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Wave equation with $p(x,t)$-Laplacian and damping term: Blow-up of solutions

Equation des ondes avec $p(x,t)$-Laplacian et un tèrme dissipatif : Blow-up des solutions

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ABSTRACT

We study the Dirichlet problem for equation

$$u_{tt} = \operatorname{div}\big(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u\big) + \alpha \triangle u_t + b(x,t)|u|^{\sigma(x,t)-2}u$$

in which α is a nonnegative constant, the coefficients $a(x,t)$, $b(x,t)$ and the exponents of nonlinearity $p(x,t)$, $\sigma(x,t)$ are given functions. Under suitable conditions on the data, we study the finite time blow-up of the solutions.

RÉSUMÉ

On considère le problème de Dirichlet pour l'équation

$$u_{tt} = \operatorname{div}\big(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u\big) + \alpha \triangle u_t + b(x,t)|u|^{\sigma(x,t)-2}u$$

où $\alpha \geqslant 0$ est une constante, $a(x,t)$, $b(x,t)$ sont des coefficients variables et $p(x,t), \sigma(x,t)$ sont des exposants nonlinéaires. Sous conditions appropriées on étudie le temps fini de *blow-up* des solutions.

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1. Introduction

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with Lipschitz-continuous boundary Γ, $Q_T = \Omega \times (0,T]$, $\Gamma_T = \partial\Omega \times (0,T)$. We consider the following initial and boundary value problem:

$$u_{tt} = \operatorname{div}\big(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u + \alpha\nabla u_t\big) + b(x,t)|u|^{\sigma(x,t)-2}u \quad \text{in } Q_T \tag{1}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega;\ u|_{\Gamma_T} = 0 \tag{2}$$

Here $\alpha \geqslant 0$ is a constant. The coefficients $a(x,t)$, $b(x,t)$ and the exponents $p(x,t)$, $\sigma(x,t)$ are given measurable functions of their arguments. Equations with variable exponents of nonlinearities are usually referred to as equations with nonstandard growth conditions.

We discuss the blow-up of solutions to problem (1)–(2) paying special attention to the specific properties caused by the variable nonlinearity of the equation.

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A study of the localization (vanishing) properties and the blow-up of solutions of elliptic and parabolic equations of the type (1) can be found in [1–4].

Equations with nonstandard growth conditions occur in the mathematical modeling of various physical phenomena such as flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, processes of filtration through a porous media and the image processing – see [1,5–8] and further references therein.

For hyperbolic equations of the form (1) with constant coefficients a, b and exponents p, σ, local and global in time existence and the blow-up behavior have been studied by many authors – [9–18]. It is to be noted here that in all papers (dealing with the case $p \neq 2$) the damping term $\alpha \triangle u_t$ plays the key role in the proof of global existence, even in the case of constant exponents p and σ. On the other hand, the presence of the damping term makes the study of the blow-up behavior more complicated.

The situation is different for hyperbolic equations with nonstandard growth conditions. The proof of existence of energy solutions to problem (1)–(2) can be found in [19,20]. A special case $a(x,t) = 1$, $b(x,t) = 0$ and $p(x,t) = p(x)$ was studied in [21]. In the case $a(x,t) = 1$, $p(x,t) = 2$, $b(x,t)|u|^{\sigma(x,t)-2}u = b(x)u^{p(x)}$, the questions of existence and blow-up of non-negative solutions of problem (1)–(2) were discussed in [22]. In this Note we study the blow-up of energy weak solutions of problem (1)–(2) ($\alpha \geqslant 0$) with nonpositive initial energy. The study is confined to the class of energy weak solutions because for the solutions of this class the energy function is well-defined (see [19–21]). The analysis relies on the methods developed in [2,3,5].

2. Energy estimates

Let us assume that

$$1 < p_- \leqslant p(x,t) \leqslant p_+ < \infty, \qquad 1 < \sigma_- \leqslant \sigma(x,t) \leqslant \sigma_+ < \infty \tag{3}$$

$$0 < a_- \leqslant a(x,t) \leqslant a_+ < \infty, \qquad |a_t| \leqslant C_a \tag{4}$$

$$0 < b_- \leqslant b(x,t) = b(x,t) < b_+ \leqslant \infty, \qquad |b_t| \leqslant C_b \tag{5}$$

$$p_t \leqslant 0, \quad 0 \leqslant \sigma_t, \qquad |p_t| \leqslant C_p, \qquad |\sigma_t| \leqslant C_\sigma \tag{6}$$

