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Variational principles in the theory of gradient plasticity

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ABSTRACT

Gradient models have been intensively discussed in the literature for the study of time-dependent or time-independent processes such as visco-plasticity, plasticity and damage. This Note is devoted to the theory of Gradient Plasticity. A general and consistent mathematical description available for common time-independent behavior is presented. Our attention is focused on the derivation of general results such as the description of the governing equations for the global response, for the rate response, the expression of the associated variational principles and the question of uniqueness in terms of the energy potential and the dissipation potential.

1. Introduction

In the two last decades, gradient theories have been much discussed in the literature, cf. for example [1–9]. In particular, the standard gradient models proposed by Frémond or Gurtin give an interesting method to introduce the gradient of the internal parameter and to derive the governing equations by the introduction of an additional virtual work equation. These models have been applied in Solid Mechanics for the description of various time-dependent or time-independent behavior in visco-plasticity, in plasticity, in phase change as in damage mechanics.

This Note is devoted to the theory of Gradient Plasticity. A general and consistent mathematical description is given for standard models. Our attention is focused on the derivation of some general theoretical results such as the governing equations of the global response, of the rate response, the associated variational principles and the question of uniqueness in terms of the energy potential and the dissipation potential.

2. Standard gradient models

In the internal variable framework, the thermo-mechanical response of a solid V in a reference configuration is described by the fields of displacement $\mathbf{u}$, of internal parameter $\mathbf{\Phi}$ and of temperature $\mathbf{T}$. The internal parameter is a scalar or a tensor and represents physically hidden parameters such as micro-displacements or phase proportions or anelastic strains, etc. For a gradient model in isothermal transformation, the set of state variables $(\nabla u, \phi, \nabla\phi)$ describes locally the material behavior.[1] The constitutive equations can be given in the following way.

2.1. Generalized forces and virtual work equation

It is first accepted that the state variables $(\nabla u, \phi, \nabla\phi)$ are associated with the generalized forces (σ, X, Y) such that a generalized virtual work equation holds:

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[1] Bold face uppercase letters as $\mathbf{\Phi}$ or $\mathbf{u}$ refer to fields whereas normal letters ϕ and u refer to local values.

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$$\begin{cases} P_i + P_j = P_e \quad \forall \delta u, \delta \phi \\ P_i = \int_V (\sigma \cdot \nabla \delta u + X \cdot \delta \phi + Y \cdot \nabla \delta \phi)\, \mathrm{d}V \\ P_j = \int_V \rho \ddot{u} \cdot \delta u \, \mathrm{d}V \\ P_e = \int_V (f_{vu} \cdot \delta u + f_{v\phi} \cdot \delta \phi)\, \mathrm{d}V + \int_{\partial V} (f_{su} \cdot \delta u + f_{s\phi} \cdot \delta \phi)\, \mathrm{d}a \end{cases} \tag{1}$$

where (f_{vu}, f_{su}) and $(f_{v\phi}, f_{s\phi})$ are respectively external body and surface forces associated with the displacement and the internal parameter. This means that the equilibrium equation and boundary condition hold for the stress

$$\begin{cases} \nabla \cdot \sigma + f_{vu} = \rho \ddot{u} & \forall x \in V \\ \sigma \cdot n = f_{su} & \forall x \in \partial V \end{cases} \tag{2}$$

and the following equations hold for the internal parameter after integration by parts:

$$\begin{cases} \nabla \cdot Y - X + f_{v\phi} = 0 & \forall x \in V \\ Y \cdot n = f_{s\phi} & \forall x \in \partial V \end{cases} \tag{3}$$

These equations are easily understood when ϕ is a micro-displacement, X is then an internal volume force and Y is a micro-stress in the same spirit as stress σ.

2.2. Energy and dissipation potentials

Standard gradient models also assume that there exist an energy potential w and a dissipation potential D per unit reference volume such that the following equations hold:

$$\begin{cases} w = w(\nabla u, \phi, \nabla \phi), \qquad D = D(\nabla \dot{u}, \dot{\phi}, \nabla \dot{\phi}, \phi) \\ \sigma = \sigma_e + \sigma_d, \qquad \sigma_e = w,_{\nabla u}, \qquad \sigma_d = D,_{\nabla \dot{u}} \\ X = X_e + X_d, \qquad X_e = w,_{\phi}, \qquad X_d = D,_{\dot{\phi}} \\ Y = Y_e + Y_d, \qquad Y_e = w,_{\nabla \phi}, \qquad Y_d = D,_{\nabla \dot{\phi}} \end{cases} \tag{4}$$

when the potentials w and D are smooth functions of its arguments. The dissipation potential can be state-dependent via the current value of the internal parameter. The relationships $X_d = D,_{\dot{\phi}}$, $Y_d = D,_{\nabla \dot{\phi}}$ describe a time-dependent behavior of the materials and are commonly discussed in Visco-Elasticity and in Visco-plasticity, in Phase change as in Damage Mechanics.

