C. R. Mecanique 340 (2012) 181–192

Contents lists available at SciVerse ScienceDirect

Comptes Rendus Mecanique

www.sciencedirect.com

Recent Advances in Micromechanics of Materials

The construction of effective relations for waves in a composite

John R. Willis

Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, Cambridge CB3 0WA, UK

ARTICLE INFO

Article history:
Available online 28 March 2012

Keywords:
Effective relations
Homogenization
Elastic waves
Metamaterials
Comparison medium

ABSTRACT

Implications of a recent general formulation for the effective dynamic response of a composite, in which "effective displacement" is defined as a weighted average, which could, for instance, be an average over the matrix material, are developed. A general formula is already known [J.R. Willis, Effective constitutive relations for waves in composites and metamaterials, Proc. R. Soc. A 467 (2011) 1865–1879], but it is expressed in terms of the Green's function of the actual composite. A corresponding formula, expressed relative to a comparison medium, is developed here. The property of self-adjointness of the problem for the actual medium is transmitted to the corresponding problem for the "effective medium". This permits, in the case of self-adjointness, variational characterizations of the effective response, both directly and in a formulation of "Hashin–Shtrikman" type relative to a comparison medium. The exposition is for waves in a viscoelastic composite but it applies equally to other physical examples, including electromagnetic waves.

1. Introduction

The current interest in metamaterials has carried with it a concern for the accurate representation of waves propagating through them. A metamaterial is no more than a composite but its distinguishing feature is that the waves that it can support can be represented by effective properties which, while not in conflict with the basic principles of physics, are nevertheless not observed in materials that occur naturally. Such properties include negative effective mass density and negative elastic constants in the context of elasticity, or simultaneously-negative dielectric constant and magnetic permeability in the case of electromagnetics. Properties such as these depend on the frequency ω of the excitation and they are in general complex, so that energy is dissipated, the purely real limit being achievable only mathematically, in the idealized case that the constituent materials display no dissipation at all. Apparently anomalous properties such as negative (real part of) mass density are associated with resonance on the scale of the microstructure and are realized at frequencies in some band which contains a resonant frequency. Liu et al. [1] made an experimental study of an acoustic medium containing resonators. They interpreted the observed spectral band gap at frequencies close to resonance as implying negative effective elastic constant but later [2] developed a model that implied that the effective density was negative. Zhikov [3] considered homogenization of the Helmholtz equation for a composite with high contrast and found negative effective moduli at some frequencies. In electromagnetics, an early discussion on the construction of lenses made from what would now be termed metamaterials appears in Chapter 19 of [4], and this includes an outline of how an array of resonators similar to present split-ring resonators could induce artificial magnetism. However, the current explosion of interest in the design of such resonating microstructures was largely stimulated by the more recent insights of J.B. Pendry [5,6].

The purpose of the present work is to develop, from rigorous definitions and a formal solution given by Willis [7], procedures for the calculation (either exactly or approximately) of the effective dynamic properties of a composite. The procedures in question are first, a formulation relative to a comparison medium, then, a variational characterization in the

E-mail address: J.R.Willis@damtp.cam.ac.uk.

1631-0721/$ – see front matter
doi:10.1016/j.crme.2012.02.001

case that the original problem is self-adjoint, and finally a variational characterization which employs a comparison medium. All are familiar in the context of homogenization in elasto- or electro-statics. However, for dynamics, not just restricted to long waves, even the form of the effective response (see Eq. (8) below) was not widely recognized until fairly recently. It was implicit in work of Willis [8,9] and was made explicit by Willis [10]. Subsequently, Milton and Willis [11] and Willis [12,7] developed a wider definition of effective properties, which allows for a weighted average of the displacement. The present work is based on this definition.

The sections that follow give the basic governing equations and the boundary-value problem associated with them, and review the definition of effective response employed in the work of Willis [7,12]. Then in Section 4, the equations and their solution are developed relative to a comparison medium. Section 5 gives the basic variational formulation in the case that the original problem is self-adjoint, and Section 6 gives the corresponding formulation relative to a comparison medium. Some concluding discussion is made in Section 7. The reasoning and the results are expressed employing a concise notation in which the problem closely resembles the corresponding problem for elastostatics. This notation also permits the immediate translation of virtually every line of reasoning to the case of electromagnetic waves, the precise correspondence being recorded in Appendix B.

2. Basic equations

The explicit context of this article is the study of waves in a composite medium whose constituents are viscoelastic. The medium occupies a domain Ω in d-dimensional space, with boundary $\partial\Omega$. It is assumed to have undergone some "transformation strain" $\eta_{ij}(x,t)$, perhaps arising from plastic deformation, thermal strain or phase transformation. The stress $\sigma_{ij}(x,t)$ in the medium is then related to the total strain $e_{ij}(x,t)$ by the constitutive relation

$$\sigma_{ij}(x,t) = \left\{C_{ijkl} * [e_{kl} - \eta_{kl}]\right\}(x,t) \equiv \int_{-\infty}^{t} C_{ijkl}\left(x, t-t'\right)\left[e_{kl}\left(x,t'\right) - \eta_{kl}\left(x,t'\right)\right] \mathrm{d}t' \tag{1}$$

The field $\eta_{ij}(x,t)$ will be regarded as prescribed. The major reason for its introduction is mathematical and will be explained later. From the physical viewpoint, however, in concentrating on the viscoelastic aspect of the problem, the present work studies only a part of a completely specified problem which should contain in addition some evolution equation for the transformation strain. The medium has mass density $\rho(x)$ and so momentum density p_i is related to velocity v_i by

$$p_i(x,t) = \rho(x) v_i(x,t) \tag{2}$$

The displacement $u_i(x,t)$ in the medium is assumed to be driven by a body-force $f_i(x,t)$ as well as by $\eta_{ij}(x,t)$ and it will be assumed that $u_i(x,t) \equiv 0$ for all $t < 0$. It will be convenient to consider the problem in the Laplace transform domain so that the transform variable s replaces the time t. The constitutive relation (1) then takes the "elastic" form

$$\sigma(x) = C(x)\left[e(x) - \eta(x)\right] \tag{3}$$

having suppressed both the suffixes and the dependence on the transform variable s. In the special case of elastic response, $C(x,t)$ has the time-dependence $\delta(t)$, and in the transform domain $C(x)$ is independent of s.[1] The momentum–velocity relation (2) similarly becomes

$$p(x) = \rho(x) v(x) \tag{4}$$

The equation of motion in the transform domain is

$$\operatorname{div}\sigma + f = sp \tag{5}$$

It is necessary also to specify boundary conditions. It will be assumed that traction $n \cdot \sigma = T$ is prescribed over a part S_T of the boundary, while displacement is prescribed over the complementary part S_u.

3. Effective response

The effective response is intended to represent some form of "averaged" response, relating "effective" stress and momentum to effective strain and velocity. Precise definitions of course are necessary. The present work is based on those advanced by Willis [7], which are now outlined.

The domain Ω is fixed but the medium within it is considered to be random. Thus, what is really represented is a family of media, any member of which is parametrized by a variable α belonging to a sample space $\mathcal{A}$, over which a probability

[1] The tensor $C(x)$ will henceforth be referred to as the elastic constant tensor even though, in the case of viscoelastic response, it depends on s as well as x.

measure p is defined. Thus, the probability that $\alpha \in \mathcal{A}_1 \subset \mathcal{A}$ is $\mathrm{p}(\mathcal{A}_1)$. The elastic constant tensor $C(x)$ then depends also on α and its mean (or expectation) value is

$$\langle C\rangle(x) = \int_{\mathcal{A}} C(x,\alpha)\mathrm{p}(\mathrm{d}\alpha) \tag{6}$$

Mean values of other variables are defined similarly. Before proceeding further, it merits emphasis that a periodic medium with period cell Q fits into this scheme, if one specified "corner" of one specified period cell has position vector y. The coordinate y serves as the sample parameter α and it is natural to assume that y is uniformly distributed over Q.