We introduce the energy function

$$E(t) = \int_\Omega \left[\frac{|u_t|^2}{2} + a(\cdot,t) \frac{|\nabla u|^{p(\cdot,t)}}{p(\cdot,t)} - b(\cdot,t) \frac{|u|^{\sigma(\cdot,t)}}{\sigma(\cdot,t)} \right] \mathrm{d}x \tag{7}$$

Using (1), we derive the energy relation

$$E'(t) + \alpha \int_\Omega \left| \nabla u_t(\cdot,t) \right|^2 \mathrm{d}x = \Lambda_1 + \Lambda_2 \tag{8}$$

where

$$\Lambda_1 = \int_\Omega \left[a_t \frac{|\nabla u|^p}{p} + \frac{a|\nabla u|^p}{p^2} \left(1 - p \ln |\nabla u|\right) |p_t| \right] \mathrm{d}x \tag{9}$$

$$\Lambda_2 = - \int_\Omega \left[\frac{b_t |u|^\sigma}{\sigma} + \frac{b|u|^\sigma}{\sigma^2} \left(1 - \sigma \ln |u|\right) \sigma_t \right] \mathrm{d}x \tag{10}$$

Lemma 2.1. *Let* (3)–(5) *be fulfilled and, in addition,*

$$a_t \leqslant 0, \quad 0 \leqslant b_t, \qquad p_t = \sigma_t = 0 \tag{11}$$

Then

$$E(t) + \alpha \int_0^t \int_\Omega |\nabla u_t|^2 \, \mathrm{d}x \, \mathrm{d}s \leqslant E(0), \quad \forall t \geqslant 0 \tag{12}$$

The inequality (12) *passes to the equality if* $a_t = b_t = 0$.

Proof. If suffices to apply formulas (8)–(10). □

Lemma 2.2. *Let conditions* (3)–(6) *be fulfilled. Then*

$$E(t)+\int_0^t\int_\Omega \alpha|\nabla u_t|^2\,dx\,ds \leqslant E(0)+Ct \tag{13}$$

with the constant $C = e\,(a_+C_p + b_+C_\sigma)|\Omega|$.

Proof. We evaluate Λ_1, Λ_2 in the following way:

$$\Lambda_1 = \int_\Omega \left[a_t\frac{|\nabla u|^p}{p} + \frac{a|\nabla u|^p}{p^2}\big(1-p\ln|\nabla u|\big)|p_t|\right]dx$$

$$\leqslant \int_{\Omega\cap(p\ln|\nabla u|\leqslant 1)} \frac{a|\nabla u|^p}{p^2}\big(1-p\ln|\nabla u|\big)|p_t|\,dx \leqslant ea_+C_p|\Omega| = C_1 \tag{14}$$

$$\Lambda_2 = -\int_\Omega \left[\frac{b_t|u|^\sigma}{\sigma} + \frac{b|u|^\sigma}{\sigma^2}\big(1-\sigma\ln|u|\big)\sigma_t\right]dx$$

$$\leqslant \int_{\Omega\cap(\sigma\ln|u|\leqslant 1)} \frac{b|u|^\sigma}{\sigma^2}\big(1-\sigma\ln|u|\big)\sigma_t\,dx \leqslant eb_+C_\sigma|\Omega| = C_2 \tag{15}$$

Integrating the energy relation (8) with respect to t, we obtain that

$$E(t)+\int_0^t\int_\Omega \alpha|\nabla u_t|^2\,dx\,ds \leqslant E(0)+tC, \quad C = C_1 + C_2 \qquad \square \tag{16}$$

3. Blow-up

Assume that

$$E(0)\leqslant 0, \quad 0<(u_0,u_1)_\Omega, \quad \exists\lambda>2:\ 2\leqslant p_-\leqslant p_+<\lambda<\sigma_- \tag{17}$$

Theorem 3.1. *Let u be an energy weak solution of problem* (1)–(2)*. Let conditions of Lemma* 2.1 *be fulfilled and let* (17) *hold. Then there exists a finite time $t_{\max}<\infty$ such that*

$$\Phi(t) = \|u(t)\|^2_{2,\Omega} + \alpha\int_0^t\int_\Omega |\nabla u|^2\,dx\,ds \to \infty \quad \text{if } t\to t_{\max} \tag{18}$$

Proof. It's easy to check that

$$\Phi' = 2(u,u_t)_\Omega + \alpha\int_\Omega |\nabla u|^2\,dx, \qquad \Phi'' = 2\|u_t\|^2_{2,\Omega} + 2\int_\Omega \big(-a|\nabla u|^p + b|u|^\sigma\big)\,dx$$