2.3. Governing equations for time-dependent processes

Eqs. (2), (3), (4) are the governing equations of a time-dependent standard gradient model. In terms of the two potentials, the governing equations for the fields of unknown **u**, **Φ** are

$$\begin{cases} \nabla \cdot (w,_{\nabla u} + D,_{\nabla \dot{u}}) + f_{vu} = \rho \ddot{u} & \forall x \in V \\ (w,_{\nabla u} + D,_{\nabla \dot{u}}) \cdot n = f_{su} & \forall x \in \partial V \\ \nabla \cdot (w,_{\nabla \phi} + D,_{\nabla \dot{\phi}}) - w,_{\phi} - D,_{\dot{\phi}} + f_{v\phi} = 0 & \forall x \in V \\ (w,_{\nabla \phi} + D,_{\nabla \dot{\phi}}) \cdot n = f_{s\phi} & \forall x \in \partial V \end{cases} \tag{5}$$

These equations describe the response of the solid from an initial position of state and velocity. The forces $f_{v\phi}$ and $f_{s\phi}$ appears as physical data. In this spirit, the condition $f_{v\phi} = 0$ and $f_{s\phi} = 0$ has been denoted as the **constitutive insulation condition** following a terminology due to Polizzotto [7]. The response of a solid under insulation condition has been discussed by several authors, cf. [2,7,8,10].

3. Time-independent processes and gradient plasticity

For time-independent processes such as plasticity or brittle damage, the dissipation potential D is convex, positively homogeneous of degree 1 of the rates:

$$D(\mu \dot{\phi}, \mu \nabla \dot{\phi}, \phi) = \mu D(\dot{\phi}, \nabla \dot{\phi}, \phi) \quad \forall \mu > 0. \tag{6}$$

Because of the loss of differentiability with respect to $(\dot{\phi}, \nabla \dot{\phi})$ when $(\dot{\phi}, \nabla \dot{\phi}) = (0, 0)$, the governing equations (5) must be specified.

3.1. Force–flux relationships

Since D is a sub-differentiable function cf. [11,12], the force–flux relationship can be written as

$$X_d = \partial_{\dot\phi} D, \qquad Y_d = \partial_{\nabla\dot\phi} D \tag{7}$$

This means that the force (X_d, Y_d) must belong to the elastic domain $C = \partial D_{(\dot\phi,\nabla\dot\phi)}(0,0,\phi)$, which is the convex set of admissible forces, and that the normality law is satisfied by the rates $(\dot\phi, \nabla\dot\phi)$.

For example, if

$$D = k(\phi)\|\dot\phi\| + \kappa(\phi)\|\nabla\dot\phi\| \tag{8}$$

then the convex of admissible forces is given by two inequalities

$$f(X_d,\phi) = \|X_d\| - k(\phi) \leqslant 0, \qquad \varphi(Y_d,\phi) = \|Y_d\| - \kappa(\phi) \leqslant 0 \tag{9}$$

and the rates $\dot\phi$, $\nabla\dot\phi$ must satisfy the normality law

$$\dot\phi = \lambda\frac{\partial f}{\partial X_d} \quad \text{with } \lambda \geqslant 0,\ \lambda f = 0, \qquad \nabla\dot\phi = \mu\frac{\partial\varphi}{\partial Y_d}, \quad \mu \geqslant 0,\ \mu\varphi = 0 \tag{10}$$

From (3), these forces must also satisfy $X_e + X_d - \nabla\cdot(Y_d + Y_e) = 0$ and appropriate boundary conditions. However, these relationships do not determine $(X_d,\ Y_d)$ when $(\dot\phi,\ \nabla\dot\phi) = (0,0)$, even if the present state is known. This indetermination is the principal difficulty of the gradient model of plasticity when the dissipation potential is gradient-dependent.

The following case has been also considered in the literature, cf. for example [6,9]:

$$D = k(\phi)\big(\|\dot\phi\|^2 + \ell^2\|\nabla\dot\phi\|^2\big)^{1/2} \tag{11}$$

and leads to a Mises-like plastic criterion and the normality law

$$\begin{cases} f = (\|X_d^p\|^2 + \frac{1}{\ell^2}\|Y_d^p\|^2)^{1/2} - k(\phi) \\ \dot\epsilon^p = \lambda\frac{\partial f}{\partial X_d}, \qquad \nabla\dot\epsilon^p = \lambda\frac{\partial f}{\partial Y_d} \\ f \leqslant 0, \qquad \lambda \geqslant 0, \qquad f\lambda = 0 \end{cases} \tag{12}$$

It is classical that the dissipation potential is obtained from the elastic domain by the maximum dissipation principle

$$D = D(\dot\phi, \nabla\dot\phi, \phi) = \max_{(X_d^*, Y_d^*)\in C_\phi} X_d^*\cdot\dot\phi + Y_d^*\cdot\nabla\dot\phi \tag{13}$$

The particular case of a model admitting a gradient-independent dissipation potential

$$D = D(\dot\phi, \phi) \tag{14}$$

is also interesting to be considered. In this case, the *force–flux* relationship is

$$X_d = \partial D(\dot\phi, \phi), \qquad Y_d = 0 \tag{15}$$

This means that the force X_d must belong to a convex domain of admissible forces $C = \partial D(0,\phi)$ and that the normality law is satisfied by the rate $\dot\phi$. There is no indetermination difficulty as in the general case since from (3), $X_d = -X_e + \nabla\cdot Y_e$ is known at the present state.

