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Exact solutions for the vibration of circumferentially stepped orthotropic circular cylindrical shells

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ABSTRACT

The combination of Flügge's shell theory, the transfer matrix approach and the Romberg integration method are used to investigate the free vibration behaviour of stepped orthotropic cylindrical shells. The hoop step on the shell surface is described by a reduced thickness over part of its circumference. Modal displacements of the shell can be described by trigonometric functions and Fourier's approach is used to separate the variables. The vibration equations of the shell are reduced to eight first-order differential equations in the circumferential coordinate, and by using the transfer matrix of the shell, these equations can be written in a matrix differential equation. The transfer matrix is derived from the non-linear differential equations of the cylindrical shells by introducing the trigonometric functions in the longitudinal direction and applying the numerical integration in the circumferential direction. The proposed model is used to get the vibration frequencies and the corresponding mode shapes for symmetrical and antisymmetrical type-modes. Computed results indicate the sensitivity of the frequency parameters and the bending deformations to the geometry of stepped shell, and also to the axial and circumferential rigidities of the shell.

1. Introduction

Circular cylindrical shells which have stepped profiles have been found in many engineering applications, such as aerospace, mechanical, civil and marine structures. The mode shapes and frequencies of vibration of stepped shells essentially depend on some determining functions such as the radius of the curvature of the neutral surface, the shell thickness, the step thickness ratio, the shape of the shell edges, and so forth. In simple cases when these functions are constant, the vibration deflection displacements occupy the entire shell surface. If the determining functions vary from point to point of the neutral surface then localization of the vibration modes lies near the weakest lines on the shell surface which has less stiffness. This kind of problems is too difficult because of closed-form or analytic solutions cannot be obtained, in general, for this class of shells, numerical or approximate techniques are necessary for their analysis. Vibration has become more of a problem in recent years since the use of high strength material requires less material for load support-structures and components have become generally more slender and vibrate-prone. The vibration response of isotropic circular cylindrical shells with constant thickness has been studied by many researchers since the basic equations for this was established by Flügge [1], Love [2] and Rayleigh [3]. The best collection of documents can be found in Lissa [4] in which more than 500 publications are analyzed and discussed in both linear and non-linear vibration cases. Other related references may be found in the well-known works, see [5–10]. Some of researchers have considerable interest in the study of vibration behaviour of circular cylindrical shells with variable thickness such as [11–18] in which their investigations have been made

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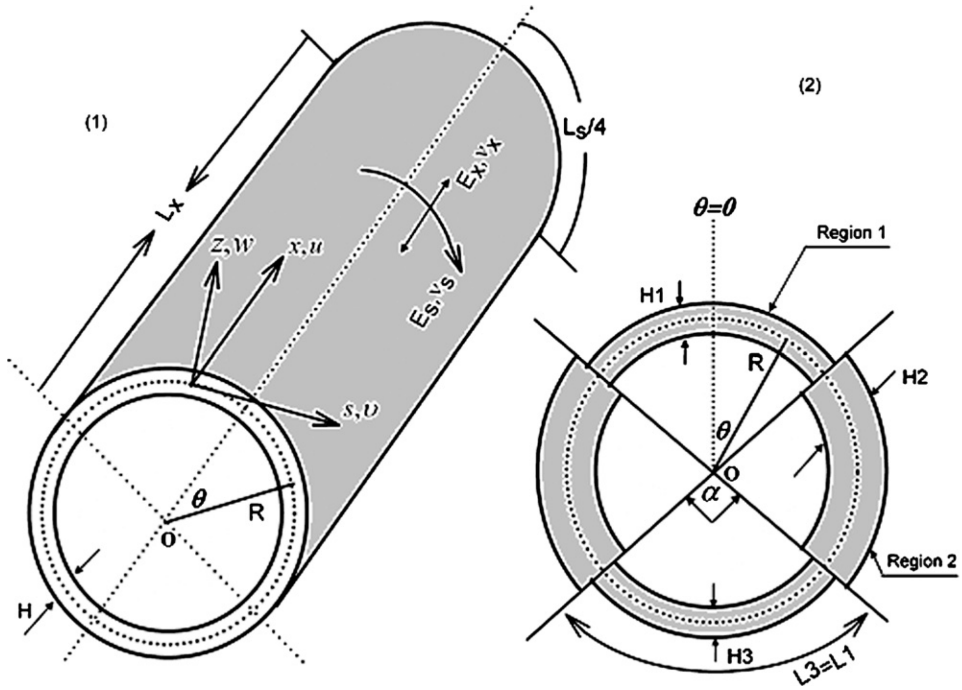

Fig. 1. Coordinate system and geometry of an orthotropic circular cylindrical shell with four-step thickness variations.

into different forms of thickness, i.e. axial, circumferential and step-wise thickness variations for isotropic shells. A few researchers have devoted their studies for vibration characteristics of orthotropic circular cylindrical shells with uniform and non-uniform thickness such as [19–26] in which their treatments have been modeled into different forms of shell theories. In contrast, the vibration study of non-uniform orthotropic cylindrical shells has received much less attention. A treatise on the use of the transfer matrix approach for mechanical science problems is presented by Tesar and Fillo [27]. However, the vibration problem of the orthotropic stepped circular cylindrical shells with step-wise thickness variations along the circumference does not appear to have been dealt with in the literature. For the first time, this Note presents the exact vibration characteristics of an orthotropic circular cylindrical shell with circumferentially step thickness, using the transfer matrix method and modeled on the Flügge's shell theory. The transfer matrix is derived from the non-linear differential equations system for the cylindrical shell by numerical integration. The method is employed to get the vibration frequencies and the corresponding deformations of the symmetrical and antisymmetrical profiles. The influences of the shell thickness ratios, locations of step-wise thickness variations, step thickness ratios, axial and circumferential rigidities parameters on the natural frequencies and mode shapes are examined. The results are cited in tabular and graphical forms for easy reference by researchers.

2. Theory and formulation of the problem

It has been mentioned in Section 1 that the problem structure is modeled by Flügge's theory. In order to have a better representation, the shell geometry and governing equations are made as separate parts. The formulation of these parts is presented below.