Using the inequality (12), we calculate

$$\Phi'' \geqslant (2+\lambda)\|u_t\|^2 + 2\int_\Omega \big((\lambda/p-1)a|\nabla u|^p + b(1-\lambda/\sigma)|u|^\sigma\big)\,dx + 2\lambda\alpha\int_0^t\int_\Omega |\nabla u_t|^2\,dx\,ds > 0 \tag{19}$$

It follows that

$$\Phi'(t)>0, \quad \text{if } \Phi'(0) = 2(u_0,u_1)_\Omega + \alpha\int_\Omega |\nabla u_0|^2\,dx > 0$$

Using the last estimates and the properties the Orlicz–Sobolev spaces (see, for example, [3]), we derive the following inequality: for every fixed t

$$0 \leqslant \Phi' = 2(u\, u_t)_\Omega + \alpha \int_\Omega |\nabla u|^2 \leqslant 2\|u(t)\| \|u_t\| + \alpha \int_\Omega |\nabla u|^2 \leqslant C\big(\Phi''\big)^{\frac{1}{\mu}}$$

where

$$1/\mu = \max(1/\sigma_- + 1/2,\ 2/p_-) < 1 \quad \text{if } \sigma_- > 2 \text{ and } \ p_- > 2$$

This ordinary differential inequality leads us to the estimate

$$\Phi'(t) \geqslant \Phi'(0)\big(1 - t(\mu-1)/C\big(\Phi'(0)\big)^{\mu-1}\big)^{-\frac{1}{\mu-1}} \to \infty$$

as

$$t \to t_{\max} = C/(\mu-1)\big(\Phi'(0)\big)^{-\mu+1} < \infty \tag{20}$$

which completes the proof of the theorem. □

Remark 1. It is noteworthy that the constants μ and C (and, respectively, $t_{\max}$) in (20) depend only on $|\Omega|, n, a_\pm, b_\pm, p_\pm, \sigma_\pm$.

Let us assume now that the exponents p, σ are weakly dependent on t, that is, the constants C_p, C_σ are small. The proof the blow-up is the same as in the previous theorem, provided that

$$E(t) + \alpha \int_0^t \int_\Omega |\nabla u_t|^2 \,dx\,ds \leqslant 0, \quad 0 \leqslant t \leqslant t_{\max} \tag{21}$$

According to Lemma 2.2 (see inequality (13))

$$E(t) + \alpha \int_0^t \int_\Omega |\nabla u_t|^2 \,dx\,ds \leqslant E(0) + t_{\max} e(a_+ C_p + b_+ C_\sigma)|\Omega| \tag{22}$$

Assuming that

$$\delta = \max(C_p, C_\sigma) \leqslant \big|E(0)\big|\big(t_{\max} e(a_+ + b_+)|\Omega|\big)^{-1}, \quad E(0) < 0 \tag{23}$$

we arrive at (21). This leads to

Theorem 3.2. *Let u be an energy weak solution of problem* (1)–(2)*. Let the conditions of Lemma* 2.2 *and inequality* (23) *with $t_{\max}$ defined in Theorem* 3.1 *be fulfilled. If*

$$2 \leqslant p_- \leqslant p_+ < \lambda < \sigma_-, \qquad (u_0, u_1)_\Omega > 0, \qquad E(0) < 0$$

then the solution u blows-up at a finite moment $t_{\max}$ (*in the sense that $\Phi(t)$ becomes unbounded as $t \to t_{\max}$*)*.*

Now we consider Eq. (1) with $\alpha = 0$, assuming that problem (1)–(2) has at least one local energy solution. Here we follow the paper [10] where was proved the blow-up for the abstract Cauchy in a Banach space which included, as an example, equation of the type (1) with the $a = b = 1$, $p = const$, $\sigma = const$.

Let us assume that

$$0 < (u_0, u_1)_\Omega, \qquad E(0) \leqslant 0,\ \exists \lambda > 2, \qquad p_+ \leqslant \lambda \leqslant \sigma_- \tag{24}$$

Following the arguments of paper [10], we prove

Theorem 3.3. *Let u be an energy weak solution to problem* (1)–(2) *with $\alpha = 0$. Let the conditions of Lemma* 2.1 *and condition* (24) *be fulfilled. Then u blows-up* (*in the sense that $\|u(t)\|^2_{2,\Omega}$ becomes unbounded*) *on the finite interval* $(0, t_{\max})$ *with $t_{\max} = 2\|u_0\|^2_{2,\Omega}/(\lambda-2)(u_0,u_1)_\Omega$.*

Theorem 3.4. *Let u be an energy weak solution of problem* (1)–(2) *with $\alpha = 0$. Let the conditions of Lemma* 2.2*, conditions* (24) *and* (23) *be fulfilled* (*with $t_{\max}$ defined in Theorem* 3.3)*. Then the solution u blows-up on the finite interval* $(0, t_{\max})$.