3.2. Governing equations

Under the insulation condition, the governing equations, given by (1), (3), (7), are

$$\begin{cases} \sigma = w_{,\nabla u}, \qquad X_e = w_{,\phi}, \qquad Y_e = w_{,\nabla\phi} \\ X = X_e + X_d, \qquad Y = Y_e + Y_d, \qquad (X_d,\ Y_d) = \partial D(\dot\phi, \nabla\dot\phi, \phi) \\ P_i + P_j = P_e \quad \forall\delta u, \delta\phi \\ P_i = \int_V (\sigma\cdot\nabla\delta u + X\cdot\delta\phi + Y\cdot\nabla\delta\phi)\,\mathrm{d}V \\ P_j = \int_V \rho\ddot u\cdot\delta u\,\mathrm{d}V \\ P_e = \int_V f_{vu}\cdot\delta u\,\mathrm{d}V + \int_{\partial V} f_{su}\cdot\delta u\,\mathrm{d}a \end{cases} \tag{16}$$

In quasi-static transformation, a variational form of the governing equations also exists and is given by the following evolutionary variational inequality, cf. for example [13,12,14]

Proposition 1. *The response of a solid* $\mathbf{U(t)} = (\mathbf{u}, \mathbf{\Phi})$ *under a quasi-static loading* $\mathbf{F(t)}$ *is a solution of the variational inequality*

$$\mathbf{W},_{\mathbf{U}} \cdot (\delta\mathbf{U} - \dot{\mathbf{U}}) + \mathbf{D}(\delta\mathbf{U}, \mathbf{U}) - \mathbf{D}(\dot{\mathbf{U}}, \mathbf{U}) - \mathbf{F} \cdot (\delta\mathbf{U} - \dot{\mathbf{U}}) \geqslant \mathbf{0} \quad \forall\delta\mathbf{U} \tag{17}$$

with

$$\begin{cases} \mathbf{W}(\mathbf{U}) = \int_V w(\nabla u, \phi, \nabla\phi)\,\mathrm{d}V, \qquad \mathbf{D}(\dot{\mathbf{U}}, \mathbf{U}) = \int_V D(\dot{\phi}, \nabla\dot{\phi}, \phi)\,\mathrm{d}V \\ \mathbf{F} \cdot \delta\mathbf{U} = \int_V f_{vu} \cdot \delta u\,\mathrm{d}V + \int_{\partial V} f_{su} \cdot \delta u\,\mathrm{d}a \end{cases} \tag{18}$$

Indeed, this variational inequality can be written in the form of a mechanical energy balance and a minimum principle

$$\begin{cases} \int_V (w,_{\nabla u} : \nabla\delta u - f_{vu} \cdot \delta u)\,\mathrm{d}V - \int_{\partial V} f_{su} \cdot \delta u\,\mathrm{d}a = 0 \\ \int_V (X_e \cdot \dot{\phi} + Y_e \cdot \nabla\dot{\phi} + D(\dot{\phi}, \nabla\dot{\phi}, \phi))\,\mathrm{d}V = 0 \\ \quad = \min_{\delta\mathbf{\Phi}} \int_V (X_e \cdot \delta\phi + Y_e \cdot \nabla\delta\phi + D(\delta\phi, \nabla\delta\phi, \phi))\,\mathrm{d}V \end{cases} \tag{19}$$

Since a solution $\mathbf{u(t)}$, $\mathbf{\Phi(t)}$ of the minimum principle must satisfy

$$\begin{cases} \text{There exists } (X_d, Y_d) \in \partial D_{(\dot{\phi}, \nabla\dot{\phi})}(\dot{\phi}, \nabla\dot{\phi}, \phi) \quad \text{such that} \\ \int_V (X_e \cdot \delta\phi + Y_e \cdot \nabla\delta\phi + X_d \cdot \delta\phi + Y_d \cdot \nabla\delta\phi)\,\mathrm{d}V = 0 \quad \forall\delta\phi \end{cases} \tag{20}$$

thus the governing equations (16) results.

3.3. Rate problem

It is, however, necessary to check that Eqs. (17) defines effectively an incremental process for a solid submitted to a loading path. At a current time t, if the present state $(\mathbf{u}, \mathbf{\Phi})$ is assumed to be known, it must be possible to determine the rate of the unknowns (displacement and internal parameter) $(\dot{\mathbf{u}}, \dot{\mathbf{\Phi}})$ in terms of the rate of the data $\dot{\mathbf{F}} = (\dot{\mathbf{f}}_{\mathbf{vu}}, \dot{\mathbf{f}}_{\mathbf{su}})$. This is a necessary condition to follow step-by-step the response of the solid along the loading path.

3.3.1. Minimum principle and admissible rates

For a given couple of energetic forces $(\mathbf{X_e}, \mathbf{Y_e})$, let us consider the solutions of the minimum problem

$$m = \min_{\mathbf{r}} \int_V \big(X_e \cdot r + Y_e \cdot \nabla r + D(r, \nabla r, \phi)\big)\,\mathrm{d}V \tag{21}$$

where $\mathbf{r}$ is an arbitrary rate of internal parameter.

It is clear that

(i) When $m = 0$, if $\mathbf{r_o} \neq \mathbf{0}$ is a solution then $a\mathbf{r_o}$ is also a solution for all number $a > 0$.
(ii) If $\mathbf{r_o} \neq \mathbf{0}$ and $\mathbf{r}'_{\mathbf{o}} \neq \mathbf{0}$ are two different solutions, then $\alpha\mathbf{r_o} + (1 - \alpha)\mathbf{r}'_{\mathbf{o}}$ is also a solution since D is a convex function.

