = [0, 0.3] \times \{0\}$ and $D = [5.7, 6] \times \{0\}$. A load $g = (0, -0.7)$ is uniformly distributed on the segment $B = [1.95, 2.05] \times \{0\}$, which is roughly equivalent to a concentrated force of $(0, -0.07)$, and a displacement $\bar{u} = (0, -1.47)$ is prescribed on $C = [3.95, 4.05] \times \{0\}$. The values of g and $\bar{u}$ have been chosen to obtain a similar displacement on the segments B and C; this is, of course, aimed at mimicking a problem with equal applied loads on the segments B and C, a case in which one has a clear picture on how a solution should look like.

As we shall be dealing with optimization via distribution of isotropic material, the material tensor E will take the form $E = \rho E_0$, where ρ is a material density continuously varying within the interval $[0, 1]$ and E_0 is a fixed isotropic linearly elastic tensor, whose Lamé coefficients are $\lambda = 0.2$ and $\mu = 0.3$:

$$E_0\, e = \lambda\, \mathrm{tr}(e)\, I + 2\mu\, e$$

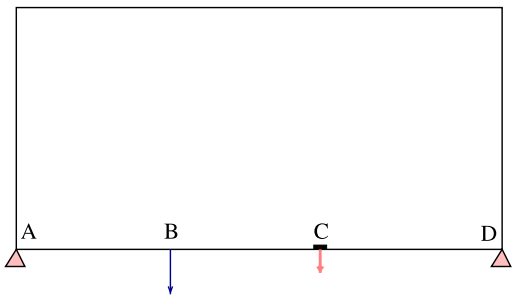

Fig. 1. Design domain and boundary conditions of the problem.

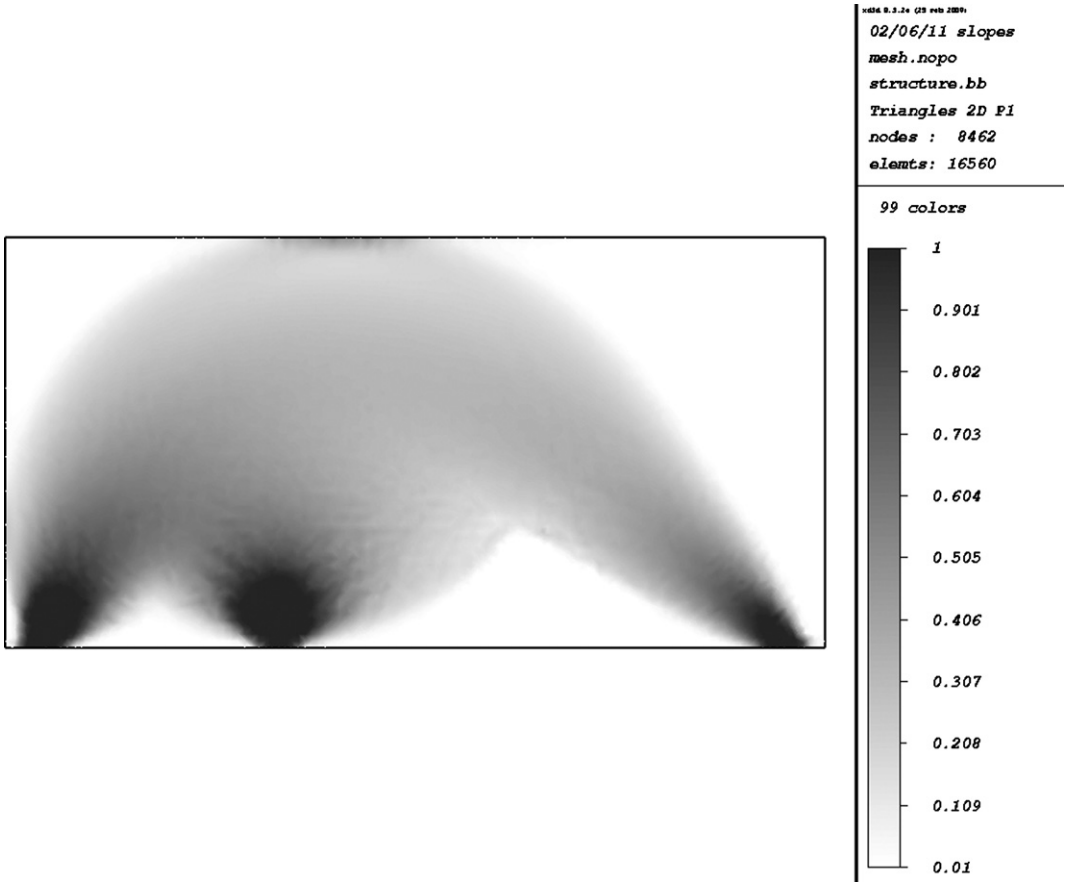

Fig. 2. Density ρ obtained by minimization of the stored energy.