2.1. Geometric formulation

We consider an orthotropic circular cylindrical shell and its material is linearly elastic. The cylindrical coordinates (x, s, z) are taken to define the position of a point on the middle surface of the shell, as shown in Fig. 1(1). The deflection displacements of the middle surface of the shell are denoted by u, υ and w in the longitudinal, circumferential and transverse directions, respectively. Its geometry is described by R, the radius of curvature of the middle surface, H, the thickness of the shell, L_x, the axial length, and L_s, the circumferential length of the middle surface of the shell. The non-uniformity of the shell is described by sections with reduced thickness along parts of its circumference and consequently the shell is composed of many regions with different step thickness, and all regions have a common circumferential centerline. The shell is of circumferentially four steps with lengths L_1, L_2, L_3 and L_4, step thicknesses H_1, H_2, H_3 and H_4. The shell has a nominal thickness H_2 of the outer-stepped regions and reduced thickness H_1 of the inner-stepped regions, which is subtended by angle α. The regions of reduced thickness are assumed to be symmetric about the axis of $\Theta = 0$ and extended throughout the entire length of the shell in the axial direction, as shown in Fig. 1(2). Suppose that the axial and circumferential directions are principal axes of the orthotropic material. Thus, there exists the relation:

$$\nu_x E_s = \nu_s E_x \tag{1}$$

where E_x, E_s are Young's moduli and ν_x, ν_s are Poisson's ratios in the axial and circumferential directions, respectively.

2.2. Governing equations

For general circular cylindrical shells and for studying the free harmonic vibrations, the equilibrium equations of forces, based on [28] can be shown to be of the forms:

$$\begin{aligned}
&N'_x + N^{\bullet}_{sx} + \rho H\omega^2 u = 0, \qquad N'_{xs} + N^{\bullet}_s + Q_s/R + \rho H\omega^2 \upsilon = 0\\
&Q'_x + Q^{\bullet}_s - N_s/R + \rho H\omega^2 w = 0, \qquad M'_x + M^{\bullet}_{sx} - Q_x = 0\\
&M'_{xs} + M^{\bullet}_s - Q_s = 0, \qquad S_s - Q_s - M'_{sx} = 0, \qquad N_{xs} - N_{sx} - M_{sx}/R = 0
\end{aligned} \tag{2}$$

where N_x, N_s and Q_x, Q_s are the normal and transverse shearing forces in the x- and s-directions, respectively, N_{sx} and N_{xs} are the in-plane shearing forces, M_x, M_s and M_{xs}, M_{sx} are the bending moment and the twisting moment, respectively, S_s is the equivalent (Kelvin–Kirchoff) shearing force, ρ is the mass density, ω is the angular frequency of vibration, $' \equiv \partial/\partial x$, and $^{\bullet} \equiv \partial/\partial s$. While H is the wall thickness, chosen to be the average thickness of the shell over the length L_s, and for the stepped shells it varies from H_1 to H_2 along the shell circumference.

The relations between strains and deflections for the cylindrical shells used here are taken from [29,30] as follows:

$$\begin{aligned}
&\varepsilon_x = u', \qquad \varepsilon_s = \upsilon^{\bullet} + w/R, \qquad \Gamma_{xs} = \upsilon' + u^{\bullet}, \qquad \Gamma_{xz} = w' + \psi_x = 0\\
&\Gamma_{sz} = w^{\bullet} + \psi_s - \upsilon/R = 0, \qquad \mathrm{K}_x = \psi'_x, \qquad \mathrm{K}_s = \psi^{\bullet}_s + \left(\upsilon^{\bullet} + w/R\right)/R\\
&\mathrm{K}_{sx} = \psi'_s, \qquad \mathrm{K}_{xs} = \psi^{\bullet}_x + u^{\bullet}/R
\end{aligned} \tag{3}$$

where ε_x and ε_s are the normal strains of the middle surface of the shell, Γ_{xs}, Γ_{xz} and Γ_{sz} are the shear strains, and the quantities K_x, K_s, K_{sx} and K_{xs} representing the change of curvature and the twist of the middle surface, ψ_x is the bending slope, and ψ_s is the angular rotation. The components of force and moment resultants in terms of Eq. (3) are given as in [28]

$$\begin{aligned}
&N_x = D_x(\varepsilon_x + \nu_s\varepsilon_s) + (K_x/R)\mathrm{K}_x, \qquad N_{xs} = D_{xs}\Gamma_{xs} + (K_{sx}/R)\mathrm{K}_{sx}\\
&N_s = D_s(\varepsilon_s + \nu_x\varepsilon_x) - (K_s/R)(\mathrm{K}_s - \varepsilon_s/R)\\
&N_{sx} = D_{sx}\Gamma_{sx} + (K_{xs}/R)\mathrm{K}_{xs}, \qquad M_{sx} = 2K_{xs}(\mathrm{K}_{xs} - \Gamma_{xs}/2R)\\
&M_x = K_x(\mathrm{K}_x + \nu_s\mathrm{K}_s + \varepsilon_x/R), \qquad M_s = K_s(\mathrm{K}_s + \nu_x\mathrm{K}_x - \varepsilon_s/R)
\end{aligned} \tag{4}$$

where the quantities D_x, D_s and D_{xs} are the extensional rigidities defined as:

$$D_x = E_x H/(1 - \nu_x\nu_s), \qquad D_s = E_s H(1 - \nu_s\nu_x) \quad \text{and} \quad D_{xs} = HG$$

and K_x, K_s, and K_{xs} are the flexural rigidities defined as:

$$K_x = E_x H^3/12(1 - \nu_x\nu_s), \qquad K_s = E_s H^3/(1 - \nu_s\nu_x) \quad \text{and} \quad K_{xs} = GH3/12$$

in terms of Young's moduli E_x and E_s, Poisson's ratios ν_x and ν_s, and the shear modulus G.

From Eqs. (2)–(4), with eliminating the variables $Q_x, Q_s, N_x, N_{xs}, M_x, M_{xs}$ and M_{sx} which are not differentiated with respect to$s$, the vibration system of the partial differential equations for the state variables $u, \upsilon, w, \psi_s, M_s, S_s, N_s$ and N_{sx} of the shell are obtained as follows:

$$\begin{aligned}
&u^{\bullet} = (1/\Theta D_{xs})N_{sx} + (\Gamma/R\Theta)\psi'_s - \upsilon', \qquad w^{\bullet} = \upsilon/r - \psi_s\\
&\upsilon^{\bullet} = N_s/D_s - w/R - \nu_x u' - \nu_x\Gamma\psi'_x - (R\Gamma/K_s)M_s\\
&\psi^{\bullet}_s = (\Theta/K_s)M_s - \nu_x\Theta\psi'_x - (1/RD_x)N_s + (\nu_x/R)u'\\
&M^{\bullet}_s = S_s - K_{xs}(\Gamma/\Theta - 4)\psi''_s - (R\Gamma/\Theta)N'_{sx}, \qquad N^{\bullet}_s = -S_s/R - N'_{sx} - \rho\omega^2 H\upsilon\\
&N^{\bullet}_{sx} = D_x(1 - \nu_x\nu_s)u'' - (D_x\nu_s/D_s)N'_s + (K_x\nu_s/RK_s)M'_s - K_x\big((1 - \nu_x\nu_s)/R\big)\psi''_x - \rho\omega^2 Hu\\
&S^{\bullet}_s = N_s/R + (\nu_s\Theta D_x/D_s)M''_s - K_x(1 - \nu_x\nu_s\Theta)w''' - (K_x/R)(1 - \nu_x\nu_s)u'' - (\nu_s D_x\Gamma/D_s)N''_s - \rho\omega^2 Hw\\
&\Theta = \Gamma + 1, \qquad \Gamma = H^2/12R^2
\end{aligned} \tag{5}$$