Now consider the boundary-value problem introduced above. Its solution depends on the realization α. Ensemble averaging the equation of motion (5) gives

$$\operatorname{div}\langle\sigma\rangle + \langle f\rangle = s\langle p\rangle \tag{7}$$

and ensemble averaging the traction boundary condition gives $n\cdot\langle\sigma\rangle = \langle T\rangle$. It is considered appropriate to define the effective stress to be $\langle\sigma\rangle$ and the effective momentum density to be $\langle p\rangle$, so that the effective variables satisfy exactly the equation of motion. It is necessary also to define effective strain and effective velocity. The most obvious way to do this is to define the effective displacement to be $\langle u\rangle$ so that effective strain is $\langle e\rangle$ and effective velocity is $\langle v\rangle$. This description is of very long standing, dating, in the case of elastostatics, back at least to the 1960's (e.g. [13,14]). In the case of dynamics, it was implicit in work of Willis [15,16,8,9] that this description lead to effective constitutive relations of the form[2]

$$\begin{pmatrix}\langle\sigma\rangle\\ \langle p\rangle\end{pmatrix} = \begin{pmatrix} C^{\mathrm{eff}} & S^{\mathrm{eff}}\\ S^{\mathrm{eff}\dagger} & \rho^{\mathrm{eff}}\end{pmatrix}\begin{pmatrix}\langle e\rangle\\ \langle v\rangle\end{pmatrix} \tag{8}$$

The terms C^{eff}, etc. are non-local operators so that, for instance,

$$\bigl\{C^{\mathrm{eff}}\langle e\rangle\bigr\}(x) = \int_{\Omega} C^{\mathrm{eff}}\bigl(x,x'\bigr)\langle e\rangle\bigl(x'\bigr)\,\mathrm{d}x' \tag{9}$$

Equations of this form were developed explicitly by Willis [10].

It is possible, however, to define effective strain and velocity more generally [11,12,7]. Just to motivate this, consider a porous medium: the displacement is at least non-unique in the pores, and it is impossible to apply body force to the pores. It seems appropriate, therefore, to allow the possibility to introduce a weighted mean displacement $\langle wu\rangle$ and to define effective strain and effective velocity to be the associated strain and velocity. The weight $w(x)$ is a random field with mean value $\langle w\rangle(x) = 1$. It could be uniform over the matrix material (not occupied by pores or, more generally, inclusions) but nothing in the mathematics requires this. It is appropriate, simultaneously, to take the body-force $f(x)$ and the prescribed surface traction $T(x)$ to have the forms $w(x)f_1(x)$ and $w(x)T_1(x)$ respectively, where $f_1(x)$ and $T_1(x)$ are sure. Following Willis [7], the displacement prescribed over S_u and the transformation strain $\eta(x)$ will be taken as sure.

Before giving more formulae, it is appropriate to introduce concise notation. Let E be the operator that takes displacement to strain, so that

$$E : u \to Eu = \frac{1}{2}\bigl[\nabla u + (\nabla u)^T\bigr] \tag{10}$$

In the transform domain, velocity is represented as $v = su$. Now define

$$\mathbf{s} = \begin{pmatrix}\sigma\\ p\end{pmatrix},\quad \mathbf{e} = \begin{pmatrix}e\\ v\end{pmatrix},\quad \mathbf{m} = \begin{pmatrix}\eta\\ 0\end{pmatrix},\quad \mathcal{E} = \begin{pmatrix}E\\ s\end{pmatrix},\quad \mathcal{D}^T = (\operatorname{div}\quad -s)\quad\text{and}\quad \mathcal{L} = \begin{pmatrix}C & 0\\ 0 & \rho I\end{pmatrix} \tag{11}$$

where I is the $d\times d$ identity. The constitutive relations (3) and (2) become

$$\mathbf{s} = \mathcal{L}(\mathbf{e} - \mathbf{m}) \tag{12}$$

where

$$\mathbf{e} = \mathcal{E}u \tag{13}$$

and the equation of motion (5) becomes

$$\mathcal{D}^T\mathbf{s} + w\langle f\rangle = 0 \tag{14}$$

(employing the form $f = wf_1$ for the body force so that $f_1 = \langle f\rangle$).

[2] Here, the transformation strain $\eta = 0$.

Willis [7] developed, for this problem, the effective relation

$$\langle \mathbf{s}\rangle = \mathcal{L}^{\text{eff}}\big[\mathcal{E}\langle wu\rangle - \mathbf{m}\big] \tag{15}$$

where

$$\mathcal{L}^{\text{eff}} = \langle \mathcal{L}\rangle - \big\langle \mathcal{L}\mathcal{E}\big(\mathcal{E}G^{\dagger}\big)^{\dagger}\mathcal{L}\big\rangle + \big\langle \mathcal{L}(\mathcal{E}G)w\big\rangle\langle wGw\rangle^{-1}\big\langle w\big(\mathcal{E}G^{\dagger}\big)^{\dagger}\mathcal{L}\big\rangle \tag{16}$$

The elastic constant tensor was taken to have the symmetries $C_{ijkl} = C_{jikl} = C_{ijlk}$ but was *not* assumed to have the symmetry $C^T = C$ (i.e. $C_{ijkl} = C_{lkji}$) that delivers self-adjointness and the associated self-adjointness of the Green's function, $G^{\dagger}(x,x') \equiv G^T(x',x) = G(x,x')$. However, when the medium does possess the self-adjoint symmetry, it follows immediately from (16) that $\mathcal{L}^{\text{eff}}$ is self-adjoint, i.e. $(\mathcal{L}^{\text{eff}})^{\dagger} = \mathcal{L}^{\text{eff}}$. The relation (15) remains true when $\eta = 0$ (so $\mathbf{m} = 0$). The mathematical consideration that lead Willis [7] to introduce $\eta \neq 0$ is that this renders all components[3] of $\mathcal{E}\langle wu\rangle - \mathbf{m}$ independent so that $\mathcal{L}^{\text{eff}}$ is unique. In the absence of η, even the effective strain $E\langle wu\rangle$ is compatible with $\langle wu\rangle$ and therefore $C'E\langle wu\rangle = 0$ for any C' for which $\partial C'_{ijkl}(x,x')/\partial x'_l = 0$ for $x' \in \Omega$ and $C'_{ijkl}(x,x')n_l(x') = 0$ for $x' \in \partial\Omega$. Thus, $\mathcal{L}^{\text{eff}}$ cannot be unique, even in the case of elastostatics. The possibilities for non-uniqueness of course are wider in the case of dynamics. See Willis [17] for further discussion, illustrated by an example involving one space dimension ($d = 1$).

This article is concerned with two issues following from the formula (16). The first is that (16) contains the Green's function G of the random medium which may be hard to calculate, even approximately. This will be circumvented by developing a formula equivalent to (16) that employs the Green's function G_0 of a fixed comparison medium. The second is that, since it is now known – even at the level of generality of employing a weighted mean displacement – that self-adjointness of the original problem induces self-adjointness of the effective medium, it is possible to develop variational formulations in the self-adjoint case, with prospects for the development of rational approximations and in some cases bounds for effective response.

