C. R. Mecanique 340 (2012) 205–218

Contents lists available at SciVerse ScienceDirect

Comptes Rendus Mecanique

www.sciencedirect.com

Recent Advances in Micromechanics of Materials

A magnetically anisotropic, ellipsoidal inclusion subjected to a non-aligned magnetic field in an elastic medium

M.H. Siboni [a], Pedro Ponte Castañeda [a,b,c,*]

[a] Department of Mechanical Engineering and Applied Mechanics, University of Pennsylvania, Philadelphia, PA 19104-6315, USA
[b] Graduate Group in Applied Mathematics and Computational Science, University of Pennsylvania, Philadelphia, PA 19104-6315, USA
[c] Madrid Institute for Advanced Studies of Materials (IMDEA Materials), 28040 Madrid, Spain

ARTICLE INFO

Article history:
Available online 30 March 2012

Keywords:
Magneto-elasticity
Single inclusion problem
Elastic stress
Maxwell stress
Infinitesimal deformations

ABSTRACT

This article deals with the problem of an isolated, rigid inclusion with linear-magnetic behavior embedded in a linear-elastic matrix. Under the hypothesis of infinitesimal deformations, an analytical expression is obtained for the equilibrium rotation of the magnetic inclusion under general magneto-mechanical loading conditions. The results show that the inclusion undergoes an 'extra' rotation due to the presence of non-aligned magnetic fields (even in the absence of mechanical loadings). Moreover, this extra rotation is found to depend on the shape of the inclusion, as well as on its magnetic anisotropy. Thus, the extra rotation increases monotonically to an asymptote with increasing magnetic anisotropy of the inclusion, while, for fixed magnetic behavior of the inclusion, the extra rotation increases up to a maximum with increasing aspect ratio, and then decays to zero.

1. Introduction

The problem of an isolated inclusion embedded in an infinite matrix, also known as the single-impurity problem, is a central problem in the theory of composite materials, and other heterogeneous media. In the context of linear elasticity, a general solution for a single ellipsoidal inclusion of one material embedded in a matrix of a second material was provided by Eshelby in a highly-cited paper (Eshelby [1]). The main feature of this solution—which was known much earlier in the context of electrostatics (e.g., Maxwell [2])—is that the stress and strain fields are uniform inside the ellipsoidal inclusion. This feature has made possible the development of dilute and effective medium theories for composites containing random distributions of such inclusions (Willis [3]). Because of this, generalizations of Eshelby's solution, and of some of the methods that are based on it, have been developed for magneto-elastic composites with piezomagnetic coupling in recent years (e.g., Benveniste [4]; Dunn and Taya [5]; Huang et al. [6]).

This work is concerned with magneto-elastic systems of a different type, involving coupled behavior through the Maxwell stress. It is well known (e.g., Landau et al. [7]) that an externally applied magnetic field gives rise to forces and torques on magnetically susceptible particles. When such particles are embedded in an elastic matrix (even if it is not itself magnetically susceptible), the externally applied magnetic field can generate stress and strain fields in the matrix material, as a consequence of the forces and torques that are transmitted by the particles to the surrounding matrix. These stresses and strains can combine with corresponding stresses and strains arising solely from purely mechanical sources to produce interesting coupled magneto-elastic behavior, such as magnetostriction. In this connection, it should be noted that the magnitude

* Corresponding author at: Department of Mechanical Engineering and Applied Mechanics, University of Pennsylvania, Philadelphia, PA 19104-6315, USA.
E-mail addresses: mortezah@seas.upenn.edu (M.H. Siboni), ponte@seas.upenn.edu (P. Ponte Castañeda).

1631-0721/$ – see front matter
doi:10.1016/j.crme.2012.02.003

of the Maxwell stresses that are generated in the matrix material are quadratic in the magnetic field and can therefore be significant for large values of the fields (even if the effect is limited by the magnetic saturation of the particles).

Magnetorheological elastomers, which consist of very stiff magnetic inclusions embedded in relatively soft elastic matrices (Ginder et al. [8,9]; Guan et al. [10]), are composite materials which attempt to exploit the above described mechanism to generate large field-induced (i.e. magnetostrictive) strains. These materials are examples of 'smart' materials, which can be useful in many applications as actuators and sensors, as well as artificial analogues of muscles. Recent papers by Borcea and Bruno [11] and Yin et al. [12] have addressed the effects of dipole forces between particles, and their implications for the constitutive behavior of such materials. In addition, Liu et al. [13] have generated variational estimates for elastic-matrix composites with a dilute concentration of a certain type of (deformable) magnetostrictive particles. However, it appears that, to date, no systematic attempts have been made to describe the effects of the magnetic torques that would be expected to develop when the particles have a geometric shape and/or magnetic anisotropy. To this end, we develop in this paper, building on the work of Eshelby and others, an analytical solution for a single, rigid inclusion with linear, anisotropic magnetic behavior that is embedded in a non-magnetic, linear-elastic matrix.

The article is organized as follows. Section 2 provides a quick review of magneto-elasticity, where we present the governing equations, as well as the constitutive equations for the two special classes of materials involved: rigid materials with linear-magnetic behavior and elastomers with linear-elastic, non-magnetic behavior. In addition, the problem for the magnetic ellipsoidal inclusion in an elastic matrix is formulated, and the general strategy for solving the problem in terms of two auxiliary problems in magnetostatics and linear elasticity is introduced. In Section 3 we address the magnetostatics problem of the single rigid inclusion with linear-magnetic behavior in the deformed configuration, by means of the integral equation formulation of Willis [14,3]. Furthermore, we obtain an explicit formula for the force and torque exerted on the inclusion by the external magnetic field. Section 4 is devoted to the elasticity problem of a single rigid inclusion in an infinite elastic matrix under the combined action of an externally applied deformation and a prescribed rotation of the inclusion (Walpole [15]). Using the results of Sections 3 and 4 for the magnetic and mechanical inclusion problems, together with the fact that the Maxwell stress in the surrounding elastic matrix is self-equilibrated, we obtain an expression in Section 5 for the equilibrium rotation of a rigid inclusion with general ellipsoidal shape and magnetic anisotropy in an infinite elastic matrix under both mechanical and magnetic loading of the system. More specifically, we show that, in the presence of an external magnetic field, the inclusion can undergo an extra rotation in addition to the Eshelby-type, mechanically driven rotation. In Section 6 an analytical expression is obtained for the in-plane rotation of the isolated inclusion under appropriate loading conditions, and the results are analyzed for some special cases. In addition, the analytical results are compared with the results of a simple finite element (FE) analysis for some two-dimensional examples. Finally, we conclude the paper in Section 7 with a brief discussion of the results and possible future applications.

Throughout this article, scalars are denoted by italic Roman, a, or Greek letters, α; vectors by boldface Roman letters, $\mathbf{b}$; second-order tensors by boldface italic Roman letters, $\boldsymbol{P}$, or Greek letters, $\boldsymbol{\alpha}$; and fourth-order tensors by barred letters, $\mathbb{S}$. Where necessary, conventional index notation is adopted, e.g., b_i, P_{ij} and S_{ijkl} are the Cartesian components of the vector $\mathbf{b}$, second-order tensor $\boldsymbol{P}$ and fourth-order tensor $\mathbb{S}$, respectively.

2. Problem formulation

Magneto-elastic materials are defined as materials exhibiting coupled magnetic and mechanical behaviors. Characterizing the constitutive behavior of magneto-elastic materials requires a model for the interaction of the magnetic fields and matter. Modeling the response of deformable bodies to magnetic fields (or in general, to electromagnetic fields) has been a challenging scientific issue (see, for example, Maxwell [2]), and, as a consequence, different formulations have been proposed in the literature, which do not seem consistent at first glance. Despite the fact that there exist different formulations, it can be shown that under certain assumptions, all these formulations are equivalent (see, for example, Hutter et al. [16]). In the following, we present a brief summary of the governing equations that are required for the purposes of this article (see Kovetz [17] and Hutter et al. [16] for more general formulations).

2.1. Governing equations

Under application of mechanical and magnetic loadings a typical magneto-elastic material deforms from its original, reference configuration to a new, current configuration. This deformation can be parametrized by a continuous map, $\mathbf{x} = \mathbf{x}(\mathbf{X})$, which is a one-to-one correspondence between the position of material points in the reference configuration, $\mathbf{X}$, and in the current configuration, $\mathbf{x}$. In the infinitesimal limit, such deformations can be characterized by the displacement gradient $\nabla\mathbf{u}$, where $\mathbf{u} := \mathbf{x} - \mathbf{X}$ denotes the infinitesimal displacement. Its symmetric part, with Cartesian components $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, and antisymmetric part, $\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$, correspond respectively to the infinitesimal strain and rotation tensors.

In the absence of electric and relativistic effects, as well as external current densities, Maxwell's equations for the case of static magnetic fields are given by

$$\operatorname{div}\mathbf{b} = 0, \quad \text{and} \quad \operatorname{curl}\mathbf{h} = \mathbf{0} \tag{1}$$

where $\mathbf{b}$ and $\mathbf{h}$ are the magnetic induction and magnetic fields, respectively. In the classical theory of electromagnetism for magnetizable materials, it is standard to introduce the magnetization, denoted by $\mathbf{m}$, such that

$$\mathbf{m} = \frac{1}{\mu_0}\mathbf{b} - \mathbf{h} \tag{2}$$

Neglecting inertial and external forces, the conservation of linear and angular momentum for a homogeneous, magneto-elastic material may be written as

$$\operatorname{div} \boldsymbol{T} = \mathbf{0}, \quad \text{and} \quad \boldsymbol{T}^T = \boldsymbol{T} \tag{3}$$

where $\boldsymbol{T}$ is the total stress, including magnetic effects. In the presence of material interfaces (between two different phases in a composite), the following set of jump conditions has to be satisfied: $[\![\mathbf{u}]\!] = \mathbf{0}$, $[\![\boldsymbol{T}]\!]\mathbf{n} = \mathbf{0}$, $[\![\mathbf{b}]\!] \cdot \mathbf{n} = 0$, and $[\![\mathbf{h}]\!] \times \mathbf{n} = \mathbf{0}$, where $\mathbf{n}$ is the unit normal to the interface in the current configuration.