In this framework, the objective functional $\mathcal{W}$, $\mathcal{E}$, or $\mathcal{C}$, is dependent on the structural parameter ρ. Formulas (13), (17) and (20), reduce to

$$\delta\mathcal{C} = \int_\Omega \delta\rho \left[-\frac{1}{2} E_0 e(u) : e(u) \right] \mathrm{d}x$$

$$\delta\mathcal{W} = \int_\Omega \delta\rho \left[-E_0 e(u) : e(p) \right] \mathrm{d}x$$

and

$$\delta\mathcal{E} = \int_\Omega \delta\rho \left[E_0 e(u) : e\left(\frac{u}{2} - p \right) \right] \mathrm{d}x$$

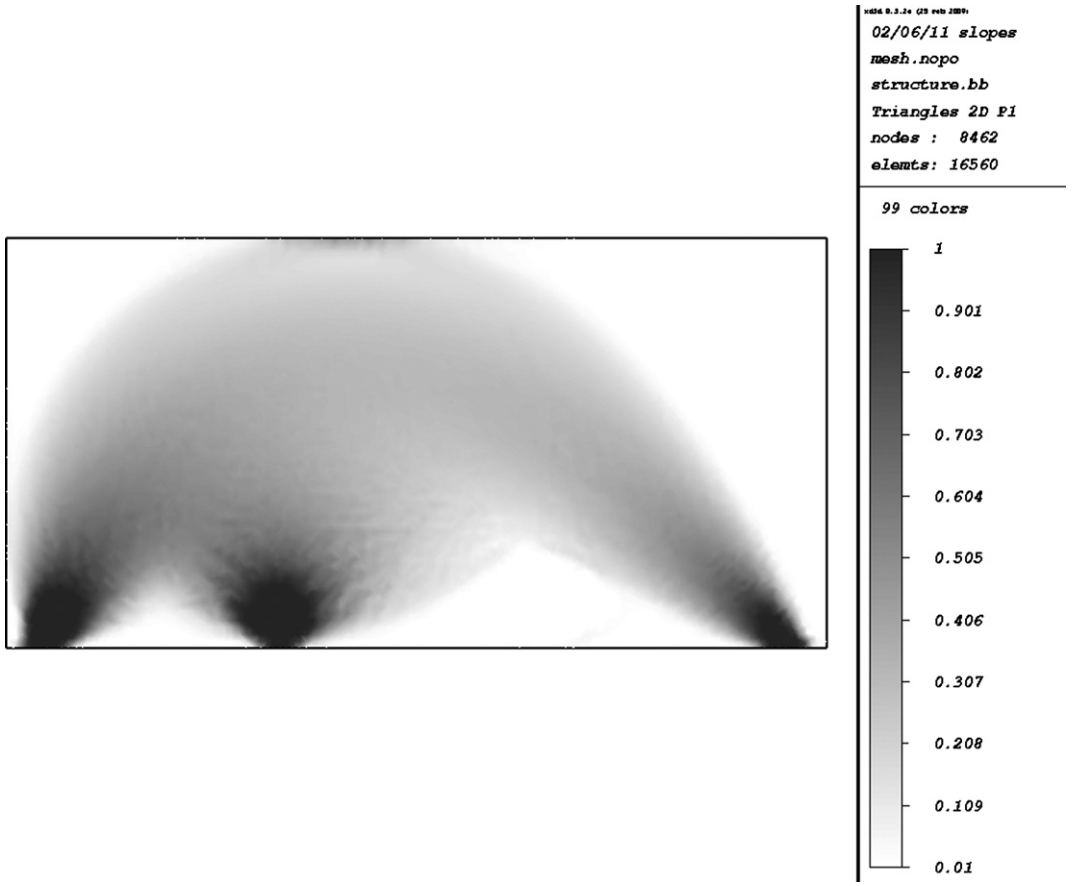

Fig. 3. Density ρ obtained by minimization of the work done by the applied loads.

respectively. The scalar function between square brackets will then be interpreted as the gradient of the objective functional with respect to ρ. Also, there is a constraint on the "volume" of material: it should fill 25% of the design domain, that is,

$$\int_{\Omega} \rho \, \mathrm{d}x = 4.5$$

obviously, the gradient of this constraint with respect to ρ is the constant function 1. Another constraint is that the pointwise "density" ρ should stay in $[0.01, 1]$. Having identified the gradients of both the objective and the constraints, a descent algorithm for constrained optimization is applied, namely the one introduced in [12] and studied in [11].