For a simply supported shell, the solution of the system (5) takes the forms:

$$\begin{aligned}
&u(x,s)=\bar{U}(s)\cos\beta x, \qquad \big(\upsilon(x,s),w(x,s)\big)=\big(\bar{V}(s),\bar{W}(s)\big)\sin\beta x, \qquad \psi_s(x,s)=\bar{\psi}_s(s)\sin\beta x\\
&\big(N_x(x,s),N_s(x,s),Q_s(x,s),S_s(x,s)\big)=\big(\bar{N}_x(s),\bar{N}_s(s),\bar{Q}_s(s),\bar{S}_s(s)\big)\sin\beta x\\
&\big(N_{xs}(x,s),N_{sx}(x,s),Q_x(x,s)\big)=\big(\bar{N}_{xs}(s),\bar{N}_{sx}(s),\bar{Q}_x(s)\big)\cos\beta x\\
&\big(M_x(x,s),M_s(x,s)\big)=\big(\bar{M}_x(s),\bar{M}_s(s)\big)\sin\beta x\\
&\big(M_{xs}(x,s),M_{sx}(x,s)\big)=\big(\bar{M}_{xs}(s),\bar{M}_{sx}(s)\big)\cos\beta x, \quad \beta=m\pi/L_x,\ m=1,2,\ldots
\end{aligned} \tag{6}$$

where m is the axial half wave number and the quantities $\bar{U}(s),\bar{V}(s),\ldots$ are the state variables which are undetermined functions of s.

3. Matrix form of the vibration equations

The differential equations as shown previously are modified to a suitable form and solved numerically. Hence, by substituting Eqs. (6) into Eqs. (5), after appropriate algebraic operations, the system of governing equations of the shell can be written in non-linear ordinary differential equations referred to the variable$s$only are obtained, in the following matrix form:

$$R\frac{\mathrm{d}}{\mathrm{d}s}\begin{Bmatrix}\tilde{U}\\ \tilde{V}\\ \tilde{W}\\ \tilde{\psi}_s\\ \tilde{M}_s\\ \tilde{S}_s\\ \tilde{N}_s\\ \tilde{N}_{sx}\end{Bmatrix}=\begin{bmatrix}0 & V_{12} & 0 & V_{14} & 0 & 0 & 0 & V_{18}\\ V_{21} & 0 & V_{23} & 0 & V_{25} & 0 & V_{27} & 0\\ 0 & V_{32} & 0 & V_{34} & 0 & 0 & 0 & 0\\ V_{41} & 0 & V_{43} & 0 & V_{45} & 0 & V_{47} & 0\\ 0 & 0 & 0 & V_{54} & 0 & V_{56} & 0 & V_{58}\\ V_{61} & 0 & V_{63} & 0 & V_{65} & 0 & V_{67} & 0\\ 0 & V_{72} & 0 & 0 & 0 & V_{76} & 0 & V_{78}\\ V_{81} & 0 & V_{83} & 0 & V_{85} & 0 & V_{87} & 0\end{bmatrix}\begin{Bmatrix}\tilde{U}\\ \tilde{V}\\ \tilde{W}\\ \tilde{\psi}_s\\ \tilde{M}_s\\ \tilde{S}_s\\ \tilde{N}_s\\ \tilde{N}_{sx}\end{Bmatrix} \tag{7}$$

By using the state vector of fundamental unknowns $Z(s)$, system (7) can be written as:

$$\begin{aligned}
&\left(R\frac{\mathrm{d}}{\mathrm{d}s}\right)\big\{Z(s)\big\}=[V]\big\{Z(s)\big\}\\
&\big\{Z(s)\big\}=\{\tilde{U},\tilde{V},\tilde{W},\tilde{\psi}_s,\tilde{M}_s,\tilde{S}_s,\tilde{N}_s,\tilde{N}_{sx}\}^T\\
&(\tilde{U},\tilde{V},\tilde{W})=k_x(\bar{U},\bar{V},\bar{W}), \qquad \tilde{\psi}_s=(k_x/\beta)\bar{\psi}_s, \qquad \tilde{M}_s=\big(1/\beta^2\big)\bar{M}_s\\
&(\tilde{S}_s,\tilde{N}_s,\tilde{N}_{sx})=\big(1/\beta^3\big)(\bar{S}_s,\bar{N}_s\bar{N}_{sx})
\end{aligned} \tag{8}$$

and the coefficients of matrix $[V]$ are given as:

$$\begin{aligned}
&V_{12}=-(m\pi/l), \qquad V_{14}=(m\pi/l)^2\big(h^2/6\big)/\Theta 1, \qquad V_{18}=\Gamma 1(m\pi/l)^3/\mu\Theta 1\\
&V_{21}=\nu_x(m\pi/l), \qquad V_{23}=1+\nu_x\Gamma 1(m\pi/l)^2, \qquad V_{25}=\Gamma 1(m\pi/l)^3/\mu\\
&V_{27}=\Gamma 1(m\pi/l)^3, \qquad V_{32}=1, \qquad V_{34}=-(m\pi/l), \qquad V_{41}=-\nu_x\\
&V_{43}=-\nu_x(m\pi/l)^2\Theta 1, \qquad V_{45}=(m\pi/l)\Theta 1/\eta, \qquad V_{47}=(m\pi/l)\Gamma 1\\
&V_{54}=\mu(m\pi/l)^2(4-\Gamma 1/\Theta 1), \qquad V_{56}=-(m\pi/l), \qquad V_{58}=(m\pi/l)^2\Gamma 1/\Theta 1\\
&V_{61}=(1-\nu_s\nu_x), \qquad V_{63}=(1-\nu_x\nu_s)(m\pi/l)^2/2-\lambda^2/\Gamma 1(m\pi/l)^3\\
&V_{65}=(m\pi/l)/\eta, \qquad V_{67}=\nu_s(m\pi/l)^2\Gamma 1/\eta, \qquad V_{72}=-\lambda^2/\Gamma 1(m\pi/l)^3\\
&V_{76}=-1, \qquad V_{78}=(m\pi/l), \qquad V_{81}=(1-\nu_x\nu_s)/\Gamma 1(m\pi/l)-\lambda^2/\Gamma 1(m\pi/l)^3\\
&V_{83}=1-\nu_x\nu_s, \qquad V_{85}=\nu_s/\eta, \qquad V_{87}=-\nu_s(m\pi/l)/\eta
\end{aligned} \tag{9}$$