The above set of governing equations is completed by appropriate constitutive relations between the pairs, $\mathbf{b}$ and $\boldsymbol{T}$, and $\mathbf{h}$ and $\nabla\mathbf{u}$. As already mentioned, in this investigation, a rigid magnetically linear inclusion is embedded in a linear-elastic, magnetically insensitive material. Their constitutive responses are described as follows (Kovetz [17]; Hutter et al. [16]).

2.1.1. Magnetically impermeable, linear-elastic materials

For elastic materials, which are not susceptible to magnetic fields, the magnetization is always zero. Therefore, for this class of materials, the constitutive behavior reduces to

$$\mathbf{b} = \mu_0\mathbf{h}, \quad \text{and} \quad \boldsymbol{T} := \boldsymbol{T}^{el} + \boldsymbol{T}^M \tag{4}$$

where μ_0 is the permeability of vacuum, and $\boldsymbol{T}^{el}$ and $\boldsymbol{T}^M$ denote the "purely mechanical" and Maxwell stresses, given respectively by the expressions

$$\boldsymbol{T}^{el} = \mathbb{C}^{(1)}\boldsymbol{\epsilon}, \quad \text{and} \quad \boldsymbol{T}^M = \frac{\mathbf{b}\otimes\mathbf{b}}{\mu_0} - \frac{\mathbf{b}\cdot\mathbf{b}}{2\mu_0}\boldsymbol{I} \tag{5}$$

In the first of these equations, $\mathbb{C}^{(1)}$ is the stiffness of the material, which in general is a fully symmetric, fourth-order tensor, but for isotropic material behavior reduces to an isotropic tensor depending only on the elastic shear modulus, μ_{el}, and the Poisson's ratio, ν.

2.1.2. Rigid magnetic materials

A rigid material can only undergo a rigid displacement (a translation and a rotation). Therefore, in the context of infinitesimal deformations, the displacement gradient in the rigid particles is given by an infinitesimal rotation, i.e.

$$\nabla\mathbf{u} = \bar{\boldsymbol{\omega}}^{(2)} \tag{6}$$

while the total stress is indeterminate.

In the current configuration, the magnetic response of the rigid material will be taken to be described by the relation

$$\mathbf{b} = \boldsymbol{\mu}^{(2)}\mathbf{h} \tag{7}$$

where $\boldsymbol{\mu}^{(2)}$ is the magnetic permeability of the material. This linear model is appropriate for diamagnetic and paramagnetic materials. However, it should be noted that the analysis may be generalized for nonlinear responses of the type that may be exhibited by a soft ferromagnetic material (with small hysteresis). We will come back to this point in the concluding remarks. It should also be noted (see e.g. Ponte Castañeda and Galipeau [18]) in the context of expression (7) that the permeability in the current configuration $\boldsymbol{\mu}^{(2)}$ is related to the corresponding permeability in the reference configuration $\boldsymbol{M}^{(2)}$ via

$$\boldsymbol{\mu}^{(2)} = \boldsymbol{M}^{(2)} + \bar{\boldsymbol{\omega}}^{(2)}\boldsymbol{M}^{(2)} - \boldsymbol{M}^{(2)}\bar{\boldsymbol{\omega}}^{(2)} \tag{8}$$

where terms of order $(\bar{\boldsymbol{\omega}}^{(2)})^2$ and higher have been neglected because of the small-deformation approximation.

2.2. Solution procedure

We consider a single rigid magnetic inclusion, denoted by Ω_0, embedded in an infinite elastic, and non-magnetic matrix, occupying $\mathbb{R}^3 \setminus \Omega_0$. For the inclusion, we assume a general ellipsoidal shape in the reference configuration $\Omega_0 = \{\mathbf{X} : |\boldsymbol{Z}_0^{-T}\mathbf{X}| \leqslant 1\}$, where $\boldsymbol{Z}_0$ is a symmetric, second-order tensor representing the shape of the inclusion. In addition, we assume a general, anisotropic, reference permeability tensor, $\boldsymbol{M}^{(2)}$, for the magnetic behavior of the inclusion. The rigid inclusion in the presence of mechanical and magnetic loadings is expected to undergo an infinitesimal rotation until reaching an equilibrium point at which the infinitesimal rotation is $\bar{\boldsymbol{\omega}}_e^{(2)}$ (see Fig. 1). It is important to note that even in the absence of the external magnetic fields the inclusion may undergo an infinitesimal rotation (see Eshelby [1]). Since the

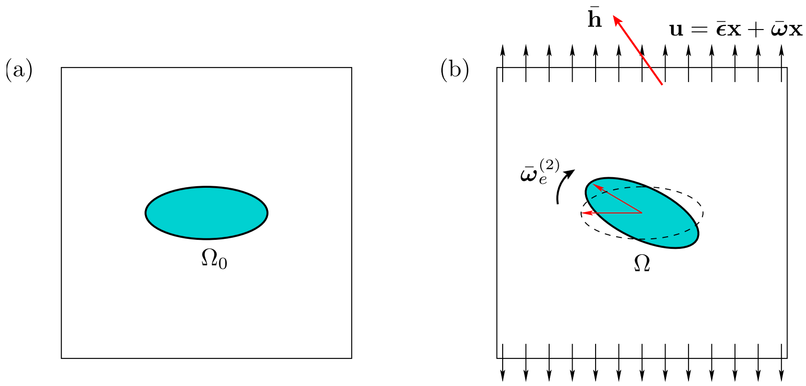

Fig. 1. Schematic of the single inclusion's magneto-elasticity problem. (a) The reference configuration, and (b) the deformed configuration.

inclusion is rigid, its shape in the deformed configuration is characterized by $\Omega = \{\mathbf{x}\colon\ |\boldsymbol{Z}^{-T}\mathbf{x}| \leqslant 1\}$, where, on account of the small rotations involved, the shape tensor $\boldsymbol{Z}$ in the current configuration is related to the corresponding reference shape tensor $\boldsymbol{Z}_0$ by

$$\boldsymbol{Z} = \boldsymbol{Z}_0 + \bar{\boldsymbol{\omega}}^{(2)}\boldsymbol{Z}_0 - \boldsymbol{Z}_0\bar{\boldsymbol{\omega}}^{(2)} \tag{9}$$

To determine the equilibrium rotation of the inclusion, we note that for static equilibrium, the global versions of the conservation laws of linear and angular momentum must be satisfied for any part of the body. In particular, for the domain Ω enclosing the rigid particle with boundary surface $\partial\Omega$, we have that the total force $\mathbf{f}$ and torque $\mathbf{l}$ on Ω must vanish identically, or

$$\mathbf{f} = \frac{1}{V}\int_{\partial\Omega} \boldsymbol{T}\mathbf{n}\,\mathrm{d}S = \mathbf{0}, \quad \text{and} \quad \mathbf{l} = \frac{1}{V}\int_{\partial\Omega} \mathbf{x}\times\boldsymbol{T}\mathbf{n}\,\mathrm{d}S = \mathbf{0} \tag{10}$$

where $\boldsymbol{T}$ is the total stress defined in the previous section (which includes *both* mechanical and magnetic contributions), V is the volume of the inclusion and $\mathbf{n}$ is the unit outward normal to $\partial\Omega$.

Noting that the inclusion Ω is surrounded by the matrix material (i.e., the surface $\partial\Omega$ in Eq. (10) is just outside the inclusion, and therefore in the matrix), the stress $\boldsymbol{T}$ in the total force and torque integrals correspond to the total stress in the matrix which is given by Eq. (4) in terms of the elastic stress, $\boldsymbol{T}^{el}$, and the Maxwell stress, $\boldsymbol{T}^{M}$, as defined by Eqs. (5). In conclusion, Eqs. (10) can be rewritten as

$$\mathbf{f}^{el} + \mathbf{f}^{mg} = \mathbf{0}, \quad \text{and} \quad \mathbf{l}^{el} + \mathbf{l}^{mg} = \mathbf{0} \tag{11}$$

where $\mathbf{f}^{el}$ and $\mathbf{f}^{mg}$, and $\mathbf{l}^{el}$ and $\mathbf{l}^{el}$ denote the "elastic" and "magnetic" components of the forces and the torques (per unit volume) acting on the inclusion due to the elastic matrix and the magnetic field, respectively. They are given by the same integrals as in (10) with $\boldsymbol{T}$ being replaced by $\boldsymbol{T}^{el}$ (or $\boldsymbol{T}^{M}$) for the elastic (or magnetic) force and torque.

Independent of the rotation of the inclusion, the elastic and magnetic forces acting on it will turn out to vanish (as will be discussed in more detail below). Thus, Eq. $(11)_1$ is automatically satisfied. On the other hand, for an arbitrary rotation of the inclusion, $\bar{\boldsymbol{\omega}}^{(2)}$, both elastic and magnetic torques in Eq. $(11)_2$ can be thought of as functions of the rotation (i.e. $\mathbf{l}^{el} = \mathbf{l}^{el}(\bar{\boldsymbol{\omega}}^{(2)})$ and $\mathbf{l}^{mg} = \mathbf{l}^{mg}(\bar{\boldsymbol{\omega}}^{(2)})$). Noting that the sum of these two torques must be zero at equilibrium, the following equation for the equilibrium rotation of the inclusion is obtained

$$\mathbf{l}^{el}\big(\bar{\boldsymbol{\omega}}^{(2)}\big) + \mathbf{l}^{mg}\big(\bar{\boldsymbol{\omega}}^{(2)}\big) = \mathbf{0}, \quad \text{for } \bar{\boldsymbol{\omega}}^{(2)} = \bar{\boldsymbol{\omega}}_e^{(2)} \tag{12}$$

As will be shown in more detail in Section 3, it is possible to obtain the magnetic fields for a given rotation of the inclusion, $\bar{\boldsymbol{\omega}}^{(2)}$, independent of the deformation field in the matrix. Having the solution for the magnetic fields, the magnetic torque can be obtained as a function of $\bar{\boldsymbol{\omega}}^{(2)}$ in terms of the Maxwell stress tensor (5), evaluated immediately outside the inclusion, by means of the integral appearing in $(10)_1$ with $\boldsymbol{T}$ replaced by the Maxwell stress $\boldsymbol{T}^{M}$.

