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Convex analysis and ideal tensegrities

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Dedicated to Michel Jean on the occasion of his seventieth birthday

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ABSTRACT

A theoretical framework based on convex analysis is formulated and developed to study tensegrity structures under steady-state loads. Many classical results for ideal tensegrities are rationally deduced from subdifferentiable models in a novel mechanical perspective. Novel energy-based criteria for rigidity and pre-stressability are provided, allowing to formulate numerical algorithms for computations.

1. Introduction

In the middle of the past century artists, architects, mathematicians and engineers have been attracted by *tensegrities*,[1] a special class of space trusses made up of struts and guys and rigidized by a state of self-stress, sharing peculiar characteristics in terms of beauty, lightness and stiffness.

At their earliest appearance,[2] tensegrity structures were addressed primarily by artists and architects. Later, from the 70s, mathematicians and engineers approached the analysis of this special class of structures, and a large amount of technical literature appeared both in mathematical [3–6] and engineering [7–12] contexts, as recently reviewed in [13–15].

Due to their unique and fascinating properties, tensegrity structures are becoming more and more common in civil (e.g., domes, bridges, towers, roofs, deployable structures) and mechanical (e.g., robots, special mechanisms) engineering [2,16]. Moreover, in recent years tensegrity-based models have been successfully applied for modeling the mechanical behavior of biological structures such as molecules and cells, and the idea that nature itself employ a tensegrity rationale was proposed [17].

Tensegrities structures exhibit a complex mechanical behavior due to the non-linear response of their structural members (cables and bars), usually modeled as equivalent one-dimensional elements, connected each other by frictionless pin-joints. If a small strain theory applies, disregarding non-linear and inelastic features, as damage, viscous and buckling effects, a bilateral linearly elastic relationship between applied force and length variation is usually assumed to be representative for bar elements. Instead, cables can be described as unilateral non-linear elements with no resistance to shortening from

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[1] The term "tensegrity" is an acronym coined by Richard B. Fuller [1] by assembling the words "tensile" and "integrity".
[2] A brief history of origin and first developments of tensegrities can be found in [2].

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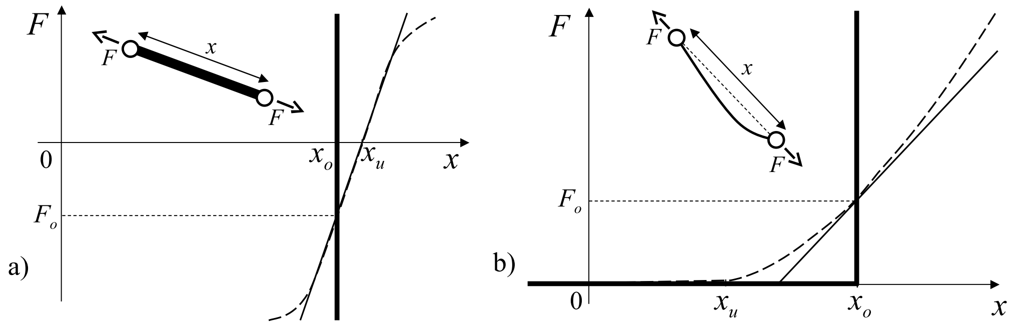

Fig. 1. Relationships between applied force F and end-to-end length x for bars (a) and cables (b). Real behavior (- - -), piecewise-linear elastic model (—), and piecewise-linear ideal model (━). As notation, x_u and x_o denote the unstressed and the reference element lengths, respectively, and F_o is the pre-stress value of F.

the unstressed configuration[3] and are characterized by a non-linear lengthening response essentially due to sag effects [18]. Therefore, cables equivalent force-extension response can be suitably described by a piecewise-linear relationship corresponding to a zero stiffness in compression and to an equivalent linear behavior in tension. Fig. 1 summarizes previous considerations.

Characterization of rigidity (absence of infinitesimal admissible nodal displacements, namely absence of admissible mechanisms) and pre-stressability (existence of a possible self-equilibrated pre-stress state, tensile in each cable) both in a reference configuration are the two main research issues in this context. These problems correspond respectively to the analysis of kinematic determinacy[4] and to the existence of a special kind of static indetermination. Since kinematic and static indeterminacy does not depend on the deformability of structural elements, an *ideal* inextensible behavior around their reference configuration, bilateral for bars and unilateral for cables, can be successfully considered (see Fig. 1). Tensegrities characterized by members with an ideal behavior are usually referred to as *ideal*, and a number of results focusing on rigidity and pre-stressability of ideal tensegrity structures have been recently provided.

Tensegrity problems can be effectively approached by using graph theory (Roth and Witheley [3]). In fact, mathematical tools of group and representation theories have been proved to allow tensegrities characterized by prescribed stability and symmetry features to be conceived [6].

A first attempt to analyze tensegrities by an energetic formulation was proposed in [4], by introducing an energy function accounting for members' deformability, and then assuming structure to be not ideal.

Obviously, energetic analysis of ideal tensegrity structures is not possible in a classical sense. Nevertheless, the definition of rigidity and pre-stressability through an energy approach would enhance the perspective on ideal tensegrity mechanics, providing a novel mechanical interpretation of some basic results.

In present work, following [19], restrictions imposed *a priori* by ideal structural members are regarded as internal constraints on the tensegrity reference configuration and they are enforced by a structural free-energy functional in the mathematical framework of convex analysis [20,21]. The definition of ideal tensegrity free-energy by means of a convex analysis approach (Section 3) allows to deduce new criteria for rigidity and pre-stressability, explicitly accounting for both kinematics and statics unilateral aspects. In Section 4 the rigidity concept is associated to a minimization problem of a non-classically differentiable convex free-energy [20,21], while in Section 5 pre-stressability is formulated in terms of energetic features. Moreover, by gathering convex analysis and Principle of Virtual Works, that is by coupling mathematical tools with mechanical interpretation, the duality between static and kinematic unilateral problems is consistently proved (Section 6). Finally, in the last part of the paper (Section 7), perspective indications to extend present approach to non-ideal and inelastic tensegrities are briefly traced.

2. Notation and preliminary background

Let E be the three-dimensional Euclidean space, let V be the vector space associated with E, endowed with the inner product $\mathbf{a}\cdot\mathbf{b}\in\mathbb{R}$ between $\mathbf{a}$, $\mathbf{b}\in\mathsf{V}$, and let $(\mathbf{a}\cdot\mathbf{a})^{1/2}=\|\mathbf{a}\|$ be the Euclidean norm of $\mathbf{a}$. Let $(O,\mathbf{i},\mathbf{j},\mathbf{k})$ be a Cartesian frame in E, being $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ an orthonormal basis.