4. Formulation relative to a comparison medium

Introduce a comparison medium with properties

$$\mathcal{L}_0 = \begin{pmatrix} C_0 & 0 \\ 0 & \rho_0 I \end{pmatrix} \tag{17}$$

and a "polarization" $\mathbf{n}$ such that

$$\mathbf{s} = \mathcal{L}(\mathbf{e} - \mathbf{m}) = \mathcal{L}_0(\mathbf{e} - \mathbf{m}) + \mathbf{n} \tag{18}$$

The equation of motion (14) then implies that

$$\mathcal{D}^T\mathcal{L}_0\mathcal{E}u - \mathcal{D}^T\mathcal{L}_0\mathbf{m} + \mathcal{D}^T\mathbf{n} + w\langle f\rangle = 0 \tag{19}$$

Green's function G_0 for the comparison medium is defined so that its adjoint $G_0^{\dagger}(x,x') = G_0^T(x',x)$ satisfies the equation

$$\mathcal{D}^T\mathcal{L}_0^T\mathcal{E}G_0^{\dagger} + I\delta\big(x - x'\big) = 0 \tag{20}$$

together with the homogeneous boundary conditions

$$n(x)\cdot C_0^T(x)EG_0^{\dagger}\big(x,x'\big) = 0, \quad x \in S_T, \qquad G_0^{\dagger}\big(x,x'\big) = 0, \quad x \in S_u \tag{21}$$

It follows[4] that

$$\begin{aligned} u\big(x'\big) = &\int_{\Omega}\big\{\big[G_0^{\dagger}\big(x,x'\big)\big]^T w(x)\langle f\rangle(x) + \big[\mathcal{E}G_0^{\dagger}\big(x,x'\big)\big]^T\big[\mathcal{L}_0(x)\mathbf{m}(x) - \mathbf{n}(x)\big]\big\}\,\mathrm{d}x \\ &+ \int_{S_T}\big[G_0^{\dagger}\big(x,x'\big)\big]^T w(x)\langle T\rangle(x)\,\mathrm{d}S - \int_{S_u}\big[EG_0^{\dagger}\big(x,x'\big)\big]^T C_0(x)\big[n(x)\otimes u(x)\big]\,\mathrm{d}S \end{aligned} \tag{22}$$

having expressed the traction over S_T as $T = w\langle T\rangle$ (matching the form of the body force). For future use, (22) is written more concisely as

$$u = G_0\big\{w\langle f\rangle, w\langle T\rangle\big\} + \big(\mathcal{E}G_0^{\dagger}\big)^{\dagger}(\mathcal{L}_0\mathbf{m} - \mathbf{n}) + u_0^* \tag{23}$$

[3] There are $d(d+1)/2$ components corresponding to strain and d to velocity.
[4] See, for instance, the Appendix of [7] for a similar derivation relative to the actual medium.

where

$$u_0^*(x) = -\int_{S_u} \{EG_0^\dagger\}^\dagger(x,x')C_0(x')\big[n(x')\otimes u(x')\big]\,\mathrm{d}S(x') \tag{24}$$

Since u is prescribed, sure, on S_u, it follows that it can be replaced in the integral in (24) by $\langle wu\rangle$. Also, the integral can be extended to the whole of $\partial\Omega$, on account of the boundary condition $(21)_1$. An application of the divergence theorem and use of the governing equation (20) for $G_0^\dagger$ then gives

$$u_0^* = \langle wu\rangle - \big(\mathcal{E}G_0^\dagger\big)^\dagger\mathcal{L}_0\mathcal{E}\langle wu\rangle \tag{25}$$

Thus, u can be represented in the form

$$u = \langle wu\rangle + G_0\big\{w\langle f\rangle, w\langle T\rangle\big\} - \big(\mathcal{E}G_0^\dagger\big)^\dagger\big[\mathcal{L}_0\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big) + \mathbf{n}\big] \tag{26}$$

Relation (26) would be equally true if $\langle wu\rangle$ were replaced by any displacement field taking the prescribed values on S_u. Taking the weighted mean of (26) shows that, for consistency, $\langle f\rangle$ and $\langle T\rangle$ must be chosen so that

$$\langle wG_0w\rangle\big\{\langle f\rangle, \langle T\rangle\big\} = \big(\mathcal{E}G_0^\dagger\big)^\dagger\mathcal{L}_0\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big) + \big\langle w\big(\mathcal{E}G_0^\dagger\big)^\dagger\mathbf{n}\big\rangle \tag{27}$$

The fact that $\mathbf{m}$ is taken to be sure has been used at this point. Eq. (27) defines $\langle f\rangle$ and $\langle T\rangle$ if $\langle wu\rangle$, $\mathbf{m}$ and $\mathbf{n}$ are prescribed. Then,

$$\begin{aligned} u = {} & \langle wu\rangle + G_0w\langle wG_0w\rangle^{-1}\big\{\big(\mathcal{E}G_0^\dagger\big)^\dagger\mathcal{L}_0\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big) + \big\langle w\big(\mathcal{E}G_0^\dagger\big)^\dagger\mathbf{n}\big\rangle\big\} \\ & - \big(\mathcal{E}G_0^\dagger\big)^\dagger\big[\mathcal{L}_0\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big) + \mathbf{n}\big] \end{aligned} \tag{28}$$

Now with a view towards the construction of an effective constitutive relation, note that, from (18),

$$\langle\mathbf{s}\rangle = \mathcal{L}_0\big(\langle\mathbf{e}\rangle - \mathbf{m}\big) + \langle\mathbf{n}\rangle \tag{29}$$

Operating on (28) with $\mathcal{E}$ and subtracting $\mathbf{m}$ gives

$$\mathbf{e} - \mathbf{m} = (\mathcal{I} - \Gamma_0\mathcal{L}_0)\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big) - \Gamma_0\mathbf{n} + \mathcal{E}G_0w\langle wG_0w\rangle^{-1}\big\{\big(\mathcal{E}G_0^\dagger\big)^\dagger\mathcal{L}_0\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big) + \big\langle w\big(\mathcal{E}G_0^\dagger\big)^\dagger\mathbf{n}\big\rangle\big\} \tag{30}$$

where

$$\Gamma_0 = \mathcal{E}\big(\mathcal{E}G_0^\dagger\big)^\dagger \tag{31}$$

and $\mathcal{I}$ is the identity operator on the space of "column vectors" of the form of $\mathcal{E}\langle wu\rangle$ and $\mathbf{m}$. The polarization $\mathbf{n}$ has still to be determined. The equation that it must satisfy is obtained by noting that (18) implies that

$$\mathbf{e} - \mathbf{m} = (\mathcal{L} - \mathcal{L}_0)^{-1}\mathbf{n} \tag{32}$$

Thus, from (30) and (32),

$$\begin{aligned} \big[(\mathcal{L} - \mathcal{L}_0)^{-1} + \Gamma_0\big]\mathbf{n} = {} & (\mathcal{I} - \Gamma_0\mathcal{L}_0)\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big) \\ & + \mathcal{E}G_0w\langle wG_0w\rangle^{-1}\big\{\big(\mathcal{E}G_0^\dagger\big)^\dagger\mathcal{L}_0\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big) + \big\langle w\big(\mathcal{E}G_0^\dagger\big)^\dagger\mathbf{n}\big\rangle\big\} \end{aligned} \tag{33}$$