From (19), it is also clear that $\dot{\mathbf{\Phi}}$ must belong to the set $\mathcal{R}$ of solutions of (21) with $m = 0$.

Since the integrants in (21) consist of linear or degree-1 functions, it is interesting to compute the value m_1 of the same functional on the set of normalized fields, for example $\int_V \|r\|\,\mathrm{d}V = 1$. It follows that if $m_1 > 0$, then $m = 0$ and $\mathbf{r} = \mathbf{0}$ is the only solution. If $m_1 = 0$, then $m = 0$ and one or many solutions $\mathbf{r_o} \neq \mathbf{0}$ exist. If $m_1 < 0$, then $m = -\infty$ and there is no solution. Thus, if $m_1 > 0$, then $\dot{\mathbf{\Phi}} = \mathbf{0}$ and the incremental behavior is purely elastic. If $m_1 < 0$, the energetic forces $(\mathbf{X_e}, \mathbf{Y_e})$ are not allowable by the constitutive equations. If $m_1 = 0$, the response $\dot{\mathbf{\Phi}}$ is to be found in $\mathcal{R}$.

The model (11) is considered here for example. Let $\mathbf{r_o} \neq \mathbf{0}$ be a solution of the minimum principle and $\mathbf{V_{r_o}}$ the loading zone associated with this solution:

$$x \in \mathbf{V_{r_o}} \quad \Leftrightarrow \quad \big(r_o(x), \nabla r_o(x)\big) \neq (0, 0) \tag{22}$$

and Ω the reunion of all loading zones associated with the solutions of the minimum principle:

$$\Omega = \bigcup_{\mathbf{r_o} \in \mathcal{S}} \mathbf{V_{r_o}} \tag{23}$$

Since the criterion (11) is strictly convex, the following proposition holds:

Proposition 2. *The dissipative forces* (X_d, Y_d) *are uniquely defined in* Ω *which is the reunion of all loading zones associated with the solutions of the minimum principle.*

The proof is quite simple as in rigid plasticity, cf. Mandel [15] for example, and based upon the strict convexity of the elastic domain C.

The set $\mathcal{R}$ can also be described in the following way:

By definition, a rate $\delta\mathbf{\Phi}$ is admissible if locally, $(\delta\phi, \nabla\delta\phi)$ belongs to the normal cone of the elastic domain C at (X_d, Y_d) for all $x \in \Omega$

$$\delta\mathbf{\Phi} \text{ admissible} \quad \Leftrightarrow \quad \big(\delta\phi(x), \nabla\delta\phi(x)\big) \in N_C(X_d, Y_d) \quad \forall x \in \Omega \tag{24}$$

For example, in terms of the criterion $f(X_d, Y_d, \phi) \leqslant 0$

$$\begin{cases} \delta\mathbf{\Phi} \text{ admissible} \quad \Leftrightarrow \quad \delta\phi = \lambda\frac{\partial f}{\partial X_d}, \qquad \nabla\delta\phi = \lambda\frac{\partial f}{\partial X_d} \\ f \leqslant 0, \quad \lambda \geqslant 0, \quad \lambda f = 0 \quad \forall x \in \Omega \end{cases} \tag{25}$$

The following statement holds:

Proposition 3. *An admissible rate is a solution of the minimum principle*

$$\delta\mathbf{\Phi} \textit{ admissible} \quad \Leftrightarrow \quad \delta\mathbf{\Phi} \in \mathcal{R} \tag{26}$$

Indeed, from (i) and (ii), it is possible to combine different solutions $\mathbf{r_o}$ to construct a solution φ such that $\varphi \neq 0$ for all $x \in \Omega$. This solution admits as loading zone Ω and the forces $(\mathbf{X_d}, \mathbf{Y_d})$ can be completed in $V - \Omega$ to obtain an admissible force system $(\mathbf{X_d}, \mathbf{Y_d})$ in the whole volume V since φ is a solution. This system of forces $(\mathbf{X_d}, \mathbf{Y_d})$ is associated with any admissible rate $\delta\mathbf{\Phi}$. It is clear that Ω can be identified as the current plastic zone, where the criterion of plasticity is satisfied $f(X_d, Y_d, \phi) = 0$.

For a model admitting the expression (9) as dissipation potential, the discussion can be done in the same spirit. It is however necessary to introduce the plastic zone Ω_X, where the criterion $f(X_d, \phi) = 0$ is satisfied, and the plastic zone Ω_Y, where the criterion $\varphi(Y_d, \phi) = 0$ is satisfied.