For $\mathbf{a}\in\mathsf{V}$, read $\mathbf{a}\leqslant 0$ componentwise, that is "each component of $\mathbf{a}$ is non-positive". Furthermore, between two sets $\mathcal{A}=\{\mathbf{a}_1,\dots,\mathbf{a}_n\}$ and $\mathcal{B}=\{\mathbf{b}_1,\dots,\mathbf{b}_n\}$ with the same cardinality n, the following symbols are defined: $\mathcal{A}\boxplus\mathcal{B}=\{\mathbf{a}_1+\mathbf{b}_1,\dots,\mathbf{a}_n+\mathbf{b}_n\}$, $\alpha\mathcal{A}=\{\alpha\mathbf{a}_1,\dots,\alpha\mathbf{a}_n\}$ with $\alpha\in\mathbb{R}$, $\langle\mathcal{A},\mathcal{B}\rangle=\sum_{j=1}^{n}\mathbf{a}_j\cdot\mathbf{b}_j$, and $\mathcal{A}\neq\mathcal{B}\Leftrightarrow\exists j$ such that $\mathbf{a}_j\neq\mathbf{b}_j$, with $\mathbf{a}_j,\mathbf{b}_j\in\mathsf{V}$. Symbol $\emptyset$ denotes the empty set and $\varnothing_n$ the set made up of n null vectors.

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[3] Some authors consider an unilateral response not only for elements bearing only traction states (that is guys, typically realized by cables), but also for structural members able to bear only compressive forces (namely struts). Nevertheless, strut function is usually obtained by bilateral bars, and therefore their unilateral behavior will be not considered here.

[4] The kinematic determinacy is addressed as either infinitesimal rigidity [3] or rigidity [13]. In present work, the term rigid instead of "kinematically determinate" or "infinitesimally rigid" will be employed.

Table 1
Kinematic and static constitutive behavior of external, internal bilateral and internal unilateral ideal constraints.

	$(j,m)\in\mathcal{E}$	$(i,j,k)\in\mathcal{I}_b$	$(i,j,h)\in\mathcal{I}_u$
Kinematics	$\omega_m(p_j)=(\mathbf{p}_j-\mathbf{p}_j^o)\cdot\mathbf{e}_m^o=0$	$\phi_k(p_i,p_j)=(\mathbf{p}_j-\mathbf{p}_i)\cdot\boldsymbol{\beta}_k-b_k^o=0$	$\theta_h(p_i,p_j)=(\mathbf{p}_j-\mathbf{p}_i)\cdot\boldsymbol{\gamma}_h-c_h^o\leqslant 0$
Statics	$\mathbf{r}_j^m=-\boldsymbol{\nu}_m=-\nu_m\mathbf{e}_m^o$	$\mathbf{r}_i^k=\boldsymbol{\lambda}_k^b=\lambda_k^b\boldsymbol{\beta}_k,\mathbf{r}_j^k=-\boldsymbol{\lambda}_k^b$	$\mathbf{r}_i^h=\boldsymbol{\lambda}_h^u=\lambda_h^u\boldsymbol{\gamma}_h,\mathbf{r}_j^h=-\boldsymbol{\lambda}_h^u,\lambda_h^u\geqslant 0$

Let $\mathcal{N}$ identify an ordered set of n_p points in E, referred to as *nodes*, $p_j\in\mathsf{E}$ be the position occupied by the node j and $\mathbf{p}_j=p_j-O$ its position vector in V. Let $p=\{\mathbf{p}_1,\dots,\mathbf{p}_{n_p}\}$ be the set which identifies a placement of $\mathcal{N}$ in the space of configurations $\mathcal{P}=\{p\mid p_j\in\mathsf{E}, j=1,\dots,n_p\}$.

Let $p^o\in\mathcal{P}$ be an *a priori* known reference configuration, and let p be the actual configuration of $\mathcal{N}$ in $\mathcal{P}$. Indicate with $\mathbf{u}_j=(\mathbf{p}_j-\mathbf{p}_j^o)\in\mathsf{V}$ the displacement of node j with respect to its reference position. Accordingly, $u=\{\mathbf{u}_1,\dots,\mathbf{u}_{n_p}\}=p\boxplus(-p^o)$ is the displacement of $\mathcal{N}$ from p^o to p, with u belonging to the displacements space

$$\mathcal{U}=\left\{u=p\boxplus\left(-p^o\right)\;\middle|\;p,p^o\in\mathcal{P};\ \|\mathbf{p}_j-\mathbf{p}_i\|\neq\left\|\mathbf{p}_j^o-\mathbf{p}_i^o\right\|\text{ for some } i,j=1\dots n_p\right\}\tag{1}$$

wherein rigid body motions (that is distance preserving displacements) are excluded.

For the aim of the present work, it is useful to introduce $\mathcal{B}_\epsilon(p^o)$ as the ϵ-neighborhood of $p^o\in\mathcal{P}$:

$$\mathcal{B}_\epsilon\left(p^o\right)=\left\{p\in\mathcal{P}\;\middle|\;\sqrt{\langle u,u\rangle}/d\leqslant\epsilon\text{ with }\epsilon\in\mathbb{R}^+\right\},\quad\text{with } d=\max_{i,j=1\dots n_p}\left\{\left\|\mathbf{p}_j^o-\mathbf{p}_i^o\right\|\right\}\tag{2}$$

Let now consider n_e constraints on the position of nodes in $\mathcal{N}$ as well as n_b bars and n_u cables constraining the relative position of couples of nodes in $\mathcal{N}$, and let define the sets:

$$\mathcal{E}=\left\{(j,m)\;\middle|\;\exists\text{ external constraint } m \text{ on node } j\right\}\tag{3}$$

$$\mathcal{I}_b=\left\{(i,j,k)\;\middle|\;\exists\text{ bar } k \text{ between nodes } i \text{ and } j, i<j\right\}\tag{4}$$

$$\mathcal{I}_u=\left\{(i,j,h)\;\middle|\;\exists\text{ cable } h \text{ between nodes } i \text{ and } j, i<j\right\}\tag{5}$$

Definition 2.1. A tensegrity $\mathcal{T}_r$ is the set of nodes collected in $\mathcal{N}$ and of constraints described in $\mathcal{E}$, $\mathcal{I}_b$ and $\mathcal{I}_u$.