in terms of the following dimensionless parameters: frequency parameter $\lambda=R\omega\sqrt{\rho(1-\nu_x\nu_s)/E_x}$, circumferential and axial orthotropic parameters $\eta=D_s/D_x=K_s/K_x=\nu_s/\nu_x$ and $\mu=D_{xs}/D_x=K_{xs}/K_x=G(1-\nu_x\nu_s)/E_x$, respectively, in addition to $l=L_x/R$, $h=H/R$, $\Theta 1=\Gamma 1+1$ and $\Gamma 1=h^2/12$.

The state vector $\{Z(\Theta)\}$ of fundamental unknowns can be easily expressed as:

$$\big\{Z(\Theta)\big\}=\big[Y(\Theta)\big]\big\{Z(0)\big\} \tag{10}$$

by using the transfer matrix $[Y(\Theta)]$ of the shell and the substitution of this expression into Eq. (8) performs to:

Table 1
Comparison of frequency parameters λ for a simply supported uniform circular cylindrical shell ($l_x = 20$, $h_1 = h_2 = 0.05$, $m = 1$).

n	Markus [7]	Swaddiwudhipong et al. [31]	Y. Xiang et al. [32]	L. Zhang el al. [16]	Present ($\eta = 1$, $\mu = 0.35$)
1	0.0161063	0.0161031	0.0161029	0.0161065	0.0161058
2	0.0392332	0.0393032	0.0392710	0.0393038	0.0393021
3	0.109477	0.109852	0.1098116	0.109853	0.1098250
4	–	–	–	–	0.1496762
5	0.209008	0.210345	0.2102773	0.210345	0.2103017

$$(\mathrm{d}/\mathrm{d}\Theta)\big[Y(\Theta)\big] = [V]\big[Y(\Theta)\big]$$
$$\big[Y(0)\big] = [I] \tag{11}$$

The governing system of vibration (11) is too complicated to obtain any closed form solution, and this problem is highly favorable for solving by numerical methods. Hence, the matrix $[Y(\Theta)]$ is obtained by using numerical integration, by use of the Romberg integration method, with the starting value $[Y(0)] = [I]$ (unit matrix) which is given by taking $\Theta = 0$ in Eq. (10), and its solution depends only on the geometric and martial properties of the shell. For a plane passing through the central axis in a shell with structural symmetry, symmetrical and antisymmetrical profiles can be obtained, and consequently, only one-half of the shell circumference is considered with the boundary conditions at the ends taken to be the symmetric or antisymmetric type of vibration modes. Therefore, the boundary conditions for symmetrical and antisymmetrical type-modes, respectively, are

$$\tilde{V} = \tilde{\psi}_s = 0, \qquad \tilde{S}_s = \tilde{N}_{sx} = 0 \quad \text{and} \quad \tilde{U} = \tilde{W} = 0, \qquad \tilde{N}_s = \tilde{M}_s = 0 \tag{12}$$

4. Vibration frequencies and mode shapes equations

The substitution of Eqs. (12) into Eq. (10) results in the frequency equations as follows:

$$\begin{bmatrix} Y_{21} & Y_{23} & Y_{25} & Y_{27} \\ Y_{41} & Y_{43} & Y_{45} & Y_{47} \\ Y_{61} & Y_{63} & Y_{65} & Y_{67} \\ Y_{81} & Y_{83} & Y_{85} & Y_{87} \end{bmatrix}_{(\pi)} \begin{Bmatrix} \tilde{U} \\ \tilde{W} \\ \tilde{M}_s \\ \tilde{N}_s \end{Bmatrix}_{(0)} = 0 \quad \text{for symmetrical vibration} \tag{13}$$

$$\begin{pmatrix} Y_{12} & Y_{14} & Y_{16} & Y_{18} \\ Y_{32} & Y_{34} & Y_{36} & Y_{38} \\ Y_{52} & Y_{45} & Y_{56} & Y_{58} \\ Y_{72} & Y_{74} & Y_{76} & Y_{78} \end{pmatrix}_{(\pi)} \begin{Bmatrix} \tilde{V} \\ \tilde{\psi}_s \\ \tilde{S}_s \\ \tilde{N}_{sx} \end{Bmatrix}_{(0)} = 0 \quad \text{for antisymmetrical vibration} \tag{14}$$

Eqs. (13) and (14) give a set of linear homogeneous equations with unknown coefficients $\{\tilde{U}, \tilde{W}, \tilde{M}_s, \tilde{N}_s\}^T_{(0)}$ and $\{\tilde{V}, \tilde{\psi}_s, \tilde{S}_s, \tilde{N}_{sx}\}^T_{(0)}$, respectively, at $\Theta = 0$. For the existence of a nontrivial solution of these coefficients, the determinant of the coefficient matrix should be vanished. The standard procedures cannot be employed for obtaining the eigenvalues of the frequency parameter λ. The nontrivial solution is found by searching the values λ in matrix $[Y(\pi)]$ which make its determinant zero by using Lagrange interpolation procedure. The mode shapes of vibration at any point of the cross-section of the shell, for each axial half mode m are determined by calculating the eigenvectors corresponding to the eigenvalues λ by using Gaussian elimination procedure.

5. Computed results and discussion

A computer program based on the analysis described herein has been developed to study the vibration behaviour for such shell, using the transfer matrix approach. The frequency parameters and the corresponding mode shapes of vibration for the shell are calculated numerically, and some of the results shown next are for cases that have not as yet been considered in the literature. Our study is divided into three parts in which the Poisson's ratio ν_s and the axial rigidity μ take the values 0.3 and 0.35, respectively.