Similarly, as will be shown in Section 4, since the matrix is not responsive to the magnetic fields, for the given rotation of the inclusion, $\bar{\boldsymbol{\omega}}^{(2)}$, the elasticity problem can be solved independent of the magnetic fields. Moreover, the elastic torque applied on the inclusion can be obtained in terms of the same integral appearing in $(10)_1$ with $\boldsymbol{T}$ replaced by the elastic stress $\boldsymbol{T}^{el} = \mathbb{C}^{(1)}\boldsymbol{\epsilon}(\mathbf{x})$ (for $\mathbf{x} \in \partial\Omega$).

In the following two sections, the magnetic and elastic problems described above will be solved independently in order to determine the elastic and magnetic torques acting on the inclusion. Then, the equilibrium rotation of the inclusion under combined mechanical and magnetic loadings will be determined by means of (12).

3. Rigid magnetizable inclusion in an external magnetic field

In this section, we address the solution of the magnetostatic problem consisting of a magnetic inclusion (denoted by Ω in the deformed configuration) placed in an external magnetic field, $\bar{\mathbf{h}}$, which is the field in the absence of the inclusion, or simply the magnetic field at infinity. Recalling Eqs. (1) and (2), the magnetostatic problem reduces to the following boundary value problem

$$\begin{aligned}&\mu_0\nabla^2\phi+\mu_0\nabla\cdot\mathbf{m}=0,\quad \mathbf{x}\in\mathbb{R}^3\\&\phi\to\bar{\mathbf{h}}\cdot\mathbf{x},\quad |\mathbf{x}|\to\infty\end{aligned}\tag{13}$$

where ϕ is the magnetic potential, such that $\mathbf{h}=\nabla\phi$, and where $\mathbf{m}$ is the magnetization, given by

$$\mu_0\mathbf{m}=\big(\boldsymbol{\mu}^{(2)}-\mu_0\boldsymbol{I}\big)\nabla\phi,\quad \text{when } \mathbf{x}\in\Omega\tag{14}$$

and zero otherwise.

There are well-known solutions for the case of isotropic permeability tensors and simple geometries, such as spheres (Reitz and Milford [19]) and ellipsoids (Landau et al. [7]). For the general case of anisotropic magnetic properties, it is possible to rewrite the governing equations of magnetostatics (i.e. the Maxwell's equations) in an integral equation form analogous to the Eshelby problem in elasticity (see Willis [14]) in order to find the magnetic fields. The main advantage of the integral equation approach is the compact form of the solution, inside and outside of the inclusion, for the most general case of anisotropic magnetic behaviors and ellipsoidal shapes of the inclusion. Therefore, by application of the procedure of Willis [14] to deal with anisotropic magnetic behavior for the inclusions, it can be shown that for ellipsoidal shapes of the inclusion, the magnetization, magnetic field and magnetic induction are constant inside the inclusion and therefore equal to their averages, i.e.

$$\bar{\mathbf{m}}^{(2)}=\boldsymbol{\alpha}^{(2)}\bar{\mathbf{h}},\qquad \bar{\mathbf{h}}^{(2)}=\boldsymbol{\beta}^{(2)}\bar{\mathbf{h}},\quad\text{and}\quad \bar{\mathbf{b}}^{(2)}=\boldsymbol{\mu}^{(2)}\boldsymbol{\beta}^{(2)}\bar{\mathbf{h}}\tag{15}$$

where $\boldsymbol{\mu}^{(2)}$ is the permeability of the inclusion, and $\boldsymbol{\alpha}^{(2)}$ and $\boldsymbol{\beta}^{(2)}$ are symmetric, second-order tensors denoting the magnetization and magnetic field concentration tensors, respectively, and are given by

$$\boldsymbol{\alpha}^{(2)}=\mu_0^{-1}\big\{\big(\boldsymbol{\mu}^{(2)}-\mu_0\boldsymbol{I}\big)^{-1}+\boldsymbol{P}\big\}^{-1},\quad\text{and}\quad \boldsymbol{\beta}^{(2)}=\big\{\boldsymbol{I}+\boldsymbol{P}\big(\boldsymbol{\mu}^{(2)}-\mu_0\boldsymbol{I}\big)\big\}^{-1}\tag{16}$$

Here the second-order, symmetric tensor $\boldsymbol{P}$ describes the magnetic microstructural tensor of the inclusion and is defined by

$$\boldsymbol{P}:=\frac{\det\boldsymbol{Z}}{4\pi\mu_0}\int_{|\boldsymbol{\xi}|=1}\boldsymbol{\xi}\otimes\boldsymbol{\xi}|\boldsymbol{Z}\boldsymbol{\xi}|^{-3}\,\mathrm{d}S\tag{17}$$

Note that it depends only on the magnetic properties of the matrix (i.e. μ_0) and shape of the inclusion. It is also important to note that the above results are given in the current configuration which differs from the reference configuration by a rigid rotation, as already mentioned in the previous section.

The net force and torque on the inclusion due to the magnetic field can be determined via the Maxwell stress tensor as explained in Section 2.2, i.e.

$$\mathbf{f}^{mg}=\frac{1}{V}\int_{\partial\Omega}\boldsymbol{T}^M\mathbf{n}\,\mathrm{d}S,\quad\text{and}\quad \mathbf{l}^{mg}=\frac{1}{V}\int_{\partial\Omega}\mathbf{x}\times\big(\boldsymbol{T}^M\mathbf{n}\big)\,\mathrm{d}S\tag{18}$$

where $\boldsymbol{T}^M$ is given by Eq. (5) with $\mathbf{b}$ denoting the magnetic field just outside the inclusion. For the single inclusion problem, it is given by (see Reitz and Milford [19] for the details)

$$\mathbf{b}(\mathbf{x})=\mu_0\bar{\mathbf{h}}-\frac{\mu_0}{4\pi}\int_{\mathbf{x}'\in\Omega}\left\{\frac{\bar{\mathbf{m}}^{(2)}}{r^3}-\frac{3(\bar{\mathbf{m}}^{(2)}\cdot\mathbf{r})\mathbf{r}}{r^5}\right\}\mathrm{d}\mathbf{x}',\quad \mathbf{r}=\mathbf{x}-\mathbf{x}'\tag{19}$$

After a long, but straightforward calculation (Landau et al. [7]), it can be shown that the force and torque (per unit volume) exerted on the inclusion by the magnetic field are given by

$$\mathbf{f}^{mg}=\mathbf{0},\quad\text{and}\quad \mathbf{l}^{mg}=\mu_0\bar{\mathbf{m}}^{(2)}\times\bar{\mathbf{h}}\tag{20}$$

Note that the magnetic force on the inclusion vanishes as expected. On the other hand, using expression $(15)_1$, and the fact that the torque is an axial vector, we can write the components of the torque in terms of a skew-symmetric, second-order tensor, $\boldsymbol{\tau}^{mg}$, such that the Cartesian components of $\mathbf{l}^{mg}$ read

$$l_i^{mg}=\epsilon_{ijk}\tau_{kj}^{mg},\quad\text{with } \boldsymbol{\tau}^{mg}=-\frac{\mu_0}{2}\big\{\big(\boldsymbol{\alpha}^{(2)}\bar{\mathbf{h}}\big)\otimes\bar{\mathbf{h}}-\bar{\mathbf{h}}\otimes\big(\boldsymbol{\alpha}^{(2)}\bar{\mathbf{h}}\big)\big\}\tag{21}$$

To make the dependence of $\boldsymbol{\tau}^{mg}$ (or equivalently $\mathbf{l}^{mg}$) on the rotation of the inclusion more explicit, we note that the magnetization concentration tensor $\boldsymbol{\alpha}^{(2)}$, which in the current configuration is given by $(16)_1$, can be written in terms of its reference counterpart, $\boldsymbol{A}^{(2)}$, and the rotation, $\bar{\boldsymbol{\omega}}^{(2)}$, as follows

$$\boldsymbol{\alpha}^{(2)} = \boldsymbol{A}^{(2)} + \bar{\boldsymbol{\omega}}^{(2)}\boldsymbol{A}^{(2)} - \boldsymbol{A}^{(2)}\bar{\boldsymbol{\omega}}^{(2)} \tag{22}$$

where

$$\boldsymbol{A}^{(2)} = \mu_0^{-1}\left\{\left(\boldsymbol{M}^{(2)} - \mu_0\boldsymbol{I}\right)^{-1} + \boldsymbol{P}_0\right\}^{-1} \tag{23}$$

and $\boldsymbol{P}_0$ is given by the same expression (17), except that $\boldsymbol{Z}$ should be replaced by $\boldsymbol{Z}_0$, as defined in (9).

Then, substituting (22) into $(21)_2$, and ignoring higher-order terms in $\bar{\boldsymbol{\omega}}^{(2)}$, we obtain the result

$$\boldsymbol{\tau}^{mg}\left(\bar{\boldsymbol{\omega}}^{(2)}\right) = \boldsymbol{\tau}_0^{mg} + \frac{\mu_0}{2}\mathbb{T}\bar{\boldsymbol{\omega}}^{(2)} \tag{24}$$

where $\boldsymbol{\tau}_0^{mg}$ denotes the magnetic torque in the reference (unrotated) configuration of the inclusion and is given by

$$\boldsymbol{\tau}_0^{mg} = -\frac{\mu_0}{2}\left\{\left(\boldsymbol{A}^{(2)}\bar{\mathbf{h}}\right)\otimes\bar{\mathbf{h}} - \bar{\mathbf{h}}\otimes\left(\boldsymbol{A}^{(2)}\bar{\mathbf{h}}\right)\right\} \tag{25}$$

and where the Cartesian components of the fourth-order tensor $\mathbb{T}$ are given by

$$T_{ijst} = \delta_{is}A_{tq}^{(2)}\bar{h}_q\bar{h}_j - \delta_{js}A_{tq}^{(2)}\bar{h}_q\bar{h}_i + \bar{h}_i\bar{h}_t A_{js}^{(2)} - \bar{h}_j\bar{h}_t A_{is}^{(2)} \tag{26}$$

Note that they satisfy the symmetries $T_{ijkl} = -T_{jikl} = -T_{ijlk}$.