Solving first for $\langle w(\mathcal{E}G_0^\dagger)^\dagger\mathbf{n}\rangle$ gives

$$\begin{aligned} \langle wG_0w\rangle^{-1}\big\langle w\big(\mathcal{E}G_0^\dagger\big)^\dagger\mathbf{n}\big\rangle = {} & \big[\langle wG_0w\rangle - \big\langle w\big(\mathcal{E}G_0^\dagger\big)^\dagger\big[(\mathcal{L} - \mathcal{L}_0)^{-1} + \Gamma_0\big]^{-1}(\mathcal{E}G_0)w\big\rangle\big]^{-1} \\ & \times \big\{\big\langle w\big(\mathcal{E}G_0^\dagger\big)^\dagger\big[(\mathcal{L} - \mathcal{L}_0)^{-1} + \Gamma_0\big]^{-1}(\mathcal{E}G_0)w\big\rangle\langle wG_0w\rangle^{-1}\big(\mathcal{E}G_0^\dagger\big)^\dagger\mathcal{L}_0\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big) \\ & + \big\langle w\big(\mathcal{E}G_0^\dagger\big)^\dagger\big[(\mathcal{L} - \mathcal{L}_0)^{-1} + \Gamma_0\big]^{-1}\big\rangle(\mathcal{I} - \Gamma_0\mathcal{L}_0)\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big)\big\} \end{aligned} \tag{34}$$

Also from (33),

$$\begin{aligned} \langle\mathbf{n}\rangle = {} & \big\langle\big[(\mathcal{L} - \mathcal{L}_0)^{-1} + \Gamma_0\big]^{-1}(\mathcal{E}G_0)w\big\rangle\langle wG_0w\rangle^{-1}\big\{\big(\mathcal{E}G_0^\dagger\big)^\dagger\mathcal{L}_0\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big) + \big\langle w\big(\mathcal{E}G_0^\dagger\big)^\dagger\mathbf{n}\big\rangle\big\} \\ & + \big\langle\big[(\mathcal{L} - \mathcal{L}_0)^{-1} + \Gamma_0\big]^{-1}\big\rangle(\mathcal{I} - \Gamma_0\mathcal{L}_0)\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big) \end{aligned} \tag{35}$$

Finally, substituting these expressions into (29) gives, after simplification,

$$\langle\mathbf{s}\rangle = \mathcal{L}^{\mathrm{eff}}\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big) \tag{36}$$

where

$$\mathcal{L}^{\mathrm{eff}} = \big\langle \big(\mathcal{I} - \mathcal{L}\Gamma_0\big[\mathcal{I} + (\mathcal{L} - \mathcal{L}_0)\Gamma_0\big]^{-1}\big)\mathcal{L}\big\rangle + \big\langle \mathcal{L}\big[\mathcal{I} + \Gamma_0(\mathcal{L} - \mathcal{L}_0)\big]^{-1}(\mathcal{E}G_0)w\big\rangle\big[\langle wG_0w\rangle - \big\langle w\big(\mathcal{E}G_0^{\dagger}\big)^{\dagger}\big[(\mathcal{L} - \mathcal{L}_0)^{-1} + \Gamma_0\big]^{-1}(\mathcal{E}G_0)w\big\rangle\big]^{-1} \times \big\langle w\big(\mathcal{E}G_0^{\dagger}\big)^{\dagger}\big[\mathcal{I} + (\mathcal{L} - \mathcal{L}_0)\Gamma_0\big]^{-1}\mathcal{L}\big\rangle \tag{37}$$

This expression is completely explicit, in the sense that it requires only the calculation of the single Green's function G_0, as opposed to the whole family of Green's functions G for the random medium. In fact, it is demonstrated in Appendix A that

$$(\mathcal{E}G) = \big[\mathcal{I} + \Gamma_0(\mathcal{L} - \mathcal{L}_0)\big]^{-1}(\mathcal{E}G_0) \tag{38}$$

$$\big(\mathcal{E}G^{\dagger}\big)^{\dagger} = \big(\mathcal{E}G_0^{\dagger}\big)^{\dagger}\big[\mathcal{I} + (\mathcal{L} - \mathcal{L}_0)\Gamma_0\big]^{-1} \tag{39}$$

$$\Gamma = \Gamma_0\big[\mathcal{I} + (\mathcal{L} - \mathcal{L}_0)\Gamma_0\big]^{-1} \tag{40}$$

and

$$G = G_0 - \big(\mathcal{E}G_0^{\dagger}\big)^{\dagger}\big[\mathcal{I} + (\mathcal{L} - \mathcal{L}_0)\Gamma_0\big]^{-1}(\mathcal{E}G_0) \tag{41}$$

Thus, the new formula (37) is precisely equivalent to (16).

5. Variational principles

Attention henceforth will be restricted to the self-adjoint case, in which the elastic constants have the symmetry $C^T = C$. The original problem then has a variational formulation (in fact, more than one) and formula (16) shows that $\mathcal{L}^{\mathrm{eff}}$ is self-adjoint so that the corresponding problem for the effective medium also has a variational formulation. The main purpose of this section is to derive the latter from the former, exposing in the process the physical interpretation of the latter. It will be convenient, at least for the sake of explicit discussion, to regard the transform variable s as real, since then the basic variational principle that will be invoked is of "minimum energy" type. In the general case of complex s, the variational principles retain the same form but are only stationary principles.

The basic principle for the original random medium is

$$\inf_u \left\langle \int_{\Omega} \left\{ \frac{1}{2}(\mathbf{e} - \mathbf{m})\mathcal{L}(\mathbf{e} - \mathbf{m}) - w\langle f\rangle u \right\} \mathrm{d}x - \int_{S_T} w\langle T\rangle u \,\mathrm{d}S \right\rangle \tag{42}$$

the infimum being taken over displacement fields $u(x,\alpha)$ that take the given (sure) values on S_u. The principle is entirely conventional, except for the inclusion of the ensemble average (i.e. the additional integration $\int_{\mathcal{A}} \dots \mathrm{p}(\mathrm{d}\alpha)$) which, as observed by Willis [9], ensures that the principle generates the solution in every realization. The minimizing field $u(x,\alpha)$ satisfies the stationarity condition

$$\left\langle \int_{\Omega} \big\{ \delta\mathbf{e}\mathcal{L}(\mathbf{e} - \mathbf{m}) - \delta u\, w\langle f\rangle \big\} \mathrm{d}x - \int_{S_T} \delta u\, w\langle T\rangle \,\mathrm{d}S \right\rangle = 0 \tag{43}$$

for all $\delta u(x,\alpha) = 0$ on S_u. The condition (43) implies the equation of motion (14) as well as the boundary condition that $n \cdot \sigma = w\langle T\rangle$ on S_T.

Now consider evaluating the infimum *sequentially*, by first evaluating the infimum conditional upon the field $\langle wu\rangle$ being given, and then minimizing the resulting expression (which depends only on $\langle wu\rangle$) with respect to $\langle wu\rangle$. The conditional minimizer $u(x,\alpha)$ satisfies (43), for all $\delta u(x,\alpha)$ that satisfy the constraint

$$\langle w\delta u\rangle \equiv \int_{\mathcal{A}} w(x,\alpha)\delta u(x,\alpha)\mathrm{p}(\mathrm{d}\alpha) = 0 \tag{44}$$

as well as $\delta u(x,\alpha) = 0$ on S_u. It follows that there must exist "Lagrange multipliers" f_1 over Ω and T_1 over S_T, both sure, such that

$$\mathcal{D}^T\mathbf{s} + wf_1 = 0, \quad x \in \Omega \tag{45}$$

and

$$n \cdot \sigma = wT_1, \quad x \in S_T \tag{46}$$

with $\mathbf{s}$ obtained from the constitutive relation (12).