5.1. Verification of solution method

Table 1 presents the frequency parameters λ for an isotropic simply supported circular cylindrical shell of a constant thickness obtained by Markus [7] who has obtained the exact 3D solutions, Swaddiwudhipong et al. [31] who has obtained the Ritz solutions based on the Sanders shell theory, Xiang et al. [32] who has obtained the state-space technique solutions based on the Goldenviezer–Novozhilov shell theory, Zhang et al. [16] who has obtained the state-space technique solutions based on the Flügge thin shell theory and the author who has obtained the transfer matrix approach solutions based on the

Table 2
The first five fundamental frequency parameters λ versus the circumferential rigidity η and step thickness ratio h_1 for two-stepped orthotropic cylindrical shells ($l_x = 4$, $h_2 = 0.02$, $l_1 = L_s/2$, $\mu = 0.35$).

h_1	Symmetric vibration				Antisymmetric vibration			
	η							
	0.1	0.5	1	2	0.1	0.5	1	2
h_2	0.032528(1)	0.069561(1)	0.077810(1)	0.089819(1)	0.325618(1)	0.069613(1)	0.077812(1)	0.089854(1)
	0.037620(1)	0.072413(1)	0.094135(1)	0.126233(1)	0.376240(1)	0.072450(1)	0.094143(1)	0.126243(1)
	0.053085(1)	0.102990(1)	0.131657(1)	0.133275(1)	0.053093(1)	0.102992(1)	0.131710(1)	0.133301(1)
	0.064258(1)	0.129203(1)	0.140892(1)	0.192190(2)	0.064332(1)	0.129209(1)	0.140896(1)	0.192463(2)
	0.084714(2)	0.145704(2)	0.173112(2)	0.195612(1)	0.084736(2)	0.145850(2)	0.173118(2)	0.195610(1)
$h_2/2$	0.023716(1)	0.049998(1)	0.060252(1)	0.069712(1)	0.023687(1)	0.050186(1)	0.060178(1)	0.069831(1)
	0.027520(1)	0.060724(1)	0.069747(1)	0.086729(1)	0.029026(1)	0.061872(1)	0.073360(1)	0.091286(1)
	0.037156(1)	0.074156(1)	0.098930(1)	0.133921(1)	0.037291(1)	0.075663(1)	0.097891(1)	0.123832(1)
	0.049800(1)	0.099425(1)	0.123270(2)	0.137550(1)	0.049743(1)	0.100102(1)	0.121823(1)	0.133890(1)
	0.056338(2)	0.105160(2)	0.128163(1)	0.144413(2)	0.056024(2)	0.104713(2)	0.122841(2)	0.145231(2)
$h_2/3$	0.018571(1)	0.039975(1)	0.048611(1)	0.057955(1)	0.018444(1)	0.040372(1)	0.048316(1)	0.057027(1)
	0.022062(1)	0.050223(1)	0.057760(1)	0.067332(1)	0.022353(1)	0.048092(1)	0.059158(1)	0.072573(1)
	0.027664(1)	0.058977(1)	0.075298(1)	0.101512(1)	0.031323(1)	0.066846(1)	0.082581(1)	0.102631(1)
	0.040134(1)	0.081625(1)	0.099879(2)	0.117052(2)	0.038413(1)	0.080298(1)	0.098560(2)	0.116742(2)
	0.043623(2)	0.084165(2)	0.110532(1)	0.132210(2)	0.042442(2)	0.084103(2)	0.105612(1)	0.120053(1)
$h_2/4$	0.015313(1)	0.033769(1)	0.040945(1)	0.050135(1)	0.015581(1)	0.035014(1)	0.041482(1)	0.048623(1)
	0.019260(1)	0.042320(1)	0.051196(1)	0.057197(1)	0.017836(1)	0.039282(1)	0.048462(1)	0.059939(1)
	0.022011(1)	0.051040(1)	0.060643(1)	0.080037(1)	0.025496(1)	0.054280(1)	0.070215(1)	0.088874(1)
	0.031846(1)	0.066578(1)	0.085190(2)	0.101623(2)	0.034170(1)	0.072569(2)	0.084489(2)	0.098839(2)
	0.035446(2)	0.071077(2)	0.090619(1)	0.108991(2)	0.035236(2)	0.074893(1)	0.090937(1)	0.105683(1)

Table 3
The first fundamental frequencies parameters λ versus the circumferential rigidity η and step thickness ratio h_1 for four-stepped orthotropic cylindrical shells ($l_x = 4$, $h_2 = 0.02$, $l_1 = L_s/4$, $\mu = 0.35$, $h_3 = h_1$).

h_1	Symmetric vibration				Antisymmetric vibration			
	η							
	0.1	0.5	1	2	0.1	0.5	1	2
$h_2/2$	0.025050(1)	0.051228(1)	0.096085(1)	0.071694(1)	0.024615(1)	0.051507(1)	0.062230(1)	0.068973(1)
	0.025399(1)	0.056501(1)	0.110574(1)	0.082285(1)	0.025319(1)	0.055997(1)	0.064663(1)	0.083165(1)
	0.035905(1)	0.073794(1)	0.124361(2)	0.114391(1)	0.037039(1)	0.074625(1)	0.098841(1)	0.135260(1)
	0.043466(1)	0.098763(1)	0.133253(2)	0.130532(1)	0.046682(1)	0.093924(1)	0.124243(1)	0.140912(1)
	0.053643(1)	0.106731(2)	0.145502(1)	0.150214(2)	0.058966(2)	0.109782(2)	0.128690(2)	0.151381(2)
$h_2/3$	0.018898(1)	0.040672(1)	0.049581(1)	0.061412(1)	0.027257(1)	0.040549(1)	0.049960(1)	0.055399(1)
	0.020984(1)	0.045561(1)	0.054437(1)	0.062368(1)	0.033712(1)	0.044540(1)	0.050217(1)	0.063252(1)
	0.028134(1)	0.063154(1)	0.076472(1)	0.092767(1)	0.045732(2)	0.057511(1)	0.074257(1)	0.100243(1)
	0.032451(1)	0.075277(1)	0.088288(1)	0.100046(1)	0.046201(2)	0.068498(1)	0.092241(1)	0.120035(1)
	0.040826(1)	0.085127(2)	0.098356(2)	0.116741(2)	0.053591(2)	0.087437(2)	0.104183(2)	0.123514(2)
$h_2/4$	0.015285(1)	0.034114(1)	0.041056(1)	0.050887(1)	0.021229(1)	0.034425(1)	0.041558(1)	0.046469(1)
	0.017803(1)	0.037772(1)	0.047249(1)	0.055171(1)	0.026275(1)	0.037352(1)	0.042355(1)	0.051731(1)
	0.024249(1)	0.057374(1)	0.066164(1)	0.079946(1)	0.036316(2)	0.046036(1)	0.058782(1)	0.078754(1)
	0.027364(1)	0.061534(1)	0.075339(1)	0.082088(1)	0.037557(2)	0.053881(1)	0.072807(1)	0.097092(1)
	0.032182(1)	0.072548(2)	0.082954(2)	0.096998(2)	0.044217(1)	0.072153(2)	0.088704(2)	0.105531(2)

Flügge's shell theory using the Romberg integration method when $\eta = 1$ and $\mu = 0.35$ that gives the isotropic cylindrical shell. We observe that the present solutions are in close agreement with these studies. The value of n in this table represents the mode sequence number of frequency parameters λ for $m = 1$. The comparison studies have confirmed the correctness of the proposed shell theory and solution method for studying vibration characteristics of the problem under consideration.