4. Rigid ellipsoidal inclusion in an infinite elastic matrix

In the previous section, we found the torque produced by an external magnetic field on the rigid inclusion Ω, as a function of its rotation (i.e., $\mathbf{l}^{mg}(\bar{\boldsymbol{\omega}}^{(2)})$). In this section, we would like to obtain the corresponding torque applied by the surrounding elastic medium on the inclusion as a function of the rotation $\bar{\boldsymbol{\omega}}^{(2)}$. Toward this goal, it is first noted that the Maxwell stress $\boldsymbol{T}^M$, as given by (5), is divergence-free for a non-magnetic matrix. This can be easily shown by means of the Maxwell's equations, given by (1), and the fact that for a non-magnetic material $\mathbf{b} = \mu_0\mathbf{h}$. Therefore, the magnetic fields do not appear explicitly in the equilibrium equation, and we can solve for the displacement field in the matrix independent of the magnetic fields. Thus, for the infinite elastic matrix with stiffness tensor $\mathbb{C}^{(1)}$, subjected to an affine displacement boundary condition at infinity, and an arbitrary rotation $\bar{\boldsymbol{\omega}}^{(2)}$ just outside the inclusion, we have the following "purely mechanical" boundary value problem (BVP) for the displacement field $\mathbf{u}$ in the elastic-matrix material:

$$\begin{aligned}
&\operatorname{div}\left(\mathbb{C}^{(1)}\nabla\mathbf{u}\right) = \mathbf{0}, \quad \mathbf{x}\in\mathbb{R}^3\setminus\Omega_0\\
&\mathbf{u}(\mathbf{x}) \to \bar{\boldsymbol{\epsilon}}\mathbf{x} + \bar{\boldsymbol{\omega}}\mathbf{x}, \quad |\mathbf{x}|\to\infty\\
&\mathbf{u}(\mathbf{x}) = \bar{\boldsymbol{\omega}}^{(2)}\mathbf{x}, \quad \mathbf{x}\in\partial\Omega_0
\end{aligned} \tag{27}$$

where $\bar{\boldsymbol{\epsilon}}$ is a constant symmetric second-order tensor denoting the macroscopic strain and $\bar{\boldsymbol{\omega}}$ is a constant antisymmetric second-order tensor representing the macroscopic rotation of the system.

For the special case of $\bar{\boldsymbol{\epsilon}} = \bar{\boldsymbol{\omega}} = \mathbf{0}$, the BVP (27) has been solved by making use of an orthogonal ellipsoidal coordinate system (Kachanov et al. [20]). On the other hand, for $\bar{\boldsymbol{\omega}} = \mathbf{0}$ and $\bar{\boldsymbol{\omega}}^{(2)}$ not prescribed, it reduces to the well-known Eshelby problem which can be solved by using the so-called simple set of imaginary cutting, straining and welding operations, as described by Eshelby [1]. The above more general problem has been addressed by Walpole [15]. For completeness, the solution for the above BVP is briefly summarized in Appendix A.

Thus, it is shown in this appendix that the "mechanical" traction $\boldsymbol{T}^{el}\mathbf{n}$ on the surface of the inclusion is given by

$$\boldsymbol{T}^{el}\mathbf{n} = (\boldsymbol{\sigma} + \boldsymbol{\tau})\mathbf{n} \tag{28}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are respectively symmetric and skew-symmetric, second-order tensors, given by

$$\begin{aligned}
\boldsymbol{\sigma} &= \mathbb{P}^{-1}\bar{\boldsymbol{\epsilon}} - \mathbb{P}^{-1}\mathbb{Q}\left\{\mathbb{S} - \mathbb{R}\mathbb{P}^{-1}\mathbb{Q}\right\}^{-1}\left(\bar{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}}^{(2)} - \mathbb{R}\mathbb{P}^{-1}\bar{\boldsymbol{\epsilon}}\right)\\
\boldsymbol{\tau} &= \left\{\mathbb{S} - \mathbb{R}\mathbb{P}^{-1}\mathbb{Q}\right\}^{-1}\left(\bar{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}}^{(2)} - \mathbb{R}\mathbb{P}^{-1}\bar{\boldsymbol{\epsilon}}\right)
\end{aligned} \tag{29}$$

where $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{S}$ are mechanical microstructural (Eshelby) tensors, defined by relations (48) and (47). Note that the stress is indeterminate inside the rigid inclusion, and hence the need for the traction on the boundary of the inclusion in order to compute the force and torque exerted by the surrounding elastic medium on the inclusion. Then, by means of expressions of the form (18) (with $\boldsymbol{T}^{mg}$ replaced by $\boldsymbol{T}^{el}$), it can be shown that the elastic force is zero (i.e. $\mathbf{f}^{el} = \mathbf{0}$) and that the torque (per unit volume) exerted by the surrounding elastic medium on the inclusion is given by

$$l_i^{el}\left(\bar{\boldsymbol{\omega}}^{(2)}\right) = \epsilon_{ijk}\tau_{kj} \tag{30}$$

where $\boldsymbol{\tau}$ is given in terms of $\bar{\boldsymbol{\omega}}^{(2)}$ by Eq. $(29)_2$.

5. Equilibrium rotation of inclusion in an infinite elastic matrix subjected to an external magnetic field

It is recalled from Section 2.1 that the magnetic and elastic torques must add up to zero at equilibrium. Thus, substituting the magnetic and elastic torque from expressions $(21)_1$ and (30), respectively, into Eq. (12), the following equation for the equilibrium rotation, $\bar{\boldsymbol{\omega}}_e^{(2)}$ is obtained

$$\boldsymbol{\tau}^{mg}\big(\bar{\boldsymbol{\omega}}^{(2)}\big)+\boldsymbol{\tau}\big(\bar{\boldsymbol{\omega}}^{(2)}\big)=\mathbf{0}, \quad \text{for } \bar{\boldsymbol{\omega}}^{(2)}=\bar{\boldsymbol{\omega}}_e^{(2)} \tag{31}$$

where $\boldsymbol{\tau}^{mg}$ and $\boldsymbol{\tau}$ are given in terms of the particle rotation $\bar{\boldsymbol{\omega}}^{(2)}$ by expressions (24) and $(29)_2$, respectively. Therefore, solving for the equilibrium rotation we obtain the result

$$\bar{\boldsymbol{\omega}}_e^{(2)}=\left(\mathbb{I}-\frac{\mu_0}{2}\mathbb{V}\mathbb{T}\right)^{-1}\big(\bar{\boldsymbol{\omega}}-\mathbb{R}\mathbb{P}^{-1}\bar{\boldsymbol{\epsilon}}+\mathbb{V}\boldsymbol{\tau}_0^{mg}\big) \tag{32}$$

where $\mathbb{V}:=\{\mathbb{S}-\mathbb{R}\mathbb{P}^{-1}\mathbb{Q}\}$ and it is recalled that $\boldsymbol{\tau}_0^{mg}$ denotes the magnetic torque exerted on the inclusion in its reference configuration. It is important to note that $\bar{\boldsymbol{\omega}}_e^{(2)}$ depends linearly on $\bar{\boldsymbol{\epsilon}}$ and $\bar{\boldsymbol{\omega}}$ and quadratically on the external magnetic field $\bar{\mathbf{h}}$.

Introducing dimensionless parameter

$$\kappa=\frac{\mu_0\bar{h}^2}{2\mu_{el}} \tag{33}$$

serving to characterize the relative strength of the magnetic versus the elastic effects, and the dimensionless variables

$$\hat{\mathbb{T}}=\mathbb{T}/\bar{h}^2, \qquad \hat{\mathbb{V}}=\mu_{el}\mathbb{V}, \quad \text{and} \quad \hat{\boldsymbol{\tau}}_0^{mg}=2\boldsymbol{\tau}_0^{mg}/\big(\mu_0\bar{h}^2\big) \tag{34}$$

the following asymptotic result is obtained for the equilibrium rotation

$$\bar{\boldsymbol{\omega}}_e^{(2)}=\bar{\boldsymbol{\omega}}-\mathbb{R}\mathbb{P}^{-1}\bar{\boldsymbol{\epsilon}}+\kappa\hat{\mathbb{V}}\big[\hat{\boldsymbol{\tau}}_0^{mg}+\hat{\mathbb{T}}\big(\bar{\boldsymbol{\omega}}-\mathbb{R}\mathbb{P}^{-1}\bar{\boldsymbol{\epsilon}}\big)\big]+\mathcal{O}\big(\kappa^2\big) \tag{35}$$

which is valid for $\kappa \ll 1$. In this connection, it should be noted that $\kappa \ll 1$ is the typical situation in applications. Indeed, in most of the recent experimental studies on two-phase, magnetorheological composites, the choices of the materials and the actuating fields are such that the largest value of κ is in the order of 0.1 (see Ginder et al. [8,9]; Guan et al. [10], where the elastic matrix is made of silicon or natural rubber with a shear modulus of 1.0–1.2 MPa, and the magnetic induction fields are of magnitude 0–1.2 T). In any case, it is useful to distinguish three different limiting cases in reference to the above expressions for the particle rotation:

Case 1. κ is small compared to $|\bar{\boldsymbol{\epsilon}}|$ and $|\bar{\boldsymbol{\omega}}|$. In this limit, terms of order one or higher in the parameter κ can be neglected, compared to the zeroth-order term, and the equilibrium rotation of the inclusion is given by the (purely mechanical) Eshelby result

$$\bar{\boldsymbol{\omega}}_e^{(2)}=\bar{\boldsymbol{\omega}}-\mathbb{R}\mathbb{P}^{-1}\bar{\boldsymbol{\epsilon}} \tag{36}$$

Clearly, this case corresponds to situations where the matrix is so stiff compared to the strength of the magnetic field that the rotation of the inclusion is independent of the magnetic field.