Let the structure $\mathcal{T}_r$ be loaded by external forces at nodes, and denote by $\mathbf{r}_j,\mathbf{f}_j\in\mathsf{V}$ the reactive (resp. active) forces resultant on node j, and by $r=\{\mathbf{r}_1,\dots,\mathbf{r}_{n_p}\}$, $f=\{\mathbf{f}_1,\dots,\mathbf{f}_{n_p}\}$ the corresponding sets for nodes in $\mathcal{N}$.

2.1. Ideal constraints and ideal tensegrities

External constraints, bars and cables in $\mathcal{T}_r$ are modeled as workless with an unilateral behavior just for cable elements.

Definition 2.2. A bilateral (unilateral) constraint is ideal if the work of its reaction forces for any admissible virtual mechanism is zero (non-negative).

In the aim of present work (namely, the analysis of kinematic and static indeterminacy), each involved constraint is assumed to be ideal.

Denoting as $\mathbf{r}_q^s$ the reaction force of constraint s on node q, the kinematic and static constitutive behavior of constraints which model physical restrictions in $\mathcal{T}_r$ are collected in Table 1 where, for $(j,m)\in\mathcal{E}$, $\mathbf{e}_m^o$ is a given unit vector, and $\boldsymbol{\beta}_k$, $\boldsymbol{\gamma}_h$ denote constraints' axes. For $(i,j,k)\in\mathcal{I}_b$, $\boldsymbol{\beta}_k=(\mathbf{p}_j-\mathbf{p}_i)/\|\mathbf{p}_j-\mathbf{p}_i\|$, $b_k^o=(\mathbf{p}_j^o-\mathbf{p}_i^o)\cdot\boldsymbol{\beta}_k^o$ and $\lambda_k^b\in\mathbb{R}$. For $(i,j,h)\in\mathcal{I}_u$, $\boldsymbol{\gamma}_h=(\mathbf{p}_j-\mathbf{p}_i)/\|\mathbf{p}_j-\mathbf{p}_i\|$, $c_h^o=(\mathbf{p}_j^o-\mathbf{p}_i^o)\cdot\boldsymbol{\gamma}_h^o$, and $\lambda_h^u\in\mathbb{R}^+\cup\{0\}$.

From the definition of ϵ given in Eq. (2), it follows that constraints kinematic behaviors in Table 1 are linear relationships in nodal displacements within $O(\epsilon^2)$ terms. Moreover, $\boldsymbol{\beta}_k^o$ and $\boldsymbol{\gamma}_h^o$ are the zeroth-order approximation in ϵ of $\boldsymbol{\beta}_k$ and $\boldsymbol{\gamma}_h$. In the following, whenever $p\in\mathcal{B}_\epsilon(p^o)$, ϵ is taken small enough to assume as exact the linearized form of the kinematic problem.

By definition, reaction forces $\boldsymbol{\nu}_m$ and $\boldsymbol{\lambda}_k^b$ belong to the spaces orthogonal to any admissible virtual mechanism, being workless. On the other hand, reaction forces of unilateral constraints belong to a dual cone of any admissible virtual mechanism. In agreement with Definition 2.2, constraints in Table 1 are ideal. Moreover, the workless hypothesis is satisfied by enforcing

$$\lambda_h^u\theta_h(p_i,p_j)=0,\quad\forall p\in\mathcal{P}\mid\theta_h(p_i,p_j)\leqslant 0\tag{6}$$

Definition 2.3. The ideal tensegrity $\mathcal{T}$ associated to the real tensegrity $\mathcal{T}_r$ is defined as:

$$\mathcal{T}=\left\{\mathcal{N},\left\{\omega_m(p_j)=0,\ \forall(j,m)\in\mathcal{E}\right\},\left\{\phi_k(p_i,p_j)=0,\forall(i,j,k)\in\mathcal{I}_b\right\},\left\{\theta_h(p_i,p_j)\leqslant 0,\ \forall(i,j,h)\in\mathcal{I}_u\right\}\right\}$$

A tensegrity structure with $n_u=0$ is usually referred to as a bar-truss.

3. Free-energy of ideal tensegrities

Let the positions of nodes in $\mathcal{N}$ be the state variables. The free-energy of $\mathcal{T}$ is defined as[5]:

$$\Psi:\mathcal{P}\mapsto\bar{\mathbb{R}},\qquad \Psi(p)=\sum_{(j,m)\in\mathcal{E}}\Omega_m(p_j)+\sum_{(i,j,k)\in\mathcal{I}_b}\Phi_k(p_i,p_j)+\sum_{(i,j,h)\in\mathcal{I}_u}\Theta_h(p_i,p_j) \tag{7}$$

where $\Omega_m(p_j)=I^{\mathcal{K}_e}(\mathbf{p}_j-\mathbf{p}_j^o)$, $\Phi_k(p_i,p_j)=I^{\mathcal{K}_b}(f_b(p))$, $\Theta_h(p_i,p_j)=I^{\mathcal{K}_u}(f_u(p))$ are the free-energy of external, internal bilateral and internal unilateral constraints, respectively, with $\mathcal{K}_e=\operatorname{span}(\mathbf{e}_m^o)^\perp$, $f_b(p)=\|\mathbf{p}_j-\mathbf{p}_i\|-b_k^o$, $\mathcal{K}_b=\{0\}$, $f_u(p)=\|\mathbf{p}_j-\mathbf{p}_i\|-c_h^o$ and $\mathcal{K}_u=\mathbb{R}^-\cup\{0\}$.

Nodal reaction $\mathbf{r}_j\in\mathsf{V}$ acting on node j is the static quantity dual to the kinematic variable p_j, and it results in:

$$\mathbf{r}_j=-\frac{\partial\Psi}{\partial\mathbf{p}_j}=-\sum_{(j,m)\in\mathcal{E}}\frac{\partial\Omega_m}{\partial\mathbf{p}_j}-\sum_{(i,j,k)\in\mathcal{I}_b}\frac{\partial\Phi_k}{\partial\mathbf{p}_j}-\sum_{(i,j,h)\in\mathcal{I}_u}\frac{\partial\Theta_h}{\partial\mathbf{p}_j} \tag{8}$$

It is herein remarked that $\Psi(p)$ is not differentiable in classical sense. To assess the well-posedness of Eq. (8) let the three right-hand terms be analyzed.