Note that we do not penalize intermediate densities: the algorithm is just set to run several iterations, until a reasonably well defined picture is obtained; one should then expect not a black-and-white design, but instead a greyscale density.

The implementation is made in the finite element object oriented language `FreeFem++` (see [13]), with a fixed mesh of 16 560 triangular elements; $\rho_0 = 0.25$. Visualization of results is made with the software `xd3d` (see [14]).

The numerical results are presented in Figs. 2–4, where the "density" ρ is represented through different levels of grey (1 is black, 0 is white). In Fig. 2, the stored energy $\mathcal{E}$ is minimized; the obtained value of $\mathcal{E}$ is 0.03417205. In Fig. 3, the work $\mathcal{W}$ is minimized; the final value of $\mathcal{W}$ is 0.0617249. In Fig. 4, the generalized compliance $\mathcal{C}$ is minimized; the value of $\mathcal{C}$ for the optimized structure is -0.0123051.

Observe how the functionals $\mathcal{W}$ and $\mathcal{E}$ seem "blind" to the presence of the non-zero prescribed displacement. Between the three functionals, the generalized compliance $\mathcal{C}$ is the only one that "takes advantage" of the non-zero prescribed displacement.

7. Conclusions

The present work proposed a physical quantity which is an adequate measure for the compliance (as opposed to "stiffness") of a structure, for general boundary conditions, that is, allowing for non-zero applied loads in simultaneous with non-zero prescribed displacements. A numerical example illustrates our claim.

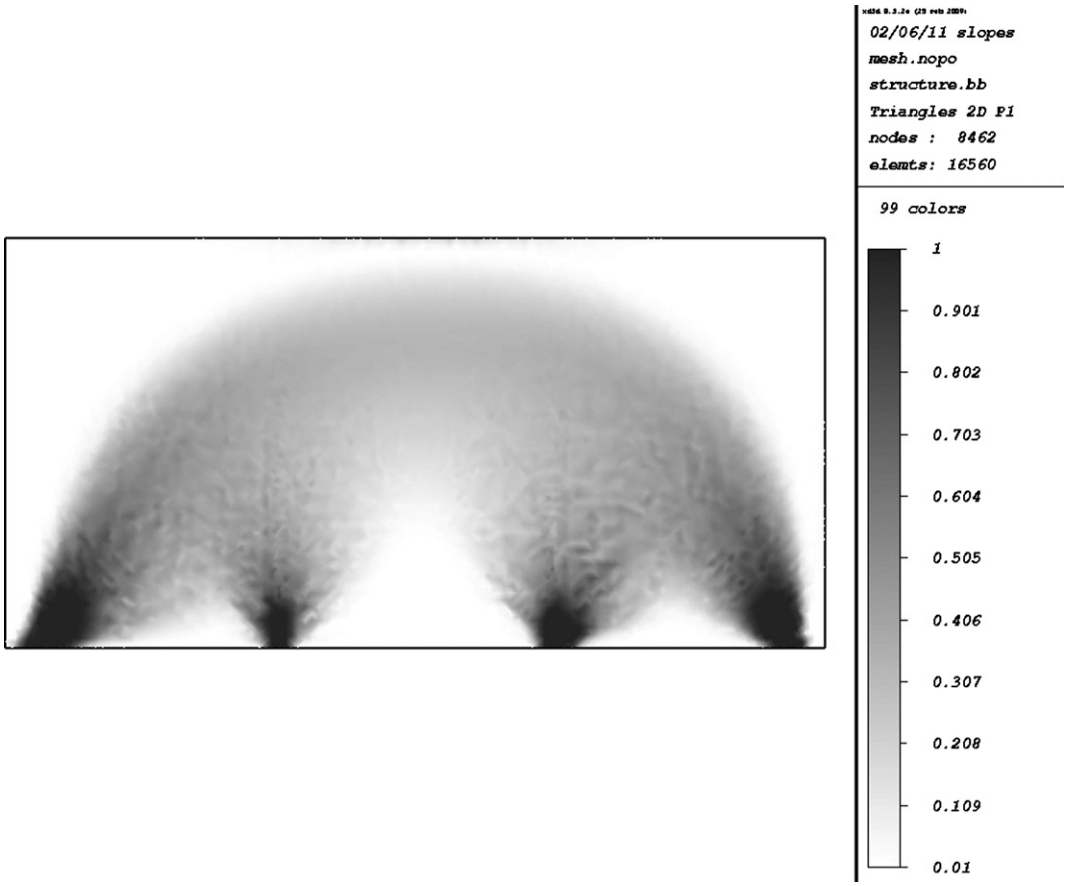

Fig. 4. Density ρ obtained by minimization of the generalized compliance.

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