Case 2. κ is of the same order as $|\bar{\boldsymbol{\epsilon}}|$ and $|\bar{\boldsymbol{\omega}}|$. In this case the second- and higher-order terms in Eq. (35) can be neglected, and the equilibrium rotation of the inclusion is given by

$$\bar{\boldsymbol{\omega}}_e^{(2)}=\bar{\boldsymbol{\omega}}-\mathbb{R}\mathbb{P}^{-1}\bar{\boldsymbol{\epsilon}}+\kappa\hat{\mathbb{V}}\hat{\boldsymbol{\tau}}_0^{mg} \tag{37}$$

It is important to emphasize that the small-strain and -rotation assumptions that are implicit in this linearized deformation approximation essentially require that κ be at most of the same order as $|\bar{\boldsymbol{\epsilon}}|$ and $|\bar{\boldsymbol{\omega}}|$, for otherwise the resulting particle rotation and associated strain distributions would violate the small-deformation approximation. In addition, it is evident from expression (37), which depends on the magnetic torque $\hat{\boldsymbol{\tau}}_0^{mg}$ in the reference configuration that in this limit the differences between the reference and deformed configuration become irrelevant, as far as the computation of the equilibrium rotation $\bar{\boldsymbol{\omega}}_e^{(2)}$ is concerned.

Case 3. *Nearly aligned magnetic fields*. For the special cases where the magnetic field is almost aligned with the inclusion (or its magnetic anisotropy axes), the equilibrium rotation of the inclusion would be infinitesimally small, independent of the parameter κ. Therefore, for these special cases the general expression (32) should be used to obtain the equilibrium rotation of the inclusion.

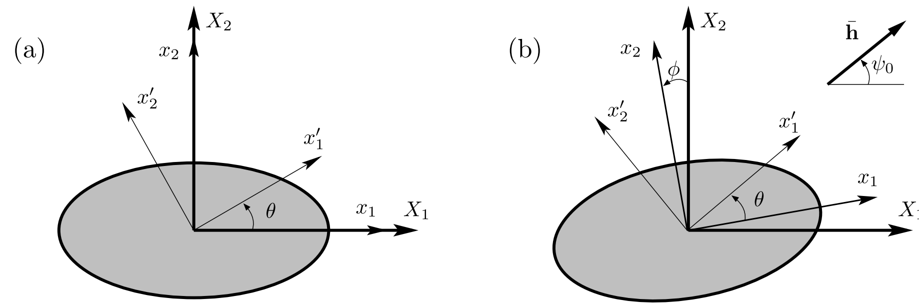

Fig. 2. Schematic representation of the in-plane rotation of the particle under magnetic loading. (a) The inclusion in the reference configuration, and (b) the inclusion in the current configuration (after application of the external magnetic field $\bar{\mathbf{h}}$). Note that here the coordinate systems $\{x_1, x_2, x_3\}$ and $\{x_1', x_2', x_3'\}$ defining the geometric and magnetic axes of the inclusion are fixed on the rigid inclusion and are such that $x_3' = x_3$.

6. Applications for in-plane particle rotations

In this section, the general results of the previous section are specialized for magnetic loading applied on a plane of symmetry of the inclusion (with $\bar{\boldsymbol{\epsilon}} = \bar{\boldsymbol{\omega}} = 0$), so that the rigid inclusion undergoes an in-plane rotation. For simplicity, the matrix is assumed to be isotropic with stiffness tensor $\mathbb{C}^{(1)}$, depending on the elastic moduli, μ_{el} and ν, and the rigid, ellipsoidal inclusion to be magnetically anisotropic with magnetic permeability $\boldsymbol{\mu}$, such that the principal permeabilities are μ_1', μ_2' and μ_3'. As depicted in Fig. 2, the geometric axes of the inclusion, defined by $\{x_1, x_2, x_3\}$, are initially aligned with the laboratory axes, characterized by $\{X_1, X_2, X_3\}$, but rotate about the X_3-axis by an angle ϕ under the magnetic loading $\bar{\mathbf{h}}$, which in turn is applied on the X_1–X_2 plane at an angle ψ_0 relative to the X_1-axis. Note that in order for the rotation of the particle to remain in the plane (so that $x_3 = X_3$), the magnetic axes of the inclusion, defined by $\{x_1', x_2', x_3'\}$, must be oriented such that $x_3' = x_3 = X_3$. For some generality, however, we assume that the magnetic axes may differ from the geometric axes by an angle θ (about the x_3-axis). Finally, we label the semi-axes of the ellipsoidal inclusion in the 1, 2 and 3 directions, a, b and c, respectively.

Under the above-mentioned hypotheses with the external magnetic fields $\bar{\mathbf{h}}$ being applied on the X_1–X_2 plane at an angle ψ_0 relative to the X_1-axis, it can be shown that the only non-zero components of the antisymmetric magnetic moment tensor, $\hat{\boldsymbol{\tau}}_0^{mg}$, in Eq. (37) are

$$\left(\hat{\tau}_0^{mg}\right)_{21} = -\left(\hat{\tau}_0^{mg}\right)_{12} = \frac{1}{2}\left(A_{11}^{(2)} - A_{22}^{(2)}\right)\sin(2\psi_0) - A_{12}^{(2)}\cos(2\psi_0) \tag{38}$$

where $A_{ij}^{(2)}$ are the components of the magnetization concentration tensor defined by expression (23). On the other hand, it follows from expression (37) for $\bar{\boldsymbol{\omega}}_e^{(2)}$ that the in-plane particle rotation $\phi = (\bar{\omega}_e^{(2)})_{21}$ is given by $\phi = 2\kappa\hat{V}_{2121}(\hat{\tau}_0^{mg})_{21}$, where $\hat{V}_{2121} = \hat{S}_{2121} - \hat{Q}_{2121}^2/\hat{P}_{2121}$ is the relevant component of the fourth-order tensor $\hat{\mathbb{V}} = \mu_{el}\{\mathbb{S} - \mathbb{R}\mathbb{P}^{-1}\mathbb{Q}\}$, defined earlier. Then, substituting expression (38) for $(\hat{\tau}_0^{mg})_{21}$ into this expression for ϕ, it can be readily shown that

$$\phi = \frac{\kappa\alpha}{\cos\xi}\sin(2\psi_0 - \xi) \tag{39}$$

where

$$\alpha = \left(A_{11}^{(2)} - A_{22}^{(2)}\right)\hat{V}_{2121}, \quad \text{and} \quad \tan\xi := \frac{A_{12}^{(2)}}{A_{11}^{(2)} - A_{22}^{(2)}} \tag{40}$$

Note that the equilibrium particle rotation ϕ is periodic on the magnetic field angle ψ_0, with period π. It is also linear on the dimensionless parameter κ, and therefore quadratic in the magnetic field $\bar{h}$ and inversely proportional to the shear modulus μ_{el}. In addition, it depends on both the "mechanical" microstructural tensors, $\mathbb{P}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{S}$ (through $\hat{\mathbb{V}}$), as well as on the corresponding "magnetic" microstructural tensor $\boldsymbol{P}_0$ (through $\boldsymbol{A}$), and therefore also depends on the Poisson's ratio ν, the magnetic permeabilities μ_1', μ_2' and μ_3', the particle shape, as determined by a, b and c, and the orientation of the magnetic axis relative to the geometric axes, as specified by θ. In an effort to elucidate the dependence of the inclusion rotation on all these variables and parameters, results will be provided next for cylindrical inclusions of elliptical cross-section aligned with the x_3-axis, and for spheroidal inclusions with their axis of symmetry lying in the rotation plane.

6.1. Cylindrical (2D) inclusions

We begin by considering cylindrical inclusions with elliptical cross-section of aspect ratio $w = a/b$, and with in-plane (principal) magnetic permeabilities, μ_1', μ_2', and generally non-aligned magnetic axes, as determined by θ. As shown in

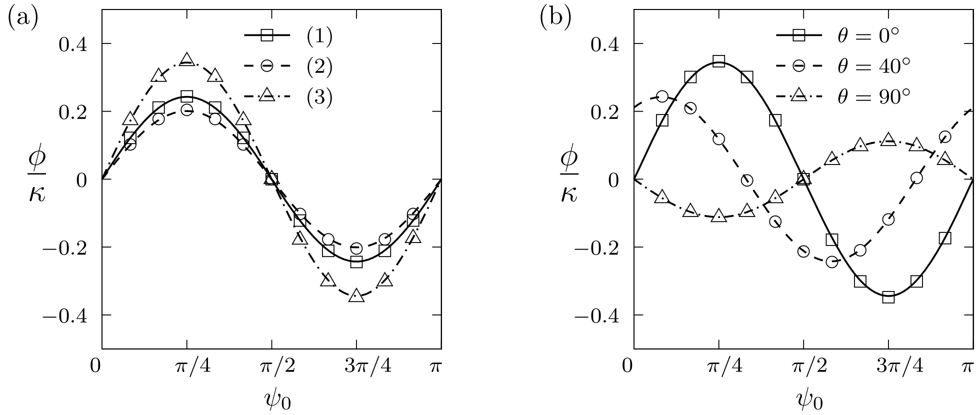

Fig. 3. Analytical predictions (lines) and FE results (symbols) for the equilibrium rotation ϕ as a function of the magnetic loading angle ψ_0. (a) Cylindrical inclusion with aligned magnetic and geometric axes ($\theta = 0°$) for three different cases: (1) circular cross-section and principal permeabilities $\mu_1'/\mu_0 = 10$ and $\mu_2'/\mu_0 = 2$, (2) elliptic cross-section with aspect ratio $w = a/b = 2$ and isotropic permeability $\mu_1'/\mu_0 = \mu_2'/\mu_0 = 10$, and (3) elliptic cross-section with aspect ratio $w = 2$ and principal permeabilities $\mu_1'/\mu_0 = 10$ and $\mu_2'/\mu_0 = 2$. (b) Cylindrical inclusion with aspect ratio $w = 2$ and principal permeabilities $\mu_1'/\mu_0 = 10$ and $\mu_2'/\mu_0 = 2$, for $\theta = 0°$, $\theta = 40°$ and $\theta = 90°$.

Fig. 3, the theoretical results are also compared to numerical results obtained using the general purpose FEM software COMSOL. These numerical simulations have been carried out for a square cell of finite length L. The length L is chosen in the following manner. We start with length L_0 ($\sim 4 \times a$ for the above specific example). Then we double the length until the condition $\frac{\phi_{n+1}-\phi_n}{\phi_n} < 0.0001$ is satisfied, where ϕ_n and ϕ_{n+1} are respectively the equilibrium rotations when $L = L_n$ and $L = L_{n+1} = 2L_n$.