The discussion is straightforward when the dissipation potential is gradient-independent. The dissipation potential $D(\dot\phi, \phi)$ leads to a convex elastic domain C in the force space X_d. The minimum principle (21) is now

$$m = \min_r \int_V \big((X_e - \nabla \cdot Y_e) \cdot r + D(r, \phi)\big)\,\mathrm{d}V + \int_{\partial V} (Y_e \cdot n) \cdot r\,\mathrm{d}a \tag{27}$$

Thus the minimum m is

$$\begin{cases} m = -\infty \quad \text{if } X_d = X_e - \nabla \cdot Y_e \notin C\ \forall x \in V \text{ or } Y_e \cdot n \neq 0\ \forall x \in \partial V \\ m = 0 \qquad \text{if } X_d = X_e - \nabla \cdot Y_e \in C\ \forall x \in V \text{ and } Y_e \cdot n = 0\ \forall x \in \partial V \end{cases} \tag{28}$$

It is clear that the set $\mathcal{R}$ of solutions is composed of admissible rates defined on the present plastic zone. For example

$$\Omega = \big\{x \in V \mid f\big(X_d(x), \phi(x)\big) = 0\big\} \tag{29}$$

when the elastic domain C is defined by the criterion $f(X_d, \phi) \leqslant 0$.

3.3.2. Rate equations

The rate equations describe the system of equations satisfied by the rate of the unknowns (displacement and internal parameter) $(\dot{\mathbf{u}}, \dot{\mathbf{\Phi}})$ in terms with the data-rate $\dot{\mathbf{F}} = (\dot{\mathbf{f}}_{\mathbf{vu}}, \dot{\mathbf{f}}_{\mathbf{su}})$. The following statement holds in a quasi-static transformation:

Proposition 4. *The rate response $\dot{\mathbf{U}} = (\dot{\mathbf{u}}, \dot{\mathbf{\Phi}})$ is a solution of the variational inequality*

$$\begin{cases} \mathbf{W}_{,\mathbf{UU}}[\dot{\mathbf{U}}, \delta\mathbf{U} - \dot{\mathbf{U}}] + \dot{\mathbf{U}} \cdot (\mathbf{D}_{,\mathbf{U}}(\delta\mathbf{U}, \mathbf{U}) - \mathbf{D}_{,\mathbf{U}}(\dot{\mathbf{U}}, \mathbf{U})) \\ -\dot{\mathbf{F}} \cdot (\delta\mathbf{U} - \dot{\mathbf{U}}) \geqslant 0 \quad \forall \textit{ admissible rates } \delta\mathbf{U} \end{cases} \tag{30}$$

This proposition shows how the rate $(\dot{\mathbf{u}}, \dot{\mathbf{\Phi}})$ can be selected among the admissible rates.

The proof of this proposition can be obtained for example from (17) by a time-discretization. The present state is assumed to be given, an increment of the response $\Delta U = \dot U \Delta t$ associated with an increment of load must satisfy (17) at the next step:

$$\begin{cases} \mathbf{W}_{,\mathbf{U}}(\mathbf{U} + \mathbf{\Delta U}) \cdot (\delta\mathbf{U} - \mathbf{\Delta U}) + \mathbf{D}(\delta\mathbf{U}, \mathbf{U} + \mathbf{\Delta U}) - \mathbf{D}(\mathbf{\Delta U}, \mathbf{U} + \mathbf{\Delta U}) \\ -\mathbf{F} + \mathbf{\Delta F} \cdot (\delta\mathbf{U} - \mathbf{\Delta U}) \geqslant 0 \quad \forall\delta\mathbf{U} \end{cases} \tag{31}$$

with

$$\begin{cases} \mathbf{W},_{\mathbf{U}}(\mathbf{U}+\boldsymbol{\Delta}\mathbf{U}) = \mathbf{W},_{\mathbf{U}}(\mathbf{U}) + \mathbf{W},_{\mathbf{UU}}(\mathbf{U})\cdot\boldsymbol{\Delta}\mathbf{U} + \text{h.o.t.} \\ \mathbf{D}(\delta\mathbf{U},\mathbf{U}+\boldsymbol{\Delta}\mathbf{U}) = \mathbf{D}(\delta\mathbf{U},\mathbf{U}) + \mathbf{D},_{\mathbf{U}}(\delta\mathbf{U},\mathbf{U})\cdot\boldsymbol{\Delta}\mathbf{U} + \text{h.o.t.} \end{cases} \tag{32}$$

The special choice ΔU and δU are admissible rates ensures that

$$\mathbf{W},_{\mathbf{U}}(\mathbf{U})\cdot\delta\mathbf{U} + \mathbf{D}(\delta\mathbf{U},\mathbf{U}) - \mathbf{F}\cdot\delta\mathbf{U} = 0 \quad \forall\delta\mathbf{U}\ \text{admissible}$$

In (31), the first order term is zero thus the second order terms must be positive and the variational inequality (30) follows.

In the case of a dissipation potential of the form $D = D(\dot{\phi}, \nabla\dot{\phi}, \phi)$, this variational inequality is

$$\begin{cases} \mathbf{W},_{\mathbf{uu}}[\dot{\mathbf{u}},\delta\mathbf{u}] + \mathbf{W},_{\mathbf{u}\boldsymbol{\Phi}}[\dot{\boldsymbol{\Phi}},\delta\mathbf{u}] - \dot{\mathbf{F}}\cdot\delta\mathbf{u} = 0 \quad \forall\delta\mathbf{u} \\ \mathbf{W},_{\boldsymbol{\Phi}\mathbf{u}}[\dot{\mathbf{u}},\delta\boldsymbol{\Phi}-\dot{\boldsymbol{\Phi}}] + \mathbf{W},_{\boldsymbol{\Phi}\boldsymbol{\Phi}}[\dot{\boldsymbol{\Phi}},\delta\boldsymbol{\Phi}-\dot{\boldsymbol{\Phi}}] + \dot{\boldsymbol{\Phi}}\cdot(\mathbf{D},_{\boldsymbol{\Phi}}(\delta\boldsymbol{\Phi},\boldsymbol{\Phi}) - \mathbf{D},_{\boldsymbol{\Phi}}(\dot{\boldsymbol{\Phi}},\boldsymbol{\Phi})) \geqslant 0 \quad \forall\delta\boldsymbol{\Phi}\ \text{admissible} \end{cases} \tag{33}$$