5.2. Vibration results

Consider the vibration of an orthotropic cylindrical shell with many reduced thickness over parts of its circumference. The study of the shell vibration is determined by finding the frequency parameters λ which equals the eigenvalues of Eqs. (13) and (14) and the corresponding deflection displacements which equals the eigenvectors of the same equations for each value of m, separately. Fundamental frequencies are of practical importance when designing shell structures. To obtain the frequencies of vibration we will search the set of all eigenvalues, and to obtain the fundamental frequencies of vibration we will search the lowest values of this set. The numerical results presented herein pertain to obtain the

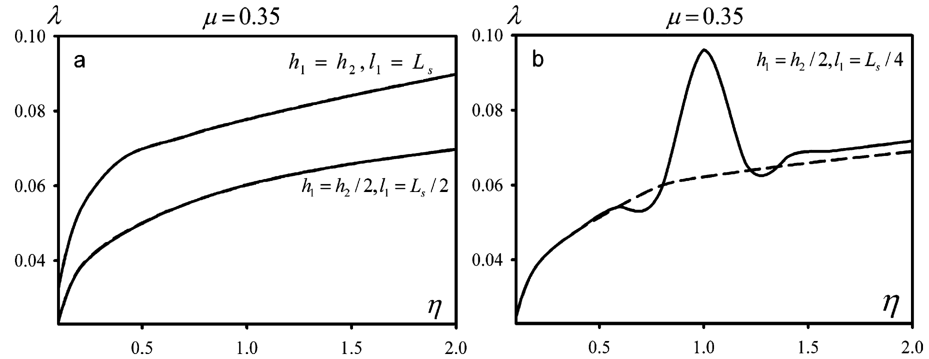

Fig. 2. Frequency parameters versus circumferential rigidity for two- and four-stepped orthotropic cylindrical shells ($l_x = 4$, $h_2 = 0.02$, $m = 1$).

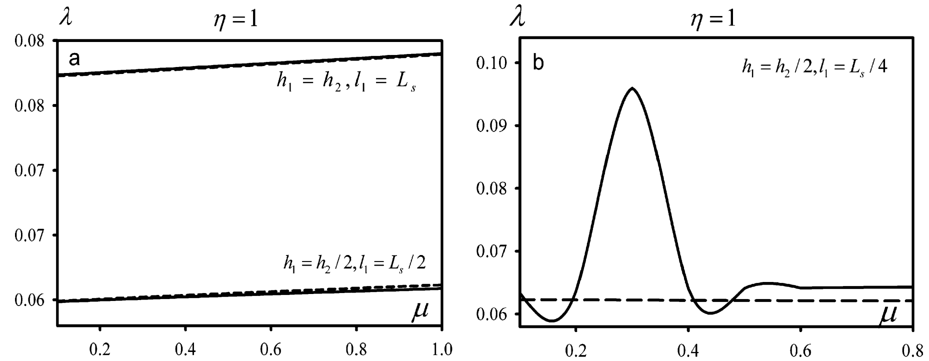

Fig. 3. Frequency parameters versus axial rigidity for two- and four-stepped orthotropic cylindrical shells ($l_x = 4$, $h_2 = 0.02, m = 1$).

fundamental frequencies in the case of symmetric and antisymmetric type-modes. In general, the frequency parameters λ decrease with a decrease of step thickness ratio for every given η value. This result confirms the fact that the effect of increasing the shell flexural rigidity becomes larger than that of increasing the shell mass when the step thickness ratio increases. For the symmetric and antisymmetric type-modes, the vibration frequencies and mode shapes are dependent on the geometry and the elastic properties of the shell. Several sets of step thickness ratio and hoop rigidity η of the stepped shell are considered when $\mu = 0.35$. For an isotropic stepped shell, $\eta = 1$ and $\mu = 0.35$. The effect of step thickness ratios on the frequencies of vibration for every given η value, Tables 2 and 3 give the fundamental frequency parameters λ for symmetric and antisymmetric vibrations modes of two- and four-stepped orthotropic shells, respectively. The numbers in the parentheses are the axial half wave numbers of the mode in the x-direction. The results presented in these tables show that the frequency parameters of symmetric and antisymmetric vibrations depend on the ranges of h_1 and L_1 as follows: (1) The symmetric and antisymmetric frequency parameters λ are found to be identical values for uniform shells ($h_1 = h_2$, $l_1 = l_s$, i.e. $\alpha = 2\pi$); (2) a decrease in the step thickness ratios h_1 for a given η value leads to a decrease in the frequency parameters; (3) an increase in the circumferential rigidity η for a given h_1 values leads to a monotonically increase in the frequency parameters λ when $\mu = 0.35$; (4) for the four-stepped shell, in the case of $h_1 = h_2/2$ and $\eta = 1$ when $\mu = 0.35$ which gives an isotropic cylindrical shell, the frequency parameter λ is higher than that for an orthotropic shell, and this indicates to the orthotropic stepped shell is more quite than the isotropic shell; (5) we observed that the axial half wave number$m$of the fundamental modes is occurred to be 2 for uniform and stepped isotropic or orthotropic cylindrical shells. Figs. 2(a) and 2(b) give the frequency parameters λ versus η when $\mu = 0.35$ of two- and four-stepped orthotropic cylindrical shells, respectively, for symmetric and antisymmetric vibration modes that be shown by solid and chain lines, respectively. The results presented in these figures show that the frequency parameter modes of symmetrical and antisymmetrical vibrations give identical values and increase regularly with an increase of η for the uniform and two-stepped shell ($h_1 = h_2/2$, $l_1 = L_s/2$, i.e. $\alpha = \pi$). For the four-stepped shell ($h_1 = h_2/2$, $l_1 = L_s/4$, i.e. $\alpha = \pi/2$), the frequency parameter mode of symmetrical vibration changes from a higher mode to a lower mode for $0.5 \leqslant \mu \leqslant 1.5$ and becomes the maximum for $\mu = 1$ that gives the isotropic stepped cylindrical shell. The frequency parameter mode of the antisymmetrical vibration is lower than that for the symmetrical vibration when $\eta \geqslant 1.5$. From Fig. 2(b), it is observed that the behaviour of vibration modes for an orthotropic cylindrical shell is more quite than that for an isotropic cylindrical shell. Figs. 3(a) and 3(b) give the frequency parameters λ versus μ when $\eta = 1$ for two- and four-stepped orthotropic cylindrical shells. It is shown from these figures that the frequency parameter modes of symmetrical and antisymmetrical vibration