In Fig. 3(a), results are shown for $\theta = 0°$ corresponding to aligned geometric and magnetic axes, such that the largest axis of the elliptical inclusion (if it is not circular) is always aligned with the largest principal permeability (if the magnetic properties of the inclusion are not isotropic). Results are presented for three cases: (1) circular cross-section ($w = 1$) and principal permeabilities $\mu_1'/\mu_0 = 10$ and $\mu_2'/\mu_0 = 2$, (2) elliptic cross-section with aspect ratio $w = a/b = 2$ and isotropic permeability $\mu_1'/\mu_0 = \mu_2'/\mu_0 = 10$, and (3) elliptic cross-section with aspect ratio $w = 2$ and principal permeabilities $\mu_1'/\mu_0 = 10$ and $\mu_2'/\mu_0 = 2$. The main observation from the results presented in this figure is that the particle always tends to align the magnetic axis with the largest permeability (if not magnetically isotropic), or its largest geometric axis (if it is isotropic) with the applied magnetic field (i.e. $\phi > 0$ for $0 < \psi_0 < \pi/2$). Note also that the particle rotation ϕ is periodic in ψ_0, with period π, as already mentioned in connection with Eq. (39), and that $\phi = 0$ when the magnetic field is aligned with the particle axes ($\psi_0 = 0$, $\pi/2$ or π). The second important observation is that the particle shape and magnetic anisotropy have a synergistic effect for the case when the largest geometric axis is aligned with the largest principal permeability. Thus, the amplitude of the particle rotation is largest for case (3) above.

In Fig. 3(b), results are shown for elliptical particles with fixed aspect ratio $w = 2$ and magnetic permeabilities $\mu_1'/\mu_0 = 10$ and $\mu_2'/\mu_0 = 2$, for three values of θ between 0 and $\pi/2$. Note that changing θ changes the components $A_{ij}^{(2)}$, and therefore, α and ξ in Eq. (39). For $\theta = 0°$, the magnetization concentration tensor $\boldsymbol{A}^{(2)}$ is diagonal (i.e. $A_{12}^{(2)} = 0$), which implies that $\xi = 0$, consistent with what is shown in the figure. For $0° < \theta < 90°$, we have $A_{12}^{(2)} \neq 0$ which results in a non-zero ξ, and therefore, a phase shift in the plot for the equilibrium rotation of the inclusion. Finally, for $\theta = 90°$, the magnetic concentration tensor is diagonal which again implies that $\xi = 0$; however, $\alpha < 0$ for the specific choice of the parameters in Fig. 3. Consequently, the inclusion no longer tends to align its largest geometric axis with the applied field, but instead prefers, for this specific choice of parameters, to align its largest magnetic axis with the applied field, resulting in $\phi < 0$ for $0 < \psi_0 < \pi/2$. In this case the magnetic and geometric effects oppose each other and the amplitude of the resulting particle rotation is also the smallest. In this connection, it should be noted that when $\theta = 90°$ it is possible to choose $w \neq 1$, μ_1' and $\mu_2' \neq \mu_1'$ such that the equilibrium rotation of the inclusion is zero for all values of ψ_0.

For a more detailed analysis of the effect of inclusion shape or magnetic behavior, we focus our attention on $\theta = 0°$ since in this case $\xi = 0$ and the equilibrium rotation simplifies to

$$\phi = \alpha\kappa \sin(2\psi_0) \tag{41}$$

Then, the effect of the shape and magnetic behavior on the rotation of the cylindrical inclusion with aspect ratio $w = a/b$, and magnetic permeabilities, μ_1', μ_2', is completely described by α. We consider the following two cases:

Case 1. *Cylindrical inclusion with circular cross-section* ($w = 1$) *and anisotropic permeability* ($\mu_1' \neq \mu_2'$). Then, the amplitude coefficient α, as given by expression $(40)_1$, simplifies to

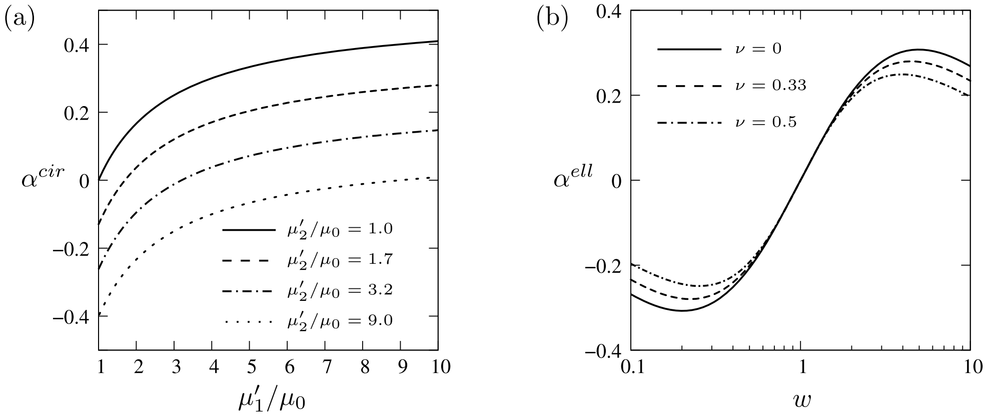

Fig. 4. The effects of magnetic anisotropy and inclusion shape on the magnitude of the particle rotation in aligned external magnetic fields. (a) The coefficient α^{cir} for different values of the parameter μ_2'/μ_0 as a function of μ_1'/μ_0, and (b) the coefficient α^{ell} for fixed permeability $\mu/\mu_0 = 10$, and different values of the Poisson's ratio of the matrix.

$$\alpha^{cir}\left(\mu_1'/\mu_0, \mu_2'/\mu_0\right) = \frac{1}{2}\left(\frac{\mu_1'/\mu_0 - 1}{\mu_1'/\mu_0 + 1} - \frac{\mu_2'/\mu_0 - 1}{\mu_2'/\mu_0 + 1}\right)$$

Note that for this special case there is no dependence on the Poisson's ratio of the matrix ν. Plots of α^{cir} as a function of μ_1'/μ_0, for fixed values of μ_2'/μ_0, are shown in Fig. 4(a). It can be seen that α^{cir} increases monotonically with increasing the magnetic anisotropy, μ_1'/μ_0, and saturates for large values of μ_1'/μ_0. In addition, note that the graphs corresponding to the larger values of μ_2'/μ_0, can be obtained by translations in the negative vertical direction.

Case 2. *Cylindrical inclusion with aspect ratio $w = a/b$ and isotropic in-plane magnetic behavior* ($\mu_1' = \mu_2' = \mu$). The coefficient α in this case reduces to

$$\alpha^{ell}(\mu/\mu_0, w, \nu) = \frac{w - w/4(1-\nu)}{(w+1)^2 - w/(1-\nu)} \frac{(w^2-1)(\mu/\mu_0 - 1)^2}{(w + \mu/\mu_0)(1 + \mu/\mu_0 w)}$$

which does depend on the Poisson's ratio ν. Plots of α^{ell} as a function of w, for $\mu/\mu_0 = 10$, are shown in Fig. 4(b). In this figure, it can be seen that α^{ell} increases up to a maximum, and then decays to zero as the aspect ratio is further increased (for $w > 1$). This is due to the fixed displacement boundary condition at infinity, and the fact that as the aspect ratio tends to infinity the rigid inclusion approaches the boundary. Note that $\alpha^{ell}(w) = -\alpha^{ell}(1/w)$, due to the symmetry of the problem.

6.2. Spheroidal (3D) inclusions

In this subsection, we consider spheroidal ($c = b$) inclusions of aspect ratio $w = a/b$ with aligned magnetic and geometric axes ($\theta = 0°$) in the plane of the rotation. In this case, the equilibrium rotation also satisfies the relation (41) with α serving to describe the amplitude of the particle rotation as a function of the various geometric and material parameters. For simplicity here, we consider the following two cases:

Case 1. *Spherical inclusion* ($w = 1$) *with anisotropic permeability* ($\mu_1' \neq \mu_2'$). In this case, the coefficient α reduces to

$$\alpha^{sph}\left(\mu_1'/\mu_0, \mu_2'/\mu_0\right) = \frac{1}{2}\left(\frac{\mu_1'/\mu_0 - 1}{\mu_1'/\mu_0 + 2} - \frac{\mu_2'/\mu_0 - 1}{\mu_2'/\mu_0 + 2}\right)$$

which is independent of the Poisson's ratio of the matrix ν. As it is shown in Fig. 5(a), similar to the cylindrical inclusion with circular cross-section case, α^{sph} increases monotonically with increasing the magnetic anisotropy, μ_1'/μ_0, and saturates for large values of μ_1'/μ_0.

Case 2. *Prolate spheroidal inclusion with aspect ratio $w = a/b > 1$ and isotropic in-plane magnetic behavior* ($\mu_1' = \mu_2' = \mu$). The coefficient α for this case can be written as

$$\alpha^{srd}(\mu/\mu_0, w, \nu) = \left(\hat{S}_{2121} - \frac{\hat{Q}_{2121}^2}{\hat{P}_{2121}}\right) \frac{(\mu/\mu_0 - 1)^2(\hat{P}_{22} - \hat{P}_{11})}{[1 + \hat{P}_{11}(\mu/\mu_0 - 1)][1 + \hat{P}_{22}(\mu/\mu_0 - 1)]}$$

where

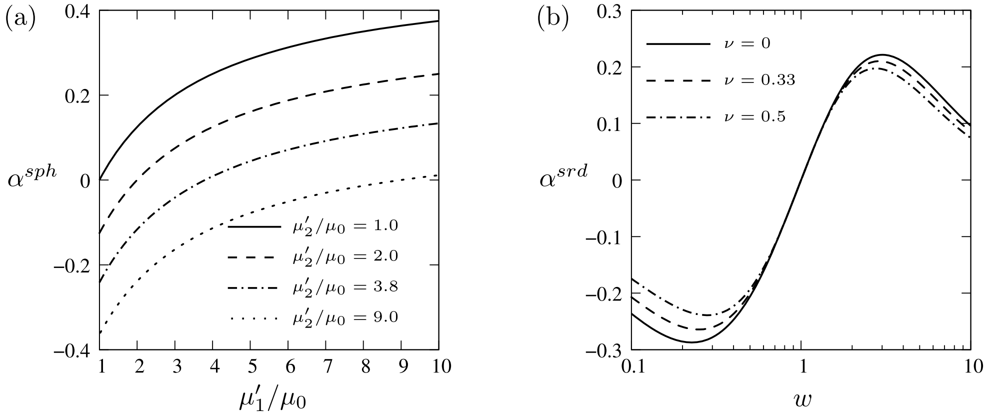

Fig. 5. The effect of anisotropy, due to the magnetic behavior and shape of the inclusion, on the response of the single inclusion to the external magnetic field. (a) The coefficient α^{sph} as a function of μ'_1/μ_0 for different values of the parameter μ'_2/μ_0, and (b) the coefficient α^{srd} for fixed permeability $\mu/\mu_0 = 10$, and different values of the Poisson's ratio of the matrix.