Proposition 3.2. *Functions Ω_m are convex in $\mathcal{P}$. Functions Φ_k are convex in $\mathcal{B}_\epsilon(p^o)$. Functions Θ_h are convex in $\mathcal{P}$.*

Proof. The convexity of functions Ω_m in $\mathcal{P}$ follows from the convexity of $\mathcal{K}_e$. For what concerns Φ_k, it is easy to check that

$$I^{\mathcal{K}_b}\big(f_b\big(p^q\big)\big)\leqslant qI^{\mathcal{K}_b}\big(f_b\big(p^1\big)\big)+(1-q)I^{\mathcal{K}_b}\big(f_b\big(p^2\big)\big) \tag{9}$$

is verified for every $p^q=qp^1\boxplus(1-q)p^2$, with $p^1,p^2\in\mathcal{B}_\epsilon(p^o)$ and $q\in(0,1)$. In fact, if either $f_b(p^1)$ or $f_b(p^2)$ is not equal to zero, Eq. (9) is trivially verified. When $f_b(p^1)=f_b(p^2)=0$, $f_b(p^q)$ is equal to zero everywhere in $\mathcal{B}_\epsilon(p^o)$, because from $p\in\mathcal{B}_\epsilon(p^o)$ it follows $f_b(p)=(\mathbf{u}_j-\mathbf{u}_i)\cdot\boldsymbol{\beta}_k^o$. Accordingly, Eq. (9) is always satisfied.

Proceeding as for $I^{\mathcal{K}_b}$, an inequality of the form (9) involving $I^{\mathcal{K}_u}(f_u)$ is satisfied for every $p^1,p^2\in\mathcal{P}$ and $q\in(0,1)$ and the convexity of Θ_h in $\mathcal{P}$ follows. □

It is worth pointing out that functions Φ_k are not convex in $\mathcal{P}$. In fact, when $f_b(p^1)=f_b(p^2)=0$, from triangular inequality follows that $f_b(p^q)\leqslant 0$, where the equality applies if and only if $\boldsymbol{\beta}_k^1=\boldsymbol{\beta}_k^2$. Thereby, when $p^1,p^2\in\mathcal{P}$ are such that $f_b(p^1)=f_b(p^2)=0$ and $f_b(p^q)<0$ for some values of q, inequality (9) is violated.

From Proposition 3.2, subdifferentiability of Ψ in $\mathcal{B}_\epsilon(p^o)$ straightly follows.

According to Eq. (8), reaction force on node j due to external constraint m is given by

$$\mathbf{r}_j^m:\mathcal{D}_m^e\mapsto\mathsf{V},\qquad \mathbf{r}_j^m=-\partial\Omega_m/\partial\mathbf{p}_j=-\nabla\Omega_m(p_j) \tag{10a}$$

[5] As it is customary in convex analysis [21], define $\bar{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$, where the regular addition is completed by the rules: $a+(+\infty)=+\infty$ ($\forall a\in\mathbb{R}$) and $+\infty+(+\infty)=+\infty$, while multiplication by positive numbers is completed by $a\times(+\infty)=+\infty$ ($\forall a\in\mathbb{R}^+$). As an example of convex function, let $I^{\mathcal{K}}(x)$ be the indicator function of the convex set $\mathcal{K}$ (that is $I^{\mathcal{K}}(x)=0$ if $x\in\mathcal{K}$ and $I^{\mathcal{K}}(x)=+\infty$ elsewhere). As a notation rule, if g is a convex function defined on a convex part X of a real vector space, indicate with $\nabla g(x)$ and $\partial g(x)$ respectively a subgradient and the subdifferential of g at point $x\in\mathsf{X}$. The following result will be used in the sequel:

Theorem 3.1. *Let g be a convex function taking only two values* (0 *or* $+\infty$)*. Function g is subdifferentiable at x if and only if $g(x)<+\infty$.*

Proof. The property

$$\partial g(x)\neq\emptyset\quad\Rightarrow\quad g(x)<+\infty$$

is true for any convex function g, $g\neq+\infty$ [21]. Conversely, to prove that

$$g(x)<+\infty\quad\Rightarrow\quad\partial g(x)\neq\emptyset$$

let us observe that $g(x)<+\infty$ implies $g(x)=0$ and

$$\forall y\in\mathsf{X},\quad g(x)\leqslant g(y)$$

or, equivalently,

$$\forall y\in\mathsf{X},\quad\langle 0,y-x\rangle+g(x)\leqslant g(y)$$

Therefore $0\in\partial g(x)$ and then $\partial g(x)\neq\emptyset$. □

with $\nabla\Omega_m(p_j) \in \partial\Omega_m(p_j) = \mathrm{span}(\mathbf{e}_m^o)$ in $\mathcal{D}_m^e$ and

$$\mathcal{D}_m^e = \{p \in \mathcal{P} \mid \mathbf{u}_j \in \mathrm{span}(\mathbf{e}_m^o)^{\perp}\} \tag{10b}$$

Moreover, reaction forces on nodes i and j due to internal bilateral constraint k are

$$\begin{aligned} \mathbf{r}_i^k, \mathbf{r}_j^k : \mathcal{D}_k^b \mapsto \mathsf{V}, \qquad & \mathbf{r}_i^k = -\partial\Phi_k/\partial\mathbf{p}_i = \nabla\Phi_k(p_i, p_j)\boldsymbol{\beta}_k^o \\ & \mathbf{r}_j^k = -\partial\Phi_k/\partial\mathbf{p}_j = -\nabla\Phi_k(p_i, p_j)\boldsymbol{\beta}_k^o \end{aligned} \tag{11a}$$

with $\nabla\Phi_k(p_i, p_j) \in \partial\Phi_k(p_i, p_j) = \mathbb{R}$ in $\mathcal{D}_k^b$ and

$$\mathcal{D}_k^b = \{p \in \mathcal{B}_\epsilon(p^o) \mid (\mathbf{u}_j - \mathbf{u}_i) \in \mathrm{span}\,(\boldsymbol{\beta}_k^o)^{\perp}\} \tag{11b}$$