The variational inequality (30) is symmetric if the following symmetry:

$$\delta\mathbf{U}\cdot\mathbf{D},_{\mathbf{U}}(\dot{\boldsymbol{\Phi}},\mathbf{U}) = \dot{\mathbf{U}}\cdot\mathbf{D},_{\mathbf{U}}(\delta\boldsymbol{\Phi},\mathbf{U}) \tag{34}$$

is satisfied. In this case, it is also equivalent to an extremum principle since the following statement holds:

Proposition 5. *Under the assumption of symmetry, the rate* $\dot{\mathbf{U}} = (\dot{\mathbf{u}}, \dot{\boldsymbol{\Phi}})$ *minimizes the rate-functional* $\mathbf{H}(\delta\mathbf{U})$ *among the admissible rates*

$$\mathbf{H}(\delta\mathbf{U}) = \frac{1}{2}\mathbf{W},_{\mathbf{UU}}[\delta\mathbf{U},\delta\mathbf{U}] + \delta\mathbf{U}\cdot\mathbf{D},_{\mathbf{U}}(\delta\mathbf{U},\mathbf{U}) - \dot{\mathbf{F}}\cdot\delta\mathbf{U} \tag{35}$$

when the following condition of positivity holds:

$$\mathbf{W},_{\mathbf{UU}}[\delta\mathbf{U},\delta\mathbf{U}] + \delta\mathbf{U}\cdot\mathbf{D},_{\mathbf{U}}(\delta\mathbf{U},\mathbf{U}) > 0 \quad \forall\delta\mathbf{U}\ \textit{admissible} \tag{36}$$

Indeed, for any admissible field $\delta\mathbf{U}$,

$$\begin{cases} \mathbf{H}(\delta\mathbf{U}) - \mathbf{H}(\dot{\mathbf{U}}) = \mathbf{W},_{\mathbf{UU}}[\delta\mathbf{U}-\dot{\mathbf{U}},\delta\mathbf{U}-\dot{\mathbf{U}}] + (\delta\mathbf{U}-\dot{\mathbf{U}})\cdot\mathbf{D},_{\mathbf{U}}(\delta\mathbf{U}-\dot{\mathbf{U}},\mathbf{U}) \\ \quad + \mathbf{W},_{\mathbf{U}}\cdot(\delta\mathbf{U}-\dot{\mathbf{U}}) + \mathbf{D}(\delta\mathbf{U},\mathbf{U}) - \mathbf{D}(\dot{\mathbf{U}},\mathbf{U}) - \mathbf{F}\cdot(\delta\mathbf{U}-\dot{\mathbf{U}}) \geqslant 0 \end{cases}$$

The rate $\dot{\mathbf{U}}$ is also unique if the quadratic form (36) is strictly positive on the linear space generated by the admissible rates. In particular, when the dissipation potential is state-independent, the strict convexity of the energy potential ensures the uniqueness of the rate response.

In the same spirit as in Classical Plasticity, the description of the rate problem leads naturally to the study of the stability of an equilibrium position and the bifurcation of a quasi-static response. For example, the same strictly positive condition on admissible rates is a criterion of stability of the present equilibrium, cf. [16,12].

From a criterion of plasticity $f(X_d, Y_d, \phi) \leqslant 0$, with the notation $\alpha = (\dot{\phi}, \nabla\dot{\phi})$ and $A = (X_d, Y_d)$, the expression

$$D(\alpha,\phi) = \max_{f(A^*,\phi)\leqslant 0} A^*\cdot\alpha = A\cdot\alpha - \lambda f(A,\phi)$$

gives

$$\begin{cases} \delta\phi\cdot D,_{\phi}(\alpha,\phi) = \delta A\cdot(\alpha - \lambda f,_{A}) - \lambda f,_{\phi}\delta\phi - \delta\lambda f \\ \quad = \mu f,_{X_d}\cdot f,_{\phi}\lambda \quad \text{with}\ \delta\phi = \mu f,_{X_d} \end{cases}$$