Eqs. (45) and (46) imply that

$$\mathcal{D}^T\mathbf{s} = w\mathcal{D}^T\langle\mathbf{s}\rangle, \quad x \in \Omega \tag{47}$$

and

$$n\cdot\sigma = w\big(n\cdot\langle\sigma\rangle\big), \quad x \in S_T \tag{48}$$

It merits emphasis that these relations apply to $\mathbf{s}$ evaluated at the conditional minimizer $u(x,\alpha)$, for any α. They allow the simplification of the conditionally-minimized variational functional, $\mathcal{F}(\langle wu\rangle)$ say, which can be written in the form

$$\begin{aligned}\mathcal{F}\big(\langle wu\rangle\big) &= \left\langle \int_\Omega \left\{\frac{1}{2}\mathbf{s}(\mathbf{e}-\mathbf{m}) - w\langle f\rangle u\right\}\mathrm{d}x - \int_{S_T} w\langle T\rangle u\,\mathrm{d}S\right\rangle \\ &\equiv \left\langle \int_\Omega \left\{-\frac{1}{2}\big(\mathcal{D}^T\mathbf{s}u + \mathbf{s}\mathbf{m}\big)\right\}\mathrm{d}x + \frac{1}{2}\int_{\partial\Omega}(n\cdot\sigma)u\,\mathrm{d}S\right\rangle - \int_\Omega \langle f\rangle\langle wu\rangle\,\mathrm{d}x - \int_{S_T}\langle T\rangle\langle wu\rangle\,\mathrm{d}S\end{aligned} \tag{49}$$

by the divergence theorem. Now use condition (47) to replace $\mathcal{D}^T\mathbf{s}$ by $w\mathcal{D}^T\langle\mathbf{s}\rangle$ over Ω. Also, use (48) to replace $n\cdot\sigma$ by $w(n\cdot\langle\sigma\rangle)$ over S_T, and note that u can be replaced by $\langle wu\rangle$ over S_u, since u is specified as sure there. Taking the ensemble mean then gives

$$\mathcal{F}\big(\langle wu\rangle\big) = \int_\Omega \left\{-\frac{1}{2}\big(\mathcal{D}^T\langle\mathbf{s}\rangle\langle wu\rangle + \langle\mathbf{s}\rangle\mathbf{m}\big) - \langle f\rangle\langle wu\rangle\right\}\mathrm{d}x + \frac{1}{2}\int_{\partial\Omega}\langle wu\rangle\big(n\cdot\langle\sigma\rangle\big)\,\mathrm{d}S - \int_{S_T}\langle T\rangle\langle wu\rangle\,\mathrm{d}S \tag{50}$$

Use of the divergence theorem in the reverse direction now gives

$$\mathcal{F}\big(\langle wu\rangle\big) = \int_\Omega \left\{\frac{1}{2}\langle\mathbf{s}\rangle\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big) - \langle f\rangle\langle wu\rangle\right\}\mathrm{d}x - \int_{S_T}\langle T\rangle\langle wu\rangle\,\mathrm{d}S \tag{51}$$

or equivalently,

$$\mathcal{F}\big(\langle wu\rangle\big) = \int_\Omega \left\{\frac{1}{2}\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big)\mathcal{L}^{\mathrm{eff}}\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big) - \langle f\rangle\langle wu\rangle\right\}\mathrm{d}x - \int_{S_T}\langle T\rangle\langle wu\rangle\,\mathrm{d}S \tag{52}$$

This final expression is of exactly the same form as the original functional in the variational principle (42). It could have been written down directly from the equations governing the problem for the effective medium. What has been gained, however, is a clear interpretation of its physical significance, as a conditional minimizer of the original functional, even when $\langle wu\rangle$ is no more than a trial field rather than the one that actually solves the problem. Another fact, of potential significance, is the deduction that

$$\int_\Omega \frac{1}{2}\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big)\mathcal{L}^{\mathrm{eff}}\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big)\,\mathrm{d}x = \inf\left\langle\int_\Omega \frac{1}{2}(\mathbf{e}-\mathbf{m})\mathcal{L}(\mathbf{e}-\mathbf{m})\,\mathrm{d}x\right\rangle \tag{53}$$

the infimum taken with $\langle wu\rangle$ prescribed over Ω and $u(x,\alpha) = \langle wu\rangle$, sure, on S_u, which will permit the development of rigorous bounds on the non-local operator $\mathcal{L}^{\mathrm{eff}}$, when s is real. As observed from the outset, the minimum principle becomes just a stationary principle when s is complex. It may be noted, however, that, when $s = -i\omega$ with ω real, Milton and Willis [18] have constructed a partial dual of the original functional which delivers a minimum principle in the case that the original medium dissipates energy.

6. Principle of Hashin–Shtrikman type

A variational principle of the type first constructed (in the case of statics) by Hashin and Shtrikman [19,20] will now be developed. It employs a comparison medium with properties defined by $\mathcal{L}_0$, as in Section 4, except that now it is assumed that $\mathcal{L}_0^T = \mathcal{L}_0$. The starting point is to note that

$$\frac{1}{2}(\mathbf{e}-\mathbf{m})(\mathcal{L}-\mathcal{L}_0)(\mathbf{e}-\mathbf{m}) \leqslant (\mathbf{e}-\mathbf{m})\mathbf{n} - \frac{1}{2}\mathbf{n}(\mathcal{L}-\mathcal{L}_0)^{-1}\mathbf{n} \tag{54}$$

so long as the quadratic forms are negative-definite,[5] with equality when Eq. (18) is satisfied. Therefore, (53) can be expressed in the form

[5] Negative-definiteness of one implies negative-definiteness of the other.

$$\int_\Omega \frac{1}{2}\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big)\mathcal{L}^{\mathrm{eff}}\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big)\,\mathrm{d}x$$
$$= \inf_{\mathbf{n}} \inf_{u} \left\langle \int_\Omega \left\{ \frac{1}{2}(\mathbf{e}-\mathbf{m})\mathcal{L}_0(\mathbf{e}-\mathbf{m}) + (\mathbf{e}-\mathbf{m})\mathbf{n} - \frac{1}{2}\mathbf{n}(\mathcal{L}-\mathcal{L}_0)^{-1}\mathbf{n}\right\}\mathrm{d}x\right\rangle \tag{55}$$

First, evaluate the infimum with respect to u, conditional on $\langle wu\rangle$ being specified, as required in (53). As in Section 5, the conditional minimizer must satisfy Eqs. (45) and (46) where f_1 and T_1 are Lagrange multipliers, except that now $\mathbf{s}$ is given by (18). Note that the problem so defined is identical to the one posed in Section 4, except that $\langle f\rangle$ and $\langle T\rangle$ are replaced by f_1 and T_1. The conditional minimizer $u(x,\alpha)$ is thus given by (28) and the corresponding $\mathbf{e}-\mathbf{m}$ is given by (30). It is helpful to note that, at the conditional minimizer $u(x,\alpha)$,

$$\left\langle \int_\Omega \left\{ \frac{1}{2}(\mathbf{e}-\mathbf{m})\mathcal{L}_0(\mathbf{e}-\mathbf{m}) + (\mathbf{e}-\mathbf{m})\mathbf{n}\right\}\mathrm{d}x\right\rangle = \left\langle \int_\Omega \left\{ \frac{1}{2}(\mathbf{e}-\mathbf{m})\mathbf{s} + \frac{1}{2}(\mathbf{e}-\mathbf{m})\mathbf{n}\right\}\mathrm{d}x\right\rangle$$
$$= \int_\Omega \left\{ \frac{1}{2}\big(\langle\mathbf{e}\rangle-\mathbf{m}\big)\langle\mathbf{s}\rangle + \frac{1}{2}\big\langle(\mathbf{e}-\mathbf{m})\mathbf{n}\big\rangle\right\}\mathrm{d}x \tag{56}$$