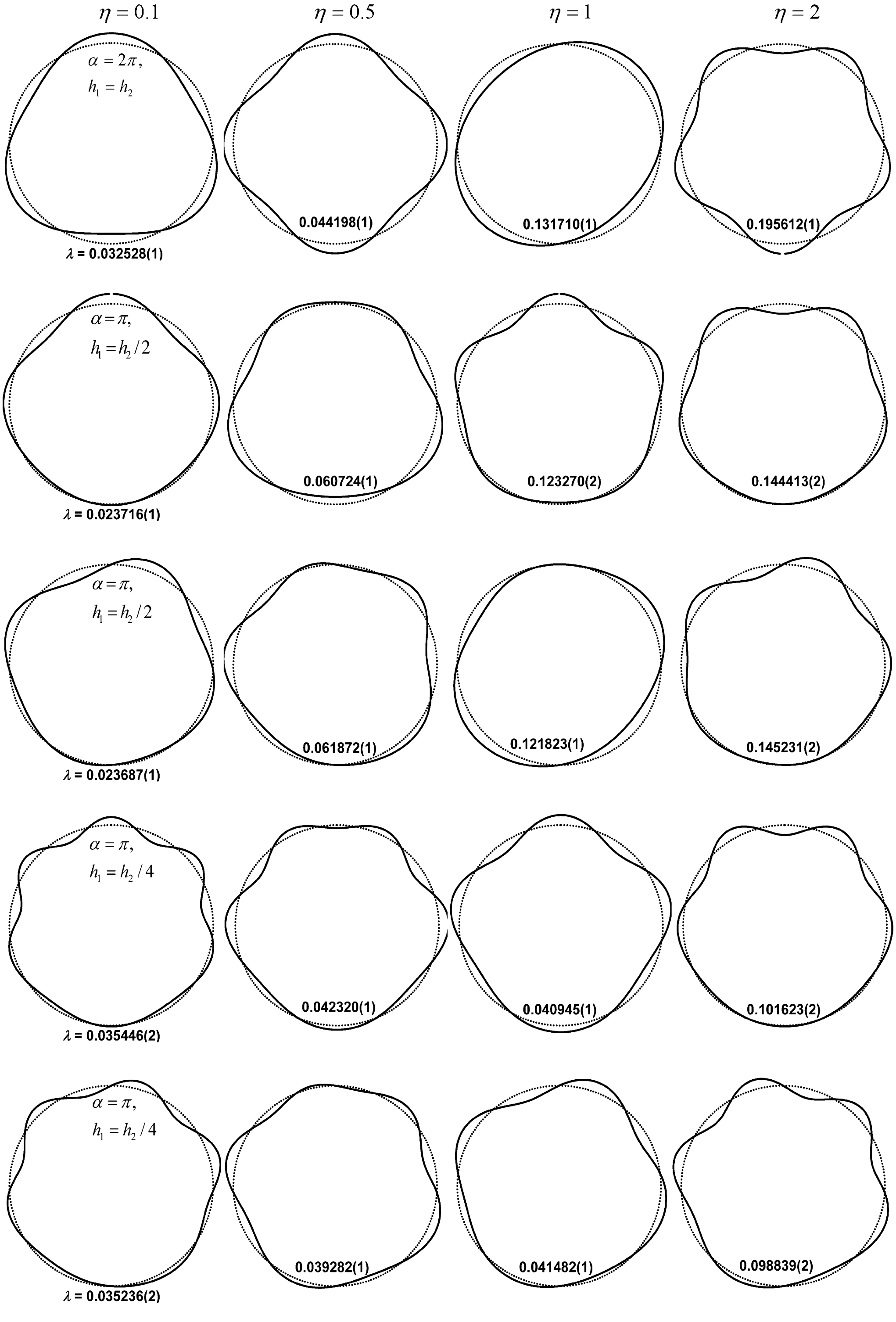

Fig. 4. The circumferential symmetric and antisymmetric vibration modes of two-stepped orthotropic cylindrical shells ($l_x = 4$, $h_2 = 0.02$, $\mu = 0.35$).

are much closed to each other and increase slightly with μ for the uniform and two-stepped shell. For the four-stepped cylindrical shell, the frequency parameter mode of symmetrical vibration changes from a higher mode to a lower mode for $0.1 \leqslant \mu \leqslant 0.5$ and becomes the maximum for $\mu = 0.35$ that gives the isotropic stepped cylindrical shell. The frequency parameter mode of the symmetrical vibration is higher than that for the antisymmetrical vibration when $\eta \geqslant 0.5$.