Finally, reaction forces on nodes i and j due to the internal unilateral constraint h are given by

$$\begin{aligned} \mathbf{r}_i^h, \mathbf{r}_j^h : \mathcal{D}_h^u \mapsto \mathsf{V}, \qquad & \mathbf{r}_i^h = -\partial\Theta_h/\partial\mathbf{p}_i = \nabla\Theta_h(p_i, p_j)\boldsymbol{\gamma}_h \\ & \mathbf{r}_j^h = -\partial\Theta_h/\partial\mathbf{p}_j = -\nabla\Theta_h(p_i, p_j)\boldsymbol{\gamma}_h \end{aligned} \tag{12a}$$

where $\nabla\Theta_h(p_i, p_j) \in \partial\Theta_h(p_i, p_j)$ and $\partial\Theta_h(p_i, p_j)$ is the multi-valued function defined as:

$$\partial\Theta_h(p_i, p_j) = \begin{cases} g & \text{for } \|\mathbf{p}_j - \mathbf{p}_i\| = c_h^o \\ 0 & \text{for } \|\mathbf{p}_j - \mathbf{p}_i\| < c_h^o \end{cases} \tag{12b}$$

with $g \in \mathbb{R}^+ \cup \{0\}$, and

$$\mathcal{D}_h^u = \{p \in \mathcal{P} \mid \|\mathbf{p}_j - \mathbf{p}_i\| \leqslant c_h^o\} \tag{12c}$$

From Eq. (12b), the workless constraint (6) follows to be automatically satisfied.

It is worth pointing out that the previously-introduced definition domains and values for the subdifferential of each free-energy contribution recover kinematic and static features, respectively, of the corresponding constraint (see Table 1).

4. Kinematic analysis: rigidity

Let $\mathcal{S}$ be the definition domain of the generalized derivative of $\Psi(p)$:

$$\mathcal{S} = \left\{\bigcap_{m=1}^{n_e} \mathcal{D}_m^e\right\} \cap \left\{\bigcap_{k=1}^{n_b} \mathcal{D}_k^b\right\} \cap \left\{\bigcap_{h=1}^{n_u} \mathcal{D}_h^u\right\} \subseteq \mathcal{B}_\epsilon(p^o) \tag{13}$$

$\mathcal{S}$ is a closed convex cone in $\mathcal{P}$ and it identifies the set of configurations in $\mathcal{B}_\epsilon(p^o)$ admissible with all the kinematic restrictions defining $\mathcal{T}$. The rigidity concept is introduced by means of:

Definition 4.1.

$$\mathcal{T} \text{ is rigid} \quad \Leftrightarrow \quad \{u \in \mathcal{U} \mid p^o \boxplus u \in \mathcal{S}\} = \emptyset$$

From Definition 4.1, the following result on rigidity reads:

$$\mathcal{T} \text{ is rigid} \quad \Leftrightarrow \quad p^o \text{ is not an accumulation point for } \mathcal{S} \tag{14}$$

Under the assumption of conservative active forces, let $E(p)$ be the total potential energy of $\mathcal{T}$:

$$E : \mathcal{P} \mapsto \bar{\mathbb{R}}, \qquad E(p) = \Psi(p) + V(p) \tag{15}$$

wherein $V(p) = -\sum_{j=1}^{n_p} \mathbf{f}_j \cdot \mathbf{p}_j$ is the potential energy of active forces. It is immediate to prove that a configuration $\hat{p}$ is an equilibrium configuration if and only if it is a stationary point for $E(p)$:

$$\left.\frac{\partial E}{\partial \mathbf{p}_j}\right|_{\hat{\mathbf{p}}_j} = \left.\frac{\partial \Psi}{\partial \mathbf{p}_i}\right|_{\hat{\mathbf{p}}_j} - \mathbf{f}_j = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{r}_j = -\left.\frac{\partial \Psi}{\partial \mathbf{p}_i}\right|_{\hat{\mathbf{p}}_j} = -\mathbf{f}_j \quad \forall \hat{p}_j \in \hat{p} \tag{16}$$

In case of null external loads ($f = \emptyset_{n_p}$), $\mathcal{T}$ is in equilibrium in $\hat{p}$ if and only if the derivative of $\Psi(p)$ in $\hat{p}$ exists and is equal to zero (i.e., $\hat{p}$ is a stationary point for $\Psi(p)$). Since $\Psi(p)$ is a two-values function (0 and $+\infty$) and its derivative does not exist for configurations p such that $\Psi(p) = +\infty$ (Theorem 3.1), $\Psi(p)$ has an absolute minimum in the equilibrium configuration $\hat{p}$. As a consequence of previous considerations, the rigidity concept can be embedded in a variational framework.

Theorem 4.2. *Let $\mathcal{T}$ be an ideal tensegrity and assume null rigid-body motions. Then*

$$\mathcal{T} \text{ is rigid} \quad \Leftrightarrow \quad \Psi(p) \text{ has an absolute isolated minimum in } p^o$$

Proof. If $\mathcal{T}$ is rigid and since Definition 4.1, then $p^o \boxplus u \notin \mathcal{S}, \forall u \in \mathcal{U}$. Accordingly, the subdifferential of $\Psi(p)$ does not exist in $p = p^o \boxplus u$ and, since Theorem 3.1, $\Psi(p^o \boxplus u) = +\infty$, $\forall u \in \mathcal{U}$. Accordingly, $\Psi(p)$ has an absolute isolated minimum in p^o.

Conversely, if $\Psi(p)$ has an absolute isolated minimum in p^o, then $\Psi(p^o \boxplus u) = +\infty$, $\forall u \in \mathcal{U}$, and therefore, employing Theorem 3.1, $(p^o \boxplus u) \notin \mathcal{S}$, that is $\mathcal{T}$ is rigid. □

Theorem 4.2 allows one to recover duality between kinematic and static concepts. From this theorem, it follows:

$$\begin{aligned} \Psi\big(p^o \boxplus u\big) = +\infty, \quad \forall u \in \mathcal{U} \quad &\Leftrightarrow \quad \nexists \mathbf{r}_j = -\frac{\partial \Psi}{\partial \mathbf{p}_j}\bigg|_{p^o \boxplus u} \quad j = 1 \ldots n_p, \ \forall u \in \mathcal{U} \\ &\Leftrightarrow \quad \mathcal{T} \text{ is rigid} \end{aligned} \tag{17}$$

Therefore, the kinematic notion of rigidity is equivalent to a dual static formulation, that is to the non-existence of an equilibrated configuration in a neighborhood of p^o. It is worth remarking that, although assuming $p \in \mathcal{B}_\epsilon(p^o)$ (and thereby $\boldsymbol{\beta}_k = \boldsymbol{\beta}_k^o$ and $\boldsymbol{\gamma}_h = \boldsymbol{\gamma}_h^o$), the equilibrium relationships written in $p = p^o \boxplus u$ differ from the ones in p^o, because non-regular functions (that is subdifferentials of indicator functions) are employed.