by an application of the divergence theorem like that used in deriving (51). It follows that[6]

$$\int_\Omega \frac{1}{2}\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big)\mathcal{L}^{\mathrm{eff}}\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big)\,\mathrm{d}x$$
$$\begin{aligned}\leqslant \int_\Omega \bigg\{ &\frac{1}{2}\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big)(\mathcal{L}_0 - \mathcal{L}_0\Gamma_0\mathcal{L}_0)\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big)\\ &+ \frac{1}{2}\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big)\mathcal{L}_0\mathcal{E}G_0\langle wG_0w\rangle^{-1}\big(\mathcal{E}G_0^\dagger\big)^\dagger\mathcal{L}_0\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big)\\ &+ \langle\mathbf{n}\rangle(\mathcal{I}-\Gamma_0\mathcal{L}_0)\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big) + \langle\mathbf{n}\mathcal{E}G_0w\rangle\langle wG_0w\rangle^{-1}\big(\mathcal{E}G_0^\dagger\big)^\dagger\mathcal{L}_0\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big)\\ &+ \frac{1}{2}\langle\mathbf{n}\mathcal{E}G_0w\rangle\langle wG_0w\rangle^{-1}\big\langle w\big(\mathcal{E}G_0^\dagger\big)^\dagger\mathbf{n}\big\rangle - \frac{1}{2}\langle\mathbf{n}\Gamma_0\mathbf{n}\rangle - \frac{1}{2}\big\langle\mathbf{n}(\mathcal{L}-\mathcal{L}_0)^{-1}\mathbf{n}\big\rangle\bigg\}\,\mathrm{d}x\end{aligned} \tag{57}$$

The condition for minimizing the right side of (57) is exactly (33). Note that the inequality is true so long as $\mathcal{L} < \mathcal{L}_0$ in the sense of quadratic forms, and s is real. Parallel reasoning shows that the inequality is reversed if $\mathcal{L} > \mathcal{L}_0$. Whether or not $\mathcal{L}-\mathcal{L}_0$ is definite, and whether s is real or complex, the stationary value of the right side of (57) coincides with the left side, and $\mathcal{L}^{\mathrm{eff}}$ is given by (37).

It is of interest now to exploit (57) to obtain bounds (or more generally, approximations) for $\mathcal{L}^{\mathrm{eff}}$. This is done by optimizing the right side, with $\mathbf{n}$ restricted to a chosen linear space. For illustration, take $\mathbf{n}(x,\alpha)$ to have the form

$$\mathbf{n}(x,\alpha) = \sum_r \mathbf{n}_r(x)\phi_r(x,\alpha) \tag{58}$$

where $\{\phi_r(x,\alpha)\}$ is the set of basis functions and $\mathbf{n}_r(x)$ are to be chosen optimally. Thus, $\{\mathbf{n}_r\}$ must satisfy

$$\sum_s A_{rs}\mathbf{n}_s = B_r\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big) \tag{59}$$

where[7]

$$A_{rs} = \big\langle\phi_r\big[(\mathcal{L}-\mathcal{L}_0)^{-1} + \Gamma_0\big]\phi_s\big\rangle - \langle\phi_r\mathcal{E}G_0w\rangle\langle wG_0w\rangle^{-1}\big\langle w\big(\mathcal{E}G_0^\dagger\big)^\dagger\phi_s\big\rangle \tag{60}$$

and

$$B_r = \langle\phi_r\rangle(\mathcal{I}-\Gamma_0\mathcal{L}_0) + \langle\phi_r\mathcal{E}G_0w\rangle\langle wG_0w\rangle^{-1}\big(\mathcal{E}G_0^\dagger\big)^\dagger\mathcal{L}_0 \tag{61}$$

The generalization of the Hashin–Shtrikman prescription as developed by Willis [9] is realized within the framework of (58) by taking $\phi_r(x,\alpha)$ to be the characteristic function of “phase r” of the composite. A scheme of the type employed by Nemat-Nasser and Srivastava [21] (with one spatial dimension, $d=1$) and Srivastava and Nemat-Nasser [22] (three

[6] Self-adjointness is exploited here.
[7] It is important to note that here $(\mathcal{L}-\mathcal{L}_0)^{-1}$ is interpreted as an operator, containing the factor $\delta(x-x')$.

dimensions, $d = 3$) for a periodic medium with period Q is obtained by sub-dividing Q by a grid. Then $\phi_{0r}(x)$ is taken to be 1 in the rth sub-division, zero elsewhere in Q and extended periodically throughout space. Finally, ϕ_r is taken to be $\phi_r(x, y) = \phi_{0r}(x - y)$, the variable $y \in Q$ playing the role of the sample parameter α. More generally, the basis functions ϕ_r need not be piecewise constant, but could, for instance, have the spatial character of a finite-element basis.

The solution of Eq. (33) may formally be written as

$$\mathbf{n} = \mathcal{N}\big(\mathcal{E}\langle wu\rangle - \mathbf{m}\big) \tag{62}$$

The resulting expression for $\mathcal{L}^{\mathrm{eff}}$ is then

$$\mathcal{L}^{\mathrm{eff}} = \mathcal{L}_0 - \mathcal{L}_0\Gamma_0\mathcal{L}_0 + \mathcal{L}_0\mathcal{E}G_0\langle wG_0w\rangle^{-1}\big(\mathcal{E}G_0^{\dagger}\big)^{\dagger}\mathcal{L}_0 + (\mathcal{I} - \mathcal{L}_0\Gamma_0)\langle\mathcal{N}\rangle + \mathcal{L}_0\mathcal{E}G_0\langle wG_0w\rangle^{-1}\big\langle w\big(\mathcal{E}G_0^{\dagger}\big)^{\dagger}\mathcal{N}\big\rangle \tag{63}$$

If an optimal approximation for $\mathcal{N}$ is found, for instance as the solution of Eq. (59), then

$$\mathcal{N} \approx \sum_r\sum_s \phi_r\big\{A^{-1}\big\}_{rs}B_s \tag{64}$$

and (63) provides an optimal approximation for $\mathcal{L}^{\mathrm{eff}}$, which is an upper or lower bound in the case that $\mathcal{L}_0$ is greater than or smaller than $\mathcal{L}$ in the sense of quadratic forms, and s is real.

Study of the full implications of this formulation will require fairly extensive computations and forms a project for the future.

7. Discussion

It should perhaps be emphasized first that the reasoning developed in this work is applicable to different types of wave propagation. In particular, it applies directly to the propagation of electromagnetic waves. Appendix B summarizes Maxwell's equations and provides definitions for $\mathcal{L}$, etc. under which every result recorded in this work applies to electromagnetic waves. In fact, the work of Willis [7] was developed primarily for Maxwell's equations, with elastodynamics treated as a corollary. The resulting effective relation in the electromagnetic case has bi-anisotropic form; see, for instance, [23] for an account of such media.