Thus the symmetry (34) is ensured and (33) can also be written as

$$\begin{cases} \int_V \nabla\delta u\cdot(w,_{\nabla u\,\nabla u}\cdot\nabla\dot{u} + w,_{\nabla u\phi}\cdot\dot{\phi})\,\mathrm{d}V - \dot{\mathbf{F}}\cdot\delta\mathbf{u} = 0 \\ \int_V (\delta\phi - \dot{\phi})\cdot(w,_{\nabla u\phi}\cdot\nabla u + w,_{\phi\phi}\cdot\dot{\phi}) + (\mu-\lambda) f,_{X_d}\cdot f,_{\phi}\lambda \geqslant 0 \\ \quad \forall\,\forall\delta\mathbf{u},\delta\boldsymbol{\Phi}\ \text{admissible},\ \delta\phi = \mu f,_{X_d} \end{cases} \tag{37}$$

and $\dot{\mathbf{U}}$ minimizes the functional $\mathbf{H}(\delta\mathbf{U})$ in $\mathcal{R}$

$$\begin{cases} \mathbf{H}(\delta\mathbf{U}) = \int_V \frac{1}{2}(\nabla\delta u\cdot w,_{\nabla u\nabla u}\cdot\nabla\delta u + 2\delta\phi\cdot w,_{\nabla u\phi}\cdot\nabla\delta u + \delta\phi\cdot w,_{\phi\phi}\cdot\delta\phi \\ \quad + f,_{X_d}\cdot f,_{\phi}\mu^2)\,\mathrm{d}V - \dot{\mathbf{F}}\cdot\delta\mathbf{u} \end{cases} \tag{38}$$

More generally, it has been established [12] that the symmetry (34) is equivalent to the symmetry of the interaction matrix $h_{ij} = f^i,_{X_d}\cdot f^j,_{\phi}$ in the case of multiple plastic criterion $f^i(X_d, Y_d, \phi) \leqslant 0$, $i = 1, n$.

3.4. Example of uniqueness of the response

The response of a solid of governing equation (17) under a given loading path and insulation condition is considered from a given initial state in isothermal transformation. Let U_i, $i = 1, 2$ denote two possible solutions, if the dissipation potential is state-independent, then the combination of the governing equations associated with these solutions gives

$$(\mathbf{W},_{\mathbf{U_2}} - \mathbf{W},_{\mathbf{U_1}}) \cdot (\dot{\mathbf{U}}_\mathbf{2} - \dot{\mathbf{U}}_\mathbf{1}) \leqslant 0$$

If the energy $\mathbf{W}(\mathbf{U})$ is a quadratic functional, since

$$\frac{\mathrm{d}}{\mathrm{d}t}((\mathbf{W}(\mathbf{U_2} - \mathbf{U_1}) = 2(\mathbf{W},_{\mathbf{U_2}} - \mathbf{W},_{\mathbf{U_1}}) \cdot (\dot{\mathbf{U}}_\mathbf{2} - \dot{\mathbf{U}}_\mathbf{1}) \leqslant 0$$

there is *a contraction of the energy distance between two responses*. It follows that

$$\mathbf{W}\big(\mathbf{U_2}(\mathbf{t}) - \mathbf{U_1}(\mathbf{t})\big) \leqslant \mathbf{W}\big(\mathbf{U_2}(\mathbf{0}) - \mathbf{U_1}(\mathbf{0})\big) = 0$$

The positivity condition

$$\mathbf{W}(\mathbf{U}) \geqslant \mathbf{a}\|\mathbf{U}\|^\mathbf{2}$$

ensures then $U_2(t) = U_1(t)$ for all $t \geqslant 0$. Thus, the following statement holds:

Proposition 6. *The uniqueness of the response is ensured if the energy potential is quadratic and strictly positive and if the dissipation pseudo-potential is state-independent.*

3.5. Example of gradient model of plasticity with isotropic and kinematic hardening

In small transformation, an interesting model of plasticity with isotropic and kinematic hardening consists of internal variable $\phi = (\epsilon^p, \gamma)$, of energy

$$w = w_e\big(\epsilon - \epsilon^p\big) + w_c\big(\epsilon^p\big) + w_i(\gamma) + w_g(\nabla\gamma)$$

and a multiple Mises-like plastic potential of the form

$$f\big(X_d^p, X_d^\gamma\big) = \big\|X_d^p\big\| + X_d^\gamma - k \leqslant 0, \qquad \varphi\big(Y_d^\gamma\big) = \big\|Y_d^\gamma\big\| - \kappa \leqslant 0$$

where k and κ are two constants. The dissipation potential and the normality law are

$$D\big(\dot\epsilon^p, \dot\gamma, \nabla\dot\gamma, \gamma\big) = \max_{f(X_d^{p*}, X_d^{\gamma*}) \leqslant 0, \varphi(Y_d^{\gamma*}) \leqslant 0} X_d^{p*} \cdot \dot\epsilon^{p*} + Y_d^{\gamma*} \cdot \nabla\dot\gamma + X_d^{\gamma*}\,\dot\gamma$$

$$\begin{cases} \dot\epsilon^p = \lambda \frac{\partial f}{\partial X_d^p} = \lambda \frac{X_d^p}{\|X_d^p\|} \\ \dot\gamma = \lambda \frac{\partial f}{\partial X_d^\gamma} = \lambda, \quad f \leqslant 0,\ \lambda \geqslant 0,\ f\lambda = 0 \\ \nabla\dot\gamma = \mu \frac{\partial \varphi}{\partial Y_d^\gamma} = \mu \frac{Y_d^\gamma}{\|Y_d^\gamma\|}, \quad \varphi \leqslant 0,\ \mu \geqslant 0,\ \varphi\mu = 0 \end{cases}$$

γ is thus the equivalent plastic strain and the dissipation is $d = X_d^p \cdot \dot\epsilon^p + Y_d^\gamma \cdot \nabla\dot\gamma + X_d^\gamma\,\dot\gamma = k\lambda + \kappa\mu$.