Moreover, Theorem 4.2 allows to face the issue of rigidity for $\mathcal{T}$ in terms of a minimization problem of a convex objective function under convex constraints and the following operative new rigidity criterion can be stated:

Criterion 1 *(Rigidity). Let $\alpha = \inf_{u \in \mathcal{U}}\{\Psi(p^o \boxplus u)\}$. Then: $\alpha = 0 \Rightarrow \mathcal{T}$ is not rigid, $\alpha = +\infty \Rightarrow \mathcal{T}$ is rigid.*

4.1. Energy and kinematics

For the sake of notation, let the function

$$\delta_h : \mathcal{U} \mapsto \mathbb{R}, \quad \delta_h(u) = (\mathbf{u}_j - \mathbf{u}_i) \cdot \boldsymbol{\gamma}_h^o, \quad \mathbf{u}_i, \mathbf{u}_j \in u \in \mathcal{U} \tag{18}$$

to be defined with $(i, j, h) \in \mathcal{I}_u$ and let $\boldsymbol{\delta}(u) = (\delta_1(u), \ldots, \delta_{n_u}(u))^t \in \mathbb{R}^{n_u}$. Accordingly, the kinematic restrictions imposed from internal unilateral constraints can be equivalently formulated as $\boldsymbol{\delta} \leqslant 0$. Moreover, let define the sets:

$$\mathcal{U}_f = \big\{u \in \mathcal{U} \mid p \in \mathcal{B}_\epsilon\big(p^o\big), \Psi\big(p^o \boxplus u\big) = 0\big\} \tag{19a}$$

$$\mathcal{U}_v = \big\{u \in \mathcal{U} \mid p \in \mathcal{B}_\epsilon\big(p^o\big), \Psi\big(p^o \boxplus u\big) = 0, \Psi\big(p^o \boxplus (-u)\big) = +\infty\big\} \tag{19b}$$

$$\bar{\mathcal{U}} = \big\{u \in \mathcal{U} \mid p \in \mathcal{B}_\epsilon\big(p^o\big), \Psi\big(p^o \boxplus u\big) = 0, \Psi\big(p^o \boxplus (-u)\big) = 0\big\} \tag{19c}$$

and note that, since $\Psi(p)$ is a two values function, $\mathcal{U}_f = \bar{\mathcal{U}} \cup \mathcal{U}_v$.

Let define $\tilde{\mathcal{T}}$ as the bar-truss derived from $\mathcal{T}$ by removing all cables. Let $\tilde{\Psi}(p)$ be the free-energy of $\tilde{\mathcal{T}}$. Denote with $\tilde{\mathcal{S}}$ the definition domain of the generalized derivative of $\tilde{\Psi}(p)$, let $\tilde{\mathcal{U}}$ be the collection of corresponding admissible displacements (i.e., $\tilde{\mathcal{U}} = \{u \in \mathcal{U} \mid p \in \mathcal{B}_\epsilon(p^o), \tilde{\Psi}(p^o \boxplus u) = 0\}$) and note that if $u \in \tilde{\mathcal{U}}$ then $p^o \boxplus u \in \tilde{\mathcal{S}}$. Moreover, let the set $\mathcal{W} = \{u \mid p \in \mathcal{B}_\epsilon(p^o), \tilde{\Psi}(p^o \boxplus u) = 0\}$ and the vector space $\mathcal{V} = \{v = \alpha w \mid \alpha \in \mathbb{R}, w \in \mathcal{W}\}$ be defined such that $\tilde{\mathcal{U}} \subset \mathcal{W} \subset \mathcal{V}$. Therefore, denoting as $\{u^1, \ldots, u^M\}$ a basis for $\mathcal{V}$, it results:

$$\forall u \in \tilde{\mathcal{U}}, \quad u = \alpha_1 u^1 \boxplus \cdots \boxplus \alpha_M u^M, \quad \alpha_j \in \mathbb{R} \tag{20}$$

Furthermore, consider the bar-truss $\bar{\mathcal{T}}$, derived from $\mathcal{T}$ by assuming as bilateral all unilateral constraints. Denote with $\bar{\Psi}(p)$ the free-energy of $\bar{\mathcal{T}}$ and with $\bar{\mathcal{S}}$ the definition domain of the generalized derivative of $\bar{\Psi}(p)$.

Lemma 4.3. *Let $\mathcal{T}$ be an ideal tensegrity and $\mathcal{U}_s = \{u \in \tilde{\mathcal{U}} \mid \boldsymbol{\delta}(u) \leqslant 0, \|\boldsymbol{\delta}(u)\| > 0\}$. Then $\mathcal{U}_s \equiv \mathcal{U}_v$.*

Proof. By definition, if $\hat{u} \in \mathcal{U}_s$, it should respect all the kinematic restrictions imposed by both bilateral and unilateral constraints and thereby $\Psi(p^o \boxplus \hat{u}) = 0$. Moreover, since the definition of $\tilde{\mathcal{U}}$ and the linearity of $\delta_h(u)$, $(-\hat{u}) \in \tilde{\mathcal{U}}$ but it does not belong to $\mathcal{U}_s$, violating the restriction imposed by at least one unilateral constraint. Accordingly, $\Psi(p^o \boxplus (-\hat{u})) = +\infty$ and $\hat{u}$ belongs to $\mathcal{U}_v$. With similar arguments, it is easy to prove that there does not exist any displacement $u \in \mathcal{U}_v$ which does not belong to $\mathcal{U}_s$. □

Lemma 4.4. *Let $\mathcal{T}$ be an ideal tensegrity and $\mathcal{U}_o = \{u \in \tilde{\mathcal{U}} \mid \boldsymbol{\delta}(u) = \mathbf{0}\}$. Then $\mathcal{U}_o \equiv \bar{\mathcal{U}}$.*

Proof. By definition, if $\hat{u} \in \mathcal{U}_o$, it should respect all the kinematic restrictions imposed by both bilateral and unilateral constraints and, thereby, $\Psi(p^o \boxplus \hat{u}) = 0$. Moreover, since the definition of $\tilde{\mathcal{U}}$ and the linearity of $\delta_h(u)$, $(-\hat{u}) \in \tilde{\mathcal{U}}$ and it belongs to $\mathcal{U}_o$, respecting the restriction imposed by all unilateral constraints. Accordingly, $\Psi(p^o \boxplus (-\hat{u})) = 0$ and $\hat{u}$ belongs to $\bar{\mathcal{U}}$. With similar arguments, it is immediate to prove that there does not exist any displacement $u \in \bar{\mathcal{U}}$ which does not belong to $\mathcal{U}_o$. □