The main message of this paper is that it is possible to develop variational descriptions of the effective dynamic response of composites, when the "effective displacement" is taken as the weighted mean, $\langle wu\rangle$. This is facilitated by the demonstration, contained in formula (16) derived by Willis [7], that the effective medium is defined by operators that are self-adjoint, so long as the components of the original composite have self-adjoint symmetry. There are, however, also insights of recent origin that apply even when the unweighted mean, $\langle u\rangle$ is employed. The main one is that effective dynamic properties are inevitably non-unique, if the intention is to develop theory to describe fields driven by any combination of body force and surface traction and/or displacement (without any "transformation strain"). Explicitly, if

$$\langle\mathbf{s}\rangle = \mathcal{L}^{\mathrm{eff}}\mathcal{E}\langle wu\rangle \tag{65}$$

then also $\langle\mathbf{s}\rangle = (\mathcal{L}^{\mathrm{eff}} + \mathcal{L}')\mathcal{E}\langle wu\rangle$, for any $\mathcal{L}'$ of the form

$$\mathcal{L}' = \begin{pmatrix} C' & S' \\ \hat{S}' & \rho' \end{pmatrix} \tag{66}$$

for which

$$\frac{\partial C'_{ijkl}(x, x')}{\partial x'_l} - sS'_{ijk}\big(x, x'\big) = 0, \quad x' \in \Omega \quad \text{and} \quad C'_{ijkl}\big(x, x'\big)n_l\big(x'\big) = 0, \quad x' \in \partial\Omega \tag{67}$$

$$\frac{\partial \hat{S}'_{ikl}(x, x')}{\partial x'_l} - s\rho'_{ik}\big(x, x'\big) = 0, \quad x' \in \Omega \quad \text{and} \quad \hat{S}'_{ikl}\big(x, x'\big)n_l\big(x'\big) = 0, \quad x' \in \partial\Omega \tag{68}$$

As already noted in Section 3, it is possible to generate tensors C' that satisfy the conditions (67) with $S' = 0$. The non-uniqueness thus remains, even in the case of elastostatics. The conditions on $\mathcal{L}'$ were given in symbolic form by Willis [7] but were not presented quite so explicitly. The implication of this non-uniqueness is that no set of experiments – or computations – can be comprehensive enough to determine $\mathcal{L}^{\mathrm{eff}}$. There is, rather, an *equivalence class* of constitutive operators, but this is not a helpful notion for practical use. What is needed, instead, is some selection procedure which ensures that only one $\mathcal{L}^{\mathrm{eff}}$ is defined, and this can be taken as representative of the whole equivalence class. Shuvalov et al. [24], for instance, found, for the propagation of free waves in a layered medium, that they could always develop a prescription in which the effective density and the "S-terms" as in (8) were functions only of frequency while the effective tensor C^{eff} was allowed to depend also on wavenumber. They did not, however, consider disturbances driven by body force. The device employed by Willis [7] was to require that the effective medium could support also an arbitrary transformation strain, the effective relation thus taking the form (36). The advantage of this is that it is applicable for any composite microgeometry, with body-force present – as well as giving the correct (unique) $\mathcal{L}^{\mathrm{eff}}$ if inelastic strain happens to be present. The problem of non-uniqueness appears first to have been confronted by Feitz and Shvets [25] in the electromagnetic context. Their concern was to deduce the components in reciprocal space of the effective operator $\mathcal{L}^{\mathrm{eff}}$ for an infinite periodic medium, by solving numerically a sufficient set of linearly independent problems, and did this by imposing, in addition to electric current, a fictitious "magnetic monopole current". The introduction of magnetization M by Willis [7] is an alternative which has the virtue that it fits exactly the pattern resulting from introducing inelastic strain in the context of (visco-)elastodynamics.

It is common practice, particularly for a matrix-inclusion composite, to develop equations relative to a comparison medium, usually chosen to be composed of matrix material. This has not been formulated previously when the chosen effective displacement is the weighted mean $\langle wu\rangle$. The advantage of including Section 4 is that it introduces this formulation in full generality, allowing both for inelastic strain and non-self-adjoint response, and provides in the process several formulae needed subsequently, in the associated variational description given in Section 6. It is possible, in fact, to implement the "comparison medium" formulation without explicit allowance for $\mathbf{m}$, although this has never previously been done for the case of a weighted mean displacement. In the context of the approximate description (59), for example, it would be natural to develop the solution operator $\mathcal{N}$ even in the absence of $\mathbf{m}$, thus providing a prescription for obtaining an approximation to a particular $\mathcal{L}^{\mathrm{eff}}$, even though it would not be unique. It is interesting that, in fact, this $\mathcal{L}^{\mathrm{eff}}$ is the one that remains correct when $\mathbf{m}$ is present.

It is remarked in closing that, in the unweighted case ($w \equiv 1$), factors involving G_0 and G_0^{-1} cancel to give the identity, with the result that Eq. (33) reduces to the familiar form

$$(\mathcal{L}-\mathcal{L}_0)^{-1}\mathbf{n} + \Gamma_0\big(\mathbf{n}-\langle\mathbf{n}\rangle\big) = \langle\mathbf{e}\rangle - \mathbf{m} \tag{69}$$

the representation (63) for $\mathcal{L}^{\mathrm{eff}}$ becomes

$$\mathcal{L}^{\mathrm{eff}} = \mathcal{L}_0 + \langle\mathcal{N}\rangle \tag{70}$$

and the inequality (57) reduces to

$$\begin{aligned}\int_\Omega \frac{1}{2}\big(\langle\mathbf{e}\rangle-\mathbf{m}\big)\mathcal{L}^{\mathrm{eff}}\big(\langle\mathbf{e}\rangle-\mathbf{m}\big)\,\mathrm{d}x \leqslant \int_\Omega \bigg\{&\frac{1}{2}\big(\langle\mathbf{e}\rangle-\mathbf{m}\big)\mathcal{L}_0\big(\langle\mathbf{e}\rangle-\mathbf{m}\big) + \langle\mathbf{n}\rangle\big(\langle\mathbf{e}\rangle-\mathbf{m}\big)\\ &+\frac{1}{2}\langle\mathbf{n}\rangle\Gamma_0\langle\mathbf{n}\rangle - \frac{1}{2}\langle\mathbf{n}\Gamma_0\mathbf{n}\rangle - \frac{1}{2}\big\langle\mathbf{n}(\mathcal{L}-\mathcal{L}_0)^{-1}\mathbf{n}\big\rangle\bigg\}\,\mathrm{d}x\end{aligned} \tag{71}$$

valid when the quadratic form associated with $\mathcal{L}-\mathcal{L}_0$ is negative-definite and s is real.

Appendix A. Some identities relating operators for the actual and the comparison medium

Green's function G for the actual medium is defined so that its adjoint $G^\dagger$ satisfies the same equations, (20) and (21), as the comparison medium, except that the suffix $_0$ is omitted. First, it is demonstrated that G also satisfies the equation

$$\mathcal{D}^T\mathcal{L}\mathcal{E}G\big(x,x''\big) + I\delta\big(x-x''\big) = 0, \quad x\in\Omega \tag{A.1}$$

and the boundary conditions

$$n\cdot\big\{C(x)\mathcal{E}G\big(x,x''\big)\big\} = 0, \quad x\in S_T \quad \text{and} \quad G\big(x,x''\big) = 0, \quad x\in S_u \tag{A.2}$$

Towards this end, call the field that satisfies (A.1) and (A.2) U rather than G. It follows from the representation (23) (appropriately specialized, with suffix $_0$ omitted) that

$$U\big(x',x''\big) = \big[G^\dagger\big(x'',x'\big)\big]^T \equiv G\big(x',x''\big) \tag{A.3}$$

from the basic definition of $G^\dagger$ as the adjoint of G.