$$\hat{P}_{11} = \frac{w}{k^2}\left\{\frac{\sinh^{-1} k}{k} - \frac{1}{w}\right\}, \qquad \hat{P}_{22} = \frac{w}{2k^2}\left\{w - \frac{\sinh^{-1} k}{k}\right\}$$
$$\hat{P}_{2121} = \frac{w}{8k^4}\left\{\frac{(1-\nu)k^4 + (2+\nu)k^2 + 3}{(1-\nu)w} - \frac{1+\nu}{1-\nu} k \sinh^{-1} k - \frac{3}{1-\nu}\frac{\sinh^{-1} k}{k}\right\}$$
$$\hat{Q}_{2121} = \frac{w}{8k^2}\left\{\frac{3\sinh^{-1} k}{k} - \frac{k^2+3}{w}\right\}, \qquad \hat{S}_{2121} = \frac{w}{8k^2}\left\{\frac{\sinh^{-1} k}{k} + \frac{k^2-1}{w}\right\}$$

In these expressions, $k = \sqrt{w^2 - 1}$, where $w = a/b$ is the aspect ratio of the spheroid. As shown in Fig. 5(b) (for $w > 1$), and similar to the elliptical particle, α^{srd} increases up to a maximum and then decays to zero with increasing the aspect ratio. However, for the spheroidal particles, the results for oblate ($w < 1$) spheroidal inclusions and the corresponding ones for prolate inclusions ($w > 1$) do *not* satisfy the symmetry relation $\alpha^{srd}(w) = -\alpha^{srd}(1/w)$ for $w < 1$, because oblate and prolate spheroids are geometrically different. Nonetheless, the general shape of the curve for oblate inclusions is similar to that of prolate inclusions, except for the sign of the rotation, of course.

7. Concluding remarks

In this paper, we have addressed the magneto-elasticity problem of a single rigid inclusion with linear-magnetic behavior embedded in a non-magnetic, and linear-elastic matrix. The main result is given by Eq. (32), which shows that the magnetic torque induced by an externally applied magnetic field will affect the overall rotation of the ellipsoidal inclusion embedded in a matrix that is being subjected to a remotely applied deformation, as long as the inclusion is not both spherical and magnetically isotropic, and provided that the magnetic field is not aligned with one of its geometric, or magnetic axes.

To shed some light on the range of validity of the resulting expression for the equilibrium rotation of the inclusion, an asymptotic expansion in terms of the dimensionless parameter $\kappa = \mu_0 \bar{h}^2/2\mu_{el}$, serving to describe the relative strengths of the magnetic and elastic fields, was obtained (see Eq. (35) in Section 5). It is shown that in order to be consistent with the assumption of infinitesimal deformations the parameter κ can be at most of the order of the displacement gradient (i.e. $\kappa \sim |\nabla \mathbf{u}|$). Because of this, it is possible to neglect second-order terms in κ in the expression for the equilibrium rotation of the inclusion.

The effect of magnetic anisotropy and inclusion shape on the particle rotation is investigated in some detail for both cylindrical (2D) and spheroidal (3D) inclusions, subjected to an in-plane magnetic field. The dependence of the particle rotation on the orientation of the applied magnetic field is found to be sinusoidal with period π, and it vanishes when the field is aligned with the symmetry axes of the particle, when both the magnetic and geometric axes coincide, and for some other intermediate orientations, when they are not. In addition, it is shown that the amplitude of the magnetically induced rotation monotonically increases and asymptotes to a constant value with increasing magnetic anisotropy of the inclusion. On the other hand, for magnetically isotropic inclusions with non-spherical shape, the equilibrium rotation increases up to a maximum, and then decays to zero, as the aspect ratio is increased.

The results of this article can be used to determine the effective behavior of magnetorheological composites consisting of a dilute concentration of rigid, magnetic inclusions distributed randomly in a non-magnetic elastomeric matrix in the small-deformation limit. The details of such a study are beyond the scope of this paper and are left for a future publication. Furthermore, the results of this paper provide some justification for the "stiff matrix" approximation in the work of Ponte Castañeda and Galipeau [18], which consists in the approximation that the particles are convected by the purely

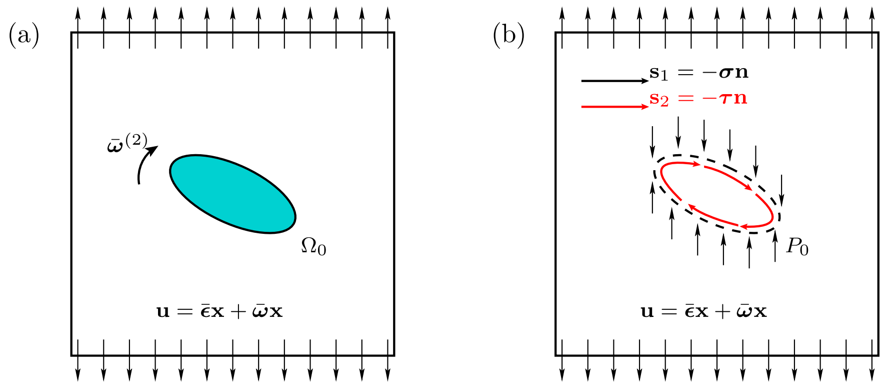

Fig. 6. Schematic of the problem given by the BVP (27). (a) The original problem, and (b) the auxiliary problem.

mechanical deformation in the limit when $\kappa \ll 1$. Indeed, when a magnetorheological composite is subjected to finite strains, the additional rotations of the particles due to the applied magnetic fields would still be expected to be of order κ, and therefore small, compared to the mechanically driven particle rotations, which can be large compared to κ.

It should also be noted that the results of this paper can be easily "translated" into corresponding results for the analogous problem of a stiff dielectric inclusion embedded in a soft dielectric matrix with a different dielectric coefficient which is assumed to be isotropic and deformation independent. Thus, identifying $\mathbf{h}$ with the electric field, $\mathbf{b}$ with the electric displacement, and μ_0 and $\boldsymbol{\mu}^{(2)}$ with the dielectric coefficients of the matrix and inclusion, respectively, expression (25)—with an appropriate reinterpretation of $\boldsymbol{A}^{(2)}$—will correspond to the electric torque of the particle, while expression (35) will provide the rotation of the particle under the combined action of electric and mechanical loadings.

Finally, it is remarked that the results of this work for the particle rotation in a linear-elastic matrix under the action of a magnetic field could be generalized in at least two ways. First, the constitutive behavior of the particles could be taken to be non-linear, corresponding to ferromagnetic behavior, and second, the constitutive response of the elastomeric matrix material could be taken to be neo-Hookean, or some other suitably chosen hyperelasticity model to account for finite strains and rotations. The first would require generalization of the magnetostatic problem described in Section 3 of this paper to include nonlinear magnetization, which can be accomplished by application of the 'linear comparison' methods developed by Ponte Castañeda [21] in the analogous context of nonlinear dielectric behavior (see also Ponte Castañeda et al. [22]). The second is a little more challenging and would require generalization of the elastic problem described in Section 4 for the torque required to produce a finite rotation of the particle in the hyperelastic matrix. This could also be accomplished, at least in principle, by suitable application of the 'second-order' homogenization methods of Ponte Castañeda and Tiberio [23] and Lopez-Pamies and Ponte Castañeda [24] for hyperelastic composites. These possible generalizations are under investigation and will be dealt with elsewhere.

Acknowledgements

This material is based upon work supported by the National Science Foundation under grants DMS-0708271 and CMMI-1068769. We thank Evan Galipeau for valuable discussions.

Appendix A. Solution of the BVP (27)

To find the solution of the BVP (27), we consider an auxiliary problem in which the whole space, $\mathbb{R}^3$, is filled with the elastic-matrix material (see Fig. 6). In this case, a part of the homogeneous body, P_0, which is geometrically the same as the inclusion but made up of the elastic-matrix phase, undergoes a constant strain and a constant rotation due to the boundary condition $(27)_2$. To account for the fact that the strain in the actual rigid inclusion vanishes, it is necessary to apply a surface traction on the boundary of P_0 as follows

$$\mathbf{s}_1 = -\boldsymbol{\sigma}\left(\mathbf{x}'\right)\mathbf{n}', \quad \mathbf{x}' \in \partial P_0 \tag{42}$$

where $\boldsymbol{\sigma}(\mathbf{x}')$ is a divergence-free *symmetric* second-order tensor defined inside P_0. On the other hand, to satisfy the boundary condition $(27)_3$ and account for the rigid body rotation of the actual inclusion, it is required to apply a body torque distribution inside P_0 (i.e., $\mathbf{l} = \mathbf{l}(\mathbf{x}')$ for $\mathbf{x}' \in P_0$). However, to make the formulation more symmetric, instead of specifying the body torque to enforce the boundary condition $(27)_3$, we define a surface traction $\mathbf{s}_2$ on the boundary of P_0 as follows

$$\mathbf{s}_2 = -\boldsymbol{\tau}\left(\mathbf{x}'\right)\mathbf{n}', \quad \mathbf{x}' \in \partial P_0 \tag{43}$$

where $\boldsymbol{\tau}(\mathbf{x}')$ is a divergence-free, *skew-symmetric*, second-order tensor defined in P_0. Therefore, finding the solution of the BVP (27), shown schematically in Fig. 6(a), is equivalent to finding the solution of the auxiliary problem depicted in Fig. 6(b) with $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ chosen such that

$$\boldsymbol{\epsilon}(\mathbf{x}) = 0, \quad \text{and} \quad \boldsymbol{\omega}(\mathbf{x}) = \bar{\boldsymbol{\omega}}^{(2)}, \quad \mathbf{x} \in P_0 \tag{44}$$

Moreover, since the boundary conditions of the general BVP (27) are satisfied, the displacement field of the auxiliary problem, described above, is guaranteed to be the solution of (27) by the uniqueness theorem of elasticity (Love [25]).