By using definition of $\tilde{\mathcal{T}}$ and $\bar{\mathcal{T}}$, Lemmas 4.3 and 4.4 can be read in a mechanical sense, providing the kinematic meaning of the energy-based sets $\mathcal{U}_v$ and $\bar{\mathcal{U}}$. In fact, if $u \in \mathcal{U}_s$ or $u \in \mathcal{U}_o$, then u is an admissible displacement for $\tilde{\mathcal{T}}$, automatically satisfying all bilateral constraints in $\mathcal{T}$. Moreover, $u \in \mathcal{U}_s$ identifies an admissible displacement for $\mathcal{T}$ where the actual end-to-end length of at least one cable in the structure is smaller than its reference one. In addition, $u \in \mathcal{U}_o$ corresponds to an admissible displacement for $\mathcal{T}$ where the actual end-to-end length of each cable does not vary. In other words, $u \in \bar{\mathcal{U}}$ is an admissible displacement for the bar-truss associated to $\mathcal{T}$ (namely, $\bar{\mathcal{T}}$).

5. Static analysis: pre-stressability

The pre-stressability concept is introduced by means of:

Definition 5.1.

$$\mathcal{T} \text{ is pre-stressable} \quad \Leftrightarrow \quad \left\{\nabla\Theta_h(p_i, p_j) \,\middle|\, r = \varnothing_{n_p}, \nabla\Theta_h(p_i, p_j) \in \mathbb{R}^+, \forall(i, j, h) \in \mathcal{I}_u\right\} \neq \emptyset$$

Pre-stressability corresponds to a special kind of static indeterminacy. In fact, considering a set of reactive forces $\hat{r}$ where each internal unilateral constraint has a strictly admissible reaction, $\mathcal{T}$ is pre-stressable if it is statically admissible (that is in equilibrium) in p^o under $f = \varnothing_{n_p}$ and $r = \hat{r}$.

Theorem 5.2. *Let $\mathcal{T}$ be an ideal tensegrity system. The following statements are equivalent*:

(i) $\mathcal{U}_v = \emptyset$,
(ii) $\mathcal{T}$ *is pre-stressable.*

In order to prove this theorem, some considerations and preliminary results have to be firstly traced. Let $\mathcal{T}'$ be the ideal tensegrity system obtained from $\mathcal{T}$ by removing cable H connecting nodes I and J. The number n_u' of cables in $\mathcal{T}'$ is equal to $(n_u - 1)$ and, without lack of generality, in the following it is assumed that $H = n_u$. Accordingly, denote with $\Psi'(p)$ the free-energy of $\mathcal{T}'$ and with $\mathcal{S}'$ the definition domain of the generalized derivative of $\Psi'(p)$. It is immediate to prove that each u such that $p^o \boxplus u \in \mathcal{S}'$ is contained in $\tilde{\mathcal{U}}$, and then the representation (20) can be employed.

Corollary 5.3. *Define C as the cone originated by the vectors*

$$\mathbf{a}_h = \left(\delta_h\left(u^1\right), \ldots, \delta_h\left(u^M\right)\right)^t \in \mathbb{R}^M$$

with $h = 1 \ldots n_u'$, and

$$\mathbf{b} = \left(-\delta_H\left(u^1\right) \ldots -\delta_H\left(u^M\right)\right) \in \mathbb{R}^M$$

If $\mathcal{U}_v = \emptyset$, then there does not exist an hyperplane separating **b** *from C.*

Proof. Let define the following problem

$$\begin{cases} \mathbf{\Delta}\boldsymbol{\alpha} \leqslant 0 \\ \mathbf{b}^t\boldsymbol{\alpha} > 0 \end{cases} \tag{21}$$

with $\mathbf{\Delta} \in \mathbb{R}^{n_u' \times M}$, $[\mathbf{\Delta}]_{ik} = \delta_i(u^k)$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_M)^t \in \mathbb{R}^M$ and $\mathbf{b} = (-\delta_H(u^1) \ldots -\delta_H(u^M)) \in \mathbb{R}^M$. Since $\delta_h(u)$ is a linear function in u, a solution $\boldsymbol{\alpha}$ of problem (21) identifies a particular $u \in \mathcal{U}_s$. Then, from Lemma 4.3 and since $\mathcal{U}_v = \emptyset$ by hypothesis, problem (21) admits no solution. This is equivalent to state that there does not exist an hyperplane separating the vector **b** from the cone C originated by the vectors $\mathbf{a}_h = (\delta_h(u^1), \ldots, \delta_h(u^M))^t$ with $h = 1 \ldots n_u'$, that is the thesis. □

Proof of Theorem 5.2. Let the statement $\mathcal{U}_v = \emptyset$ be verified and assign on $\mathcal{T}'$ the set of self-equilibrated active forces f defined as:

$$\mathbf{f}_J = -\boldsymbol{\lambda}_H^u = -\lambda_H^u \boldsymbol{\gamma}_H^o, \qquad \mathbf{f}_I = \boldsymbol{\lambda}_H^u, \quad \text{with } \lambda_H^u = 1$$
$$\mathbf{f}_n = \mathbf{0} \quad \forall n \in \{1, \ldots, n_p\} \quad \text{with } n \neq I, J$$

$\boldsymbol{\lambda}_H^u$ being the unit reaction force of the Hth internal unilateral constraint in $\mathcal{T}$ acting on nodes I and J. Due to the Principle of Virtual Works, there exists in $\mathcal{T}'$ a set of reactive forces in equilibrium with f if and only if

$$\mathbf{f}_J \cdot \mathbf{u}_J + \mathbf{f}_I \cdot \mathbf{u}_I = -\boldsymbol{\lambda}_H^u \cdot (\mathbf{u}_J - \mathbf{u}_I) = \sum_{h=1}^{n'_u} \lambda_h^u \delta_h(u) \quad \forall u \mid p^o \boxplus u \in \mathcal{S}' \tag{22}$$

or, equivalently, if and only if

$$\sum_{h=1}^{n'_u} \lambda_h^u \delta_h(u) = -\delta_H(u) \quad \forall u \mid p^o \boxplus u \in \mathcal{S}' \tag{23}$$

As a consequence of representation (20) and of the linearity of $\delta_h(u)$, relationship (23) is verified if there exists a set of non-negative real numbers λ_h^u such that:

$$\sum_{h=1}^{n'_u} \lambda_h^u \mathbf{a}_h = \mathbf{b}, \quad 0 \leqslant \lambda_1^u, \ldots, \lambda_{n'_u}^u \in \mathbb{R}, \text{ with } \sum_{h=1}^{n'_u} \lambda_h^u > 0 \tag{24}$$

Nevertheless, Corollary 5.3 proves that vector **b** is not separated by an hyperplane from the convex cone C. From Farkas' lemma [22], it follows that vector **b** belongs to C and, thereby, a set of non-negative reactions λ_h^u satisfying problem (24) exists.