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References

[1] S.N. Antontsev, S.I. Shmarev, Elliptic Equations with Anisotropic Nonlinearity and Nonstandard Growth Conditions, Handbook of Differential Equations, Stationary Partial Differential Equations, vol. 3, Elsevier, 2006, Chapter 1, pp. 1–100.

[2] S.N. Antontsev, S.I. Shmarev, Extinction of solutions of parabolic equations with variable anisotropic nonlinearities, in: Proceedings of the Steklov Institute of Mathematics, vol. 268, Moscow, Russia, 2008, pp. 2289–2301.

[3] S.N. Antontsev, S.I. Shmarev, Blow-up of solutions to parabolic equations with non-standard growth conditions, J. Comput. Appl. Math. 234 (2010) 2633–2645.

[4] S.N. Antontsev, S.I. Shmarev, Vanishing solutions of anisotropic parabolic equations with variable nonlinearity, J. Math. Anal. Appl. 361 (2010) 371–391.

[5] S.N. Antontsev, J.I. Díaz, S. Shmarev, Energy Methods for Free Boundary Problems: Applications to Non-Linear PDEs and Fluid Mechanics, Progress in Nonlinear Differential Equations and Their Applications, vol. 48, Bikhäuser, Boston, 2002.

[6] S.N. Antontsev, J.F. Rodrigues, On stationary thermo-rheological viscous flows, Ann. Univ. Ferrara, Sez. VII Sci. Mat. 52 (2006) 19–36.

[7] K. Rajagopal, M. Růžička, Mathematical modeling of electro-rheological fluids, Cont. Mech. Therm. 13 (2001) 59–78.

[8] M. Růžička, Electrorheological Fluids: Modeling and Mathematical Theory, Lecture Notes in Mathematics, vol. 1748, Springer, Berlin, 2000.

[9] A. Benaissa, S. Mokeddem, Decay estimates for the wave equation of p-Laplacian type with dissipation of m-Laplacian type, Math. Methods Appl. Sci. 30 (2007) 237–247.

[10] V.A. Galaktionov, S.I. Pohozaev, Blow-up and critical exponents for nonlinear hyperbolic equations, Nonlinear Anal. 53 (2003) 453–466.

[11] H. Gao, T.F. Ma, Global solutions for a nonlinear wave equation with p-Laplacian operator, EJQTDE (1999) 1–13.

[12] T. Kato, Blow up of solutions of some nonlinear hyperbolic equations, Manuscripta Math. 28 (1980) 235–268.

[13] S.A. Messaoudi, B. Said Houari, Global non-existence of solutions of a class of wave equations with non-linear damping and source terms, Math. Methods Appl. Sci. 27 (2004) 1687–1696.

[14] È. Mitidieri, S.I. Pokhozhaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities, Tr. Mat. Inst. Steklova 234 (2001) 1–384.

[15] J. Serrin, G. Todorova, E. Vitillaro, Existence for a nonlinear wave equation with damping and source terms, Differential Integral Equations 16 (2003) 13–50.

[16] Z. Yang, Existence and asymptotic behavior of solutions for a class of quasi-linear evolution equations with non-linear damping and source terms, Math. Methods Appl. Sci. 25 (2002) 795–814.

[17] Z. Yang, G. Chen, Global existence of solutions for quasi-linear wave equations with viscous damping, J. Math. Anal. Appl. 285 (2003) 604–618.

[18] Y. Zhijian, Existence and asymptotic behavior of solutions for a class of quasi-linear evolution equations with non-linear damping and source terms, Math. Methods Appl. Sci. 25 (2002) 795–814.

[19] S. Antontsev, Wave equation with $p(x,t)$-Laplacian and damping term: Local and global existence, RACSAM, Rev. R. Acad. Cien. Serie A. Mat., pp. 1–15, submitted for publication.

[20] S. Antontsev, Wave equation with $p(x,t)$-Laplacian and damping term: Existence and blow-up, in: Abstracts of the International Congress "Nonlinear Models in Partial Differential Equations", Toledo, Spain, June 14–17, 2011.

[21] J. Haehnle, A. Prohl, Approximation of nonlinear wave equations with nonstandard anisotropic growth conditions, Math. Comp. 79 (2010) 189–208.

[22] J.P. Pinasco, Blow-up for parabolic and hyperbolic problems with variable exponents, Nonlinear Anal. 71 (2009) 1094–1099.