Using the linearity of the problem, the total displacement field for the above auxiliary problem can be divided into three parts. The first part is an affine displacement due to the boundary condition $(27)_3$ at infinity. Then, there is the displacement field due to the surface traction $\mathbf{s}_1$ which can be found by integration (over ∂P_0) from the solution of the concentrated force problem in an infinite elastic matrix (or the Green's tensor function of elasticity). Finally, there is the displacement field due to the traction $\mathbf{s}_2$. Noting that the surface traction $\mathbf{s}_2$ has a similar structure to $\mathbf{s}_1$, its contribution to the total displacement can be found by following the same procedure. It is important to note that, due to its singular character, the Green's function in the above computation has to be treated as a generalized function. However, to avoid the difficulties of dealing with such functions, it is useful to deploy the plane-wave decomposition of the delta function to generate alternative representations (in terms of the acoustic tensor) for the Green's function (see e.g. Willis [14,3]). Details for the calculations involved can be found in the works of Willis [3] and Walpole [15]; the important results are summarized next.

Thus, it can be shown that the constraints (44) for inside the inclusion can be satisfied for constant $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$, i.e.

$$\boldsymbol{\sigma}(\mathbf{x}') = \boldsymbol{\sigma}, \quad \text{and} \quad \boldsymbol{\tau}(\mathbf{x}') = \boldsymbol{\tau}, \quad \mathbf{x}' \in P_0 \tag{45}$$

Therefore, the infinitesimal strain and rotation tensors for $\mathbf{x} \in P_0$ are uniform and equal to

$$\boldsymbol{\epsilon} = \bar{\boldsymbol{\epsilon}} - \mathbb{P}\boldsymbol{\sigma} - \mathbb{Q}\boldsymbol{\tau}, \quad \text{and} \quad \boldsymbol{\omega} = \bar{\boldsymbol{\omega}} - \mathbb{R}\boldsymbol{\sigma} - \mathbb{S}\boldsymbol{\tau} \tag{46}$$

where the fourth-order microstructural tensors $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{S}$ are defined by

$$P_{ijpq} = X_{(ij)(pq)}, \qquad R_{ijpq} = X_{[ij](pq)}, \qquad Q_{ijpq} = X_{(ij)[pq]}, \quad \text{and} \quad S_{ijpq} = X_{[ij][pq]} \tag{47}$$

in terms of

$$X_{ijpq} = \frac{\det \boldsymbol{Z}_0}{4\pi} \int\limits_{|\boldsymbol{\xi}|=1} \xi_q \xi_j K_{ip}^{-1}(\boldsymbol{\xi}) |\boldsymbol{Z}_0 \boldsymbol{\xi}|^{-3} \, \mathrm{d}S \tag{48}$$

and depend only on the shape of the inclusion, through $\boldsymbol{Z}_0$, and on the elastic properties of the matrix, through the acoustic tensor $K_{ip}(\boldsymbol{\xi}) = C^{(1)}_{ijpq} \xi_j \xi_q$. Note that $\mathbb{P}$ and $\mathbb{R}$ are the usual Eshelby tensors characterizing the particle rotations in the small-strain/small-rotation limit, in the absence of external torque. On the other hand, $\mathbb{Q}$ and $\mathbb{S}$, which are such that $Q_{ijkl} = R_{klij}$, serve to account for the additional rotation of the inclusion due to the external torque.

Applying the constraints (44), the following system of equations is obtained

$$\bar{\boldsymbol{\epsilon}} = \mathbb{P}\boldsymbol{\sigma} + \mathbb{Q}\boldsymbol{\tau}, \quad \text{and} \quad \bar{\boldsymbol{\omega}} - \bar{\boldsymbol{\omega}}^{(2)} = \mathbb{R}\boldsymbol{\sigma} + \mathbb{S}\boldsymbol{\tau} \tag{49}$$

which can be inverted for $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ to yield the results in expressions (29). Having obtained expressions for $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$, it is now a simple matter to compute the total traction needed to produce the above-mentioned deformation in the auxiliary problem via $\mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2 = -(\boldsymbol{\sigma} + \boldsymbol{\tau})\mathbf{n}$. This is of course the negative of the "mechanical" traction required to produce the needed rotation $\bar{\boldsymbol{\omega}}^{(2)}$ of the rigid particle in the original problem, as is given by expression (28) in terms of $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$.

References

[1] J.D. Eshelby, The determination of the elastic field of an ellipsoidal inclusion, and related problems, Proc. R. Soc. Lond. A 241 (1957) 367–396, doi:10.1098/rspa.1957.0133.

[2] J.C. Maxwell, A Treatise on Electricity and Magnetism, Clarendon Press, Oxford, 1873, Article 111.

[3] J.R. Willis, Variational and related methods for the overall properties of composites, Adv. Appl. Mech. 21 (1981) 1–78, doi:10.1016/S0065-2156(08)70330-2.

[4] Y. Benveniste, The determination of the elastic and electric fields in a piezoelectric inhomogeneity, J. Appl. Phys. 72 (1992) 1086–1095, doi:10.1063/1.351784.

[5] M.L. Dunn, M. Taya, An analysis of piezoelectric composite materials containing ellipsoidal inhomogeneities, Proc. R. Soc. Lond. A 443 (1993) 265–267, doi:10.1098/rspa.1993.0145.

[6] J.H. Huang, Y.-L. Chiu, H.-K. Liu, Magneto–electro-elastic Eshelby tensors for a piezoelectric–piezomagnetic composite reinforced by ellipsoidal inclusions, J. Appl. Phys. 83 (2009) 5364–5370, doi:10.1063/1.367365.

[7] L.D. Landau, E.M. Lifshitz, L.P. Pitaevskii, Electrodynamics of Continuous Media, 2nd ed., Pergamon Press, Great Britain, 1984.

[8] J.M. Ginder, M.E. Nichols, L.D. Elie, J.L. Tardiff, Magnetorheological elastomers: Properties and applications, in: Proc. SPIE, vol. 3675, 1999, pp. 131–138, doi:10.1117/12.352787.

[9] J.M. Ginder, S.M. Clark, W.F. Schlotter, M.E. Nichols, Magnetostrictive phenomena in magnetorheological elastomers, Int. J. Modern Phys. B 16 (2002) 2412–2418, doi:10.1142/S021797920201244X.

[10] X. Guan, X. Dong, J. Ou, Magnetostrictive effect of magnetorheological elastomer, J. Magn. Magn. Mater. 320 (2008) 158–163, doi:10.1016/j.jmmm.2007.05.043.

[11] L. Borcea, O. Bruno, On the magneto-elastic properties of elastomer–ferromagnet composites, J. Mech. Phys. Solids 49 (2001) 2877–2919, doi:10.1016/S0022-5096(01)00108-9.

[12] H.M. Yin, L.Z. Sun, J.S. Chen, Magneto-elastic modeling of composites containing chain-structured magnetostrictive particles, J. Mech. Phys. Solids 54 (2006) 975–1003, doi:10.1016/j.jmps.2005.11.007.

[13] L.P. Liu, R.D. James, P.H. Leo, Magnetostrictive composites in the dilute limit, J. Mech. Phys. Solids 54 (2006) 951–974, doi:10.1016/j.jmps.2005.11.006.
[14] J.R. Willis, Bounds and self-consistent estimates for the overall properties of anisotropic composites, J. Mech. Phys. Solids 25 (1977) 185–202, doi:10.1016/0022-5096(77)90022-9.
[15] L.J. Walpole, A rotated rigid ellipsoidal inclusion in an infinite elastic medium, Proc. R. Soc. Lond. A 433 (1991) 179–207, doi:10.1098/rspa.1991.0042.
[16] K. Hutter, A.A.F. Van de Ven, A. Ursescu, Electromagnetic field matter interactions in thermodynamic solids and viscous fluids, in: R. Beig, et al. (Eds.), Lect. Notes Phys., vol. 710, Springer, Berlin, 2006.
[17] A. Kovetz, The Principles of Electromagnetic Theory, 1st ed., Cambridge University Press, Cambridge, 1990.
[18] P. Ponte Castañeda, E. Galipeau, Homogenization-based constitutive models for magnetorheological elastomers at finite strains, J. Mech. Phys. Solids 59 (2011) 194–215, doi:10.1016/j.jmps.2010.11.004.
[19] J.R. Reitz, F.J. Milford, Foundations of Electromagnetism Theory, Addison–Wesley Publishing Company, Reading, MA, 1962.
[20] M. Kachanov, E. Karapetian, N. Sevostianov, Elastic space containing a rigid ellipsoidal inclusion subjected to translation and rotation, in: T.-J. Chuang, J.W. Rudnicki (Eds.), Multiscale Deformation and Fracture in Materials and Structures, Kluwer Academic, Netherlands, 2001, pp. 123–143.
[21] P. Ponte Castañeda, Bounds and estimates for the properties of nonlinear heterogeneous systems, Phil. Trans. R. Soc. Lond. A 340 (1992) 531–567, doi:10.1098/rsta.1992.0079.
[22] P. Ponte Castañeda, G. deBotton, G. Li, Effective properties of nonlinear inhomogeneous dielectrics, Phys. Rev. B 64 (1992) 4387–4394, doi:10.1103/PhysRevB.46.4387.
[23] P. Ponte Castañeda, E. Tiberio, A second-order homogenization procedure in finite elasticity and applications to black-filled elastomers, J. Mech. Phys. Solids 48 (2000) 1389–1411, doi:10.1016/S0022-5096(99)00087-3.
[24] O. Lopez-Pamies, P. Ponte Castañeda, On the overall behavior, microstructure evolution, and macroscopic stability in reinforced rubbers at large deformations: I—Theory, J. Mech. Phys. Solids 54 (2006) 807–830, doi:10.1016/j.jmps.2005.10.006.
[25] A.E.H. Love, The Mathematical Theory of Elasticity, 4th ed., Dover Publications, New York, 1944.