For example, in linear isotropic and kinematic hardening with

$$\begin{cases} w_e = 1/2(\epsilon - \epsilon^p) : L : (\epsilon - \epsilon^p), \qquad w_c = h/2\epsilon^p : \epsilon^p, \\ w_g = \frac{g}{2}\nabla\epsilon^p \cdot \nabla\epsilon^p, \qquad w_i = \frac{I}{2}\gamma^2 \\ f = (\|X_d^\phi\|^2 + \frac{1}{\ell^2}\|Y_d^\phi\|^2)^{1/2} + X_d^\gamma - k \leqslant 0 \end{cases}$$

the response in stress and strain of a solid from a given initial state under a given loading path is unique when $g > 0$, $h = I = 0$, in contrast with the well known phenomenon of localization obtained in perfect plasticity (where $g = h = I = 0$). A discussion on existence and uniqueness of the quasi-static response has been recently given by Giacomini and Musesti [17] in linear isotropic hardening ($h = 0$, $g = 0$, $I > 0$).

At finite strain, several models of gradient plasticity have been proposed, cf. [18], using the classical multiplicative decomposition $\nabla u = F_e F_p$ where F_e and F_p are the elastic and plastic transformation gradients, the internal parameter is F_p:

$$w(\nabla u, F_p, \nabla F_p) = w_e(\nabla u, F_p) + w_c(F_p) + w_g(F_p, \operatorname{Curl} F_p)$$

Some results on existence and uniqueness have also been given for the particular case of gradient-independent dissipation potentials, cf. Mainik and Mielke [19].

3.6. Energy regularization

The following approach has been proposed to avoid the difficulty on the determination of the current plastic zone Ω when the dissipation potential is gradient-dependent. The idea is to go back to the particular case of gradient-independent dissipation by *energy regularization*, thanks to the introduction of an additional internal parameter β and of an additional energy. For example, an additional term of the form $1/2r\|\beta - \nabla\phi\|^2$ can be included in the energy while $\nabla\dot{\phi}$ is replaced by $\dot{\beta}$ in the dissipation potential. This leads to a model of internal parameters ϕ, β and of potentials

$$\begin{cases} w(\nabla u, \phi, \nabla\phi, \beta) = w(\nabla u, \phi, \nabla\phi) + \frac{1}{2}r\|\beta - \nabla\phi\|^2 \\ D = D(\dot{\phi}, \dot{\beta}, \phi) \end{cases} \tag{39}$$

For the regularized model, the governing equations (17) lead to the same elastic domain C in the force space (X_d^ϕ, X_d^β) and the normality law for $(\dot{\phi}, \dot{\beta})$.

For example, the model (11) leads to

$$\begin{cases} f = (\|X_d^\phi\|^2 + \frac{1}{\ell^2}\|X_d^\beta\|^2)^{1/2} - k(\phi) \\ \dot{\phi} = \lambda\frac{\partial f}{\partial X_d^\phi}, \qquad \dot{\beta} = \lambda\frac{\partial f}{\partial X_d^\beta}, \quad f \leqslant 0,\ \lambda \geqslant 0,\ f\lambda = 0 \\ X_d^\phi = -X_e^\phi + \nabla\cdot Y_e^\phi + r\Delta\phi - r\nabla\cdot\beta, \qquad X_d^\beta = -r(\beta - \nabla\gamma) \end{cases}$$

Thus β approaches $\nabla\gamma$ and X_d^β plays the role of Y_d^γ when the coefficient of rigidity r is high enough.

The gradient model of isotropic-kinematic hardening of energy and criterion of plasticity

$$\begin{cases} w = w_e(\epsilon - \epsilon^p) + w_c(\epsilon^p) + w_i(\gamma) + 1/2g\,\|\nabla\gamma\|^2 \\ f(X_d^p, X_d^\gamma) = \|X_d^p\| + X_d^\gamma - k \leqslant 0, \qquad \varphi(Y_d^\gamma) = \|Y_d^\gamma\| - \kappa \leqslant 0 \end{cases}$$

can be regularized as a model of energy and criterion

$$\begin{cases} w = w_e(\epsilon - \epsilon^p) + w_c(\epsilon^p) + w_i(\gamma) + 1/2g\|\nabla\gamma\|^2 + \frac{1}{2}r\|\beta - \nabla\gamma\|^2 \\ f(X_d^p, X_d^\gamma) = \|X_d^p\| + X_d^\gamma - k \leqslant 0, \qquad \varphi(X_d^\beta) = \|X_d^\beta\| - \kappa \leqslant 0 \\ X_d^p = \sigma - \frac{\partial w_c}{\partial\epsilon^p}, \qquad X_d^\gamma = -w_i' + (g + r)\Delta\gamma - r\nabla\cdot\beta, \qquad X_d^\beta = -r(\beta - \nabla\gamma) \end{cases}$$

4. Conclusion

In this Note, the constitutive equations of standard gradient models are given from the expressions of the energy and the dissipation potentials. Attention is focused on the derivation of the governing equations of the global response, on the determination of the rates and associated variational principles for time-independent processes such as incremental plasticity and brittle damage. In particular, the theory of Gradient Plasticity is presented in a consistent description with some new arguments.

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