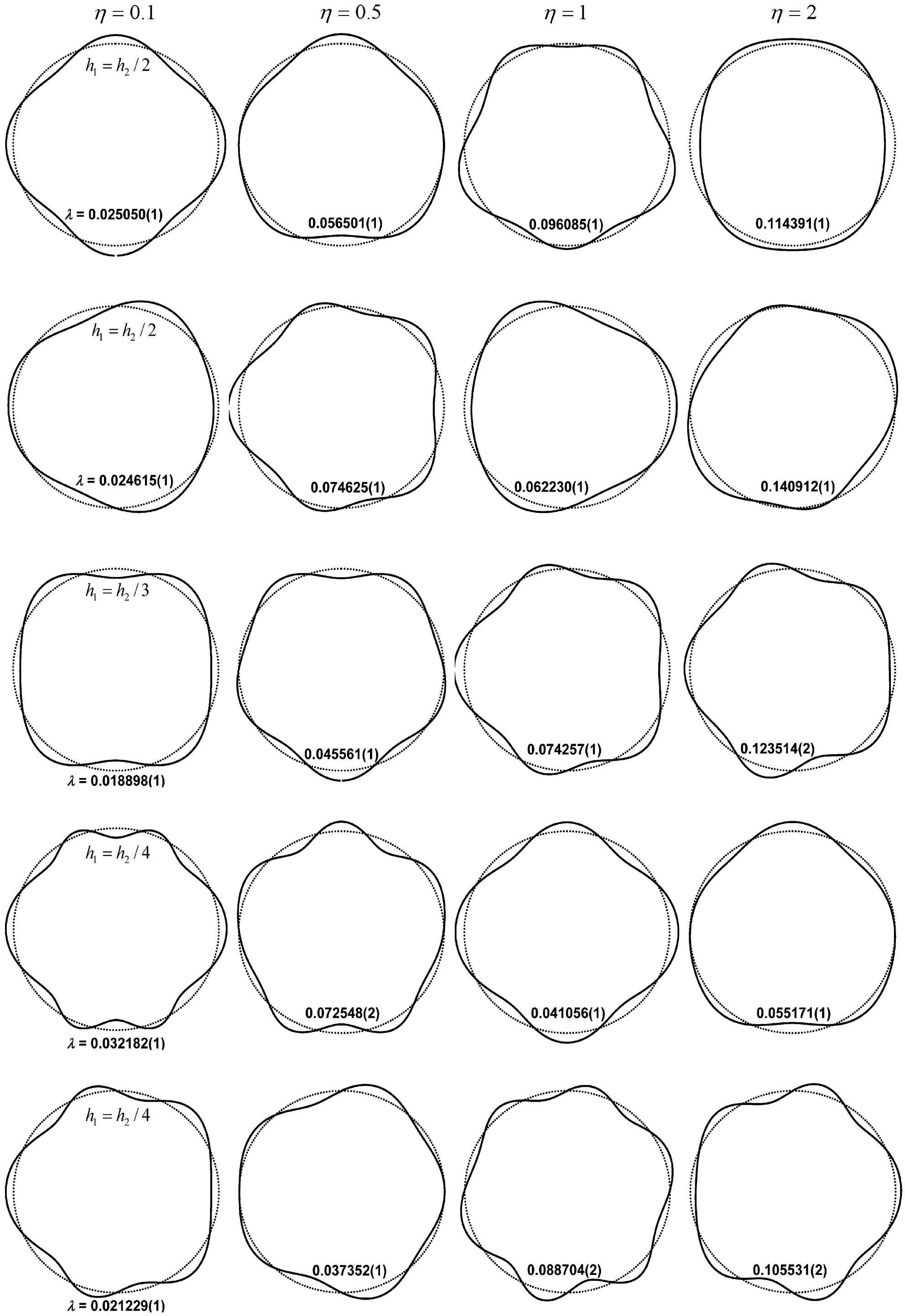

Fig. 5. The circumferential symmetric and antisymmetric vibration modes of four-stepped orthotropic cylindrical shells ($l_x = 4$, $h_2 = 0.02$, $\alpha = \pi/2$, $\mu = 0.35$).

5.3. Mode shapes of vibration

A vibration mode has been defined as the deflection of a structure at a particular frequency. To obtain the symmetric and antisymmetric mode shapes of vibration corresponding to a particular frequency parameter λ of the shell, we substitute by λ into Eqs. (13) and (14) and set the fourth unknown variables of the initial state vector in the same equations equal to 1, and therefore we get a set of linear non-homogeneous equations. By solving this new system for the remaining unknown initial state variables, the state vector at any point of the cross-section, *mode shapes of vibration*, can be obtained by further transfer matrix approach procedure. Generally, the vibration mode is located at the weakest generatrix of the shell and become pronounced over the sections with reduced thicknesses where the shell has less stiffness. Figs. 4 and 5 show the circumferential vibration modes of an orthotropic circular cylindrical shell with two and four step-wise thickness variations, respectively, for symmetric and antisymmetric type-modes. The thick lines show the composition of the circumferential and transverse deflections on the shell surface while the dotted lines show the original shell shape before the vibration case. The numbers in the parentheses are the axial half wave numbers corresponding to the frequency parameters λ listed in Tables 2 and 3. There are considerable differences between the modes of uniform and stepped shell for symmetrical and antisymmetrical modes. For ($h_1 = h_2$, $\alpha = 2\pi$), the vibration modes are distributed regularly over the shell surface, but for ($h_1 < h_2$, $\alpha = \pi, \pi/2$), the majority of symmetric and antisymmetric vibration-modes at the reduced region 1, with less stiffness, are larger than those at the region 2, with more stiffness, i.e. the vibration modes are localized near the weakest lines on the shell surface. However, for a given η value, the deflections of the vibration mode in the reduced regions become more pronounced with lower η, and the deformation is the maximum at $\Theta = 0$ for the two-stepped cylindrical shell but it is the maximum at $\Theta = 0, \pi$ for the four-stepped shell. It is also shown by Fig. 4 that very big difference is observed as $h_1 = h_2/2$, where the orthotropic mode shape of vibration is more quite than that for the isotropic mode shape which corresponds to $\eta = 1$. From these figures, one may be observed that the mode shapes of vibration are more sensitivity to the locations of step-wise thickness variations and step thickness rations than the axial and circumferential rigidities of the shell. Also it is found that the mode shapes are similar in the sets of the vibration modes having $m = 1, 2$ and 3 for given η and μ values.

6. Conclusions

Using the transfer matrix approach, an approximate analysis for studying the vibration behavior of an orthotropic circular cylindrical shell having circumferentially many reduced thicknesses over parts of its circumference is presented. For two- and four-stepped shell, the vibration frequencies were found to decrease and the mode shapes become more pronounced when step thickness ratios decrease. For two-stepped shell, the frequency parameters increase regularly with an increase of axial or circumferential rigidities and become the higher for an isotropic four-stepped shell. The frequency parameters increase rapidly with the circumferential rigidity when $\mu = 0.35$ and increase slightly with the axial rigidity when $\eta = 1$ for two-stepped orthotropic shells and they are very close to each other. The study showed that the deformations corresponding frequency parameters are located and concentrated at the less stiffened zones of the shell surface, reduced regions, and become more pronounced with lower step thickness ratios. The behavior of the orthotropic stepped shell is more quite than the isotropic shell for four step-wise thickness variations when $h_1 = h_3 = h_2/2$. Finally, it is worth to mention that the shells of stepped thickness over part of its circumference will undergo large deformation than the uniform shells, due to their low stiffness of reduced parts, and the geometry of stepped shells makes the frequency parameters are reduced and the mode shapes are pronounced.

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