These arguments can be applied for each internal unilateral constraint in $\mathcal{T}$ and therefore, by definition (12b) and by the superposition principle, when $\mathcal{U}_v = \emptyset$ a set of non-null $\nabla\Theta_h(p_i, p_j)$ (for each $h = 1, \ldots, n_u$) exists such that the equilibrium condition $r = \varnothing_{n_p}$ is verified, that is statement (ii) holds.

On the contrary, when statement (ii) holds, Eq. (12b) prescribes that $\|\mathbf{p}_j - \mathbf{p}_i\| = c_h^o$ or, in its linearized form, all admissible displacements u respect $(\mathbf{u}_j - \mathbf{u}_i) \cdot \boldsymbol{\gamma}_h^o = \delta_h(u) = 0$ (for each $h = 1, \ldots, n_u$), or equivalently, from Lemma 4.4, $u \in \bar{\mathcal{U}}$. Then $\mathcal{U}_f = \bar{\mathcal{U}}$ and $\mathcal{U}_v = \emptyset$ (that is statement (i) holds). □

Finally, the following new energy-based criterion for pre-stressability holds.

Criterion 2 *(Pre-stressability).*

$$\{u \in \mathcal{U} \mid \Psi(p^o \boxplus u) = 0, \Psi(p^o \boxplus (-u)) = +\infty\} = \emptyset \quad \Leftrightarrow \quad \mathcal{T} \text{ is pre-stressable}$$

6. Kinematic–static duality

Duality between kinematic and static concepts is established on the basis of:

Theorem 6.1. *Let $\mathcal{T}$ be an ideal tensegrity, then*

$$\mathcal{T} \text{ is rigid} \quad \Leftrightarrow \quad \bar{\mathcal{T}} \text{ is rigid} \wedge \mathcal{T} \text{ is pre-stressable}$$

Proof. Theorem 4.2 proves that $\mathcal{U}_f = \emptyset$ (or equivalently $\bar{\mathcal{U}} = \emptyset$ and $\mathcal{U}_v = \emptyset$) if and only if $\mathcal{T}$ is rigid. Accordingly, Lemma 4.4 proves that $\bar{\mathcal{U}} = \emptyset$ if and only if there does not exist any admissible displacement for $\bar{\mathcal{T}}$, that is $\bar{\mathcal{T}}$ is rigid. Moreover, Theorem 5.2 states that $\mathcal{U}_v = \emptyset$ if and only if $\mathcal{T}$ is pre-stressable. □

Theorem 6.1 employs energy-based arguments to recover a classical result for tensegrities. To the best of authors' knowledge, the proposed proof is alternative with respect to the specialized literature and this novel contribution clearly provides the mechanical interpretation of results obtained by using convex analysis. It is worth pointing out that the kinematic issue of rigidity for $\mathcal{T}$ involves inequalities and it generally results in though algebraic solutions. On the other hand, to verify that $\bar{\mathcal{T}}$ is rigid, the solution of linear equalities in nodal displacements is needed, because only bilateral relationships on $\mathcal{B}_\epsilon(p^o)$ are involved. Moreover, pre-stressability condition requires to find admissible solutions of a system of $3n_p$ equilibrium equations. Thereby, Theorem 6.1 allows to move towards the creation of algorithms more effective to evaluate rigidity. In a forthcoming paper, as a consequence of Theorem 6.1, the rigidity problem will be formulated in algebraic terms within the mathematical framework of quadratic optimization.

7. Conclusions and perspective

In present work, behavior of ideal tensegrity structures has been addressed by means of an energy approach formulated in the framework of convex analysis. A novel expression for the free-energy of ideal tensegrities has been proposed, providing a special perspective on their mechanics. The well-known dual relationship between kinematics and statics has been proved in an original way which allows a clear mechanical interpretation.

Tensegrity rigidity and pre-stressability problems have been focused accounting for non-smooth restrictions on both kinematic and static behavior of ideal constraints. Novel energy-based criteria have been provided. Present work clearly proves that the issue of rigidity reduces to a minimization problem (Criterion 1), and that it represents a sufficient condition for pre-stressability (see Theorem 6.1). Moreover, a necessary and sufficient condition for pre-stressability is provided (Criterion 2) in terms of structural free-energy.

In the future, present approach could be generalized to non-ideal structures. In fact, elastic behavior can be simply included and analyzed by means of a generalization of the overall structural free-energy:

$$\Psi : \mathcal{P} \mapsto \bar{\mathbb{R}}, \qquad \Psi(p) = \Psi^{\mathrm{D}}(p) + \Psi^{\mathrm{I}}(p) \tag{25}$$

where $\Psi^{\mathrm{D}}(p)$ represents structure deformation energy and $\Psi^{\mathrm{I}}(p)$ is the contribute to the free-energy deriving from the ideal constraints within the structure. For instance, a linearly elastic cable $(i, j, h) \in \mathcal{I}_u$ with stiffness k_h gives a contribution to Ψ^{I} equal to zero and to Ψ^{D} equal to $0.5k_h(c - c_h)^2 H(c - c_h)$, being $H(x - x_o)$ the Heaviside function centered in x_o, $c = \|\mathbf{p}_j - \mathbf{p}_i\|$, and c_h a measure of the cable unstressed length. Moreover, the definition of a structural dissipative pseudo-potential Φ^{D} would allow to relate reactive forces with time-dependent causes [19]:

$$\mathbf{r}_j = -\frac{\partial \Psi}{\partial \mathbf{p}_j} - \frac{\partial \Phi^{\mathrm{D}}}{\partial \dot{\mathbf{p}}_j} \tag{26}$$

$\dot{x}$ being the time derivative of x. Therefore, inelastic features (such as viscous behavior) could be incorporated in the same theoretical framework [20,21].

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