Next, G is expressed relative to the comparison medium. By noting that the stress $\Sigma = C(\mathcal{E}G)$ and momentum $P = \rho(sG)$ associated with G can be written as

$$\begin{pmatrix}\Sigma\\ P\end{pmatrix} = \mathcal{L}(\mathcal{E}G) \equiv \mathcal{L}_0(\mathcal{E}G) + (\mathcal{L}-\mathcal{L}_0)(\mathcal{E}G) \tag{A.4}$$

it follows immediately from the representation (23) that

$$G = G_0 - \big(\mathcal{E}G_0^\dagger\big)^\dagger(\mathcal{L}-\mathcal{L}_0)(\mathcal{E}G) \tag{A.5}$$

Also, by interchanging the roles of $\mathcal{L}$ and $\mathcal{L}_0$,

$$G_0 = G + (\mathcal{E}G)^\dagger(\mathcal{L}-\mathcal{L}_0)(\mathcal{E}G_0) \tag{A.6}$$

Therefore, from (A.5),

$$(\mathcal{E}G) = (\mathcal{E}G_0) - \Gamma_0(\mathcal{L} - \mathcal{L}_0)(\mathcal{E}G) \tag{A.7}$$

which immediately implies (38). Also, from (A.6),

$$\big(\mathcal{E}G^\dagger\big)^\dagger = \big(\mathcal{E}G_0^\dagger\big)^\dagger - \big(\mathcal{E}G^\dagger\big)^\dagger(\mathcal{L} - \mathcal{L}_0)\Gamma_0 \tag{A.8}$$

which implies (39). Relation (40) follows by application of $\mathcal{E}$ to (39) (or to (A.8)). Finally, Eq. (41) is obtained from combining (A.5) and (38).

Appendix B. Notation for electromagnetic waves

Maxwell's equations (in the Laplace transform domain) are

$$\operatorname{curl} H - sD - j = 0 \tag{B.1}$$

$$\operatorname{curl} E + sB = 0 \tag{B.2}$$

which are to be solved in conjunction with the constitutive relations

$$D = \varepsilon E \quad \text{and} \quad B = \mu H + M \tag{B.3}$$

The unconventional feature here is the presence of the "magnetization" M relative to the medium: it serves the same purpose as the inelastic strain η introduced into the (visco)elastic constitutive relation (3). The second Maxwell equation, (B.2), is satisfied identically if a vector potential A is introduced, such that

$$B = \operatorname{curl} A \quad \text{and} \quad E = -sA \tag{B.4}$$

Now, in analogy with Eqs. (11), define

$$\mathbf{s} = \begin{pmatrix} H \\ D \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} B \\ E \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} M \\ 0 \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} \operatorname{curl} \\ -s \end{pmatrix}, \quad \mathcal{D}^T = (\operatorname{curl} \quad -s) \quad \text{and} \quad \mathcal{L} = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \varepsilon \end{pmatrix} \tag{B.5}$$

The Maxwell system can then be written in the form

$$\mathcal{D}^T \mathbf{s} - j = 0 \tag{B.6}$$

in conjunction with the constitutive relation

$$\mathbf{s} = \mathcal{L}(\mathbf{e} - \mathbf{m}), \quad \text{where } \mathbf{e} = \mathcal{E}A \tag{B.7}$$

For more detail, including specification of boundary conditions, the reader is referred to Willis [7]. He primarily discussed the Maxwell system and afterwards deduced the corresponding result for elasticity. Correspondingly, certain different symbols (**b** instead of **e**, and $\mathcal{B}$ instead of $\mathcal{E}$, for example) were employed, but otherwise the structure is identical. The point is that the notation introduced in (B.5) places the Maxwell system in exact correspondence with the (visco)elastic system, so that every line of the reasoning developed in the main text is applicable directly also to electromagnetic waves.

References

[1] Z. Liu, X. Zhang, Y. Mao, Y.Y. Zhu, Z. Yang, C.T. Chan, P. Sheng, Locally resonant sonic materials, Science 289 (2000) 1734–1736.
[2] Z. Liu, C.T. Chan, P. Sheng, Analytic model of phononic crystals with local resonances, Phys. Rev. B 71 (2005) 014103.
[3] V.V. Zhikov, On an extension of the method of two-scale convergence and its applications, Sb. Math. 191 (2000) 973–1014.
[4] S.A. Schelkunoff, H.J. Friis, Antennas: Theory and Practice, J. Wiley, New York, 1952.
[5] J.B. Pendry, A.J. Holden, W.J. Stewart, I. Youngs, Extremely low frequency plasmons in metallic mesostructures, Phys. Rev. Lett. 76 (1996) 4773–4776.
[6] J.B. Pendry, A.J. Holden, D.J. Robbins, W.J. Stewart, Magnetism from conductors and enhanced nonlinear phenomena, IEEE Trans. Microwave Theory Tech. 47 (1999) 2075–2084.
[7] J.R. Willis, Effective constitutive relations for waves in composites and metamaterials, Proc. R. Soc. A 467 (2011) 1865–1879.
[8] J.R. Willis, Variational principles for waves in inhomogeneous elastic media, Wave Motion 3 (1981) 1–11.
[9] J.R. Willis, Elasticity theory of composites, in: H.G. Hopkins, M.J. Sewell (Eds.), Mechanics of Solids. The Rodney Hill 60th Anniversary Volume, Pergamon, Oxford, 1982, pp. 653–686.
[10] J.R. Willis, Dynamics of composites, in: P. Suquet (Ed.), Continuum Micromechanics, in: CISM Lecture Notes, Springer, Wien/New York, 1997, pp. 265–290.
[11] G.W. Milton, J.R. Willis, On modifications of Newton's second law and linear continuum elastodynamics, Proc. R. Soc. A 463 (2007) 855–880.
[12] J.R. Willis, Exact effective relations for dynamics of a laminated body, Mech. Mater. 41 (2009) 385–393.
[13] E. Kröner, Elastic moduli of perfectly disordered composite materials, J. Mech. Phys. Solids 15 (1967) 319–329.
[14] M.J. Beran, Statistical Continuum Theories, Interscience, New York, 1968.
[15] J.R. Willis, A polarization approach to the scattering of elastic waves. I. Scattering by a single inclusion, J. Mech. Phys. Solids 28 (1980) 287–305.
[16] J.R. Willis, A polarization approach to the scattering of elastic waves. II. Scattering by multiple inclusions, J. Mech. Phys. Solids 28 (1980) 307–327.
[17] J.R. Willis, A comparison of two formulations for effective relations for waves in composites, Mechanics of Materials 47 (2012) 51–60.

[18] G.W. Milton, J.R. Willis, Minimum variational principles for time-harmonic waves in a dissipative medium and associated variational principles of Hashin–Shtrikman type, Proc. R. Soc. A 466 (2010) 3013–3032.
[19] Z. Hashin, S. Shtrikman, On some variational principles in anisotropic and inhomogeneous elasticity, J. Mech. Phys. Solids 10 (1962) 335–342.
[20] Z. Hashin, S. Shtrikman, A variational approach to the theory of the elastic behaviour of multiphase materials, J. Mech. Phys. Solids 11 (1963) 127–140.
[21] S. Nemat-Nasser, A. Srivastava, Overall dynamic constitutive relations of layered elastic composites, J. Mech. Phys. Solids 59 (2011) 1953–1965.
[22] A. Srivastava, S. Nemat-Nasser, Overall dynamic properties of three-dimensional periodic elastic composites, Proc. R. Soc. A 468 (2012) 269–287.
[23] A. Serdyukov, I. Semchenko, S. Tratyakov, A. Sihvola, Electromagnetics of Bi-anisotropic Materials, Gordon and Breach, Amsterdam, 2001.
[24] A.L. Shuvalov, A.A. Kutsenko, A.N. Norris, O. Poncelet, Effective Willis constitutive equations for periodically stratified anisotropic elastic media, Proc. R. Soc. A 467 (2011) 1749–1769.
[25] C. Feitz, G. Shvets, Current-driven metamaterial homogenization, Phys. B: Condens. Matter 405 (2010) 2930–2934.