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A more applicable notion of effective stability for Hamiltonian systems

Une notion plus applicable de stabilité effective pour les systèmes hamiltoniens

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ABSTRACT

The purpose of this Note is to give a result of effective stability for perturbations of integrable Hamiltonian systems, which we believe is more suitable for applications to concrete Hamiltonian systems.

RÉSUMÉ

Le but de cette Note est de donner un résultat de stabilité effective pour les perturbations de systèmes hamiltoniens intégrables, que nous pensons être plus adapté à des systèmes hamiltoniens concrets.

1. Introduction

1.1. Let $n \geqslant 2$ be an integer, $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ and $B = B_R$ be an open ball in $\mathbb{R}^n$ of radius $R > 0$ with respect to the supremum norm. Small analytic perturbations of an integrable Hamiltonian system can be described by an analytic Hamiltonian function $H : \mathbb{T}^n \times B \to \mathbb{R}$ of the form

$$H(\theta, I) = h(I) + \varepsilon f(\theta, I), \quad (\theta, I) \in \mathbb{T}^n \times B, \quad \varepsilon \geqslant 0 \tag{1}$$

The case $\varepsilon = 0$ corresponds to the integrable approximation, and motions are stable: solutions $(\theta(t), I(t))$ are defined for all times, the action variables $I(t)$ are first integrals and for any initial condition (θ_0, I_0), the motion restricted to the invariant torus $\mathbb{T}^n \times \{I_0\}$ is quasi-periodic. Now for $\varepsilon > 0$, motions can become chaotic but some stability properties persist, at least if ε is sufficiently small and h satisfies a non-degeneracy assumption. First there is a result by Kolmogorov ([1], this is now part of KAM theory) on the existence of a "large" set of invariant tori carrying quasi-periodic, and hence stable, motions. This set is large in the sense of measure, this is a probabilistic result of stability: if one picks an initial condition at random, then with a large probability the corresponding solution is quasi-periodic. However this set has no interior, and from a practical point of view it is very hard, or even hopeless, to determine in advance if an initial condition belongs to it or not. A second result, due to Nekhoroshev [2], is more effective in the sense that it concerns all solutions, but of course conclusions are weaker: for all initial conditions (θ_0, I_0), there exist constants $c_1, c_2 > 0$ and $0 < a, b < 1$ such that the stability estimate $|I(t) - I_0| \leqslant c_1\varepsilon^b$ is valid for $|t| \leqslant \exp(c_3\varepsilon^{-a})$. Hence one can only show that the action variables remain almost constant, during an interval of time which is finite but almost exponentially large with respect to the inverse of the

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size of the perturbation. In this Note we shall be concerned with such stability properties, that we call "effective" since, from a physical point of view, we believe it is more relevant than the "probabilistic" stability properties given by KAM theory.

1.2. Perturbations of integrable Hamiltonian systems do appear naturally in many fields of physics, especially in classical and statistical mechanics. Perhaps one of the most important examples, and the one we are most interested in, is the planetary problem [3]. There the mathematical issue of the effective stability translates into the fundamental problem of determining if the actual orbit of a planet resembles the elliptical orbit given by the Keplerian approximation, and for how long. However, for such concrete problems, theoretical results are extremely hard to apply. In addition to computations which are usually quite involved, one faces at least two difficulties.

The first one is that the non-degeneracy assumption has to be satisfied. In practice, either it is difficult to prove or it is not satisfied (the integrable system is degenerate). However, at least concerning the problem we have in mind, one can use further properties of the perturbation to remove this degeneracy.

The second one, which we believe is much more important, is that one needs to check that the size of the perturbation ε is sufficiently small, that is a condition $\varepsilon \leqslant \varepsilon_0$ has to hold true, where ε_0 is a positive constant depending only on the system. From a mathematical point of view, ε is considered as a free parameter that one has the freedom to choose small enough so that the condition $\varepsilon \leqslant \varepsilon_0$ is satisfied and the theorem can be applied. But in practice ε is fixed, the constant ε_0 is very difficult to estimate and with realistic parameters, the condition $\varepsilon \leqslant \varepsilon_0$ is very far from being satisfied.

1.3. In this Note, we shall state a simple result that hopefully can be used for concrete systems in order to circumvent the smallness of the threshold. The basic idea is to introduce a free parameter $\rho > 0$ to make the result weaker but more flexible. This parameter will be restricted by a condition $\rho \leqslant \rho_0$, but in general ρ_0 will be large so that ρ itself can be taken large. Then, for a fixed ρ, we can find a threshold $\varepsilon_0(\rho)$ such that for any $\varepsilon \leqslant \varepsilon_0(\rho)$, the estimate $|I(t) - I_0| \leqslant \rho$ is valid for $|t| \leqslant T(\rho, \varepsilon)$ with $T(\rho, \varepsilon) > 0$. The result is weaker since variations are only of order ρ, and choosing ρ too large the result might even become irrelevant. However the interest is that $\varepsilon_0(\rho)$ increases when ρ increases, so that taking ρ large the threshold $\varepsilon \leqslant \varepsilon_0(\rho)$ is more likely to be satisfied. Hence our constant ρ is an external parameter that can be used, at the theoretical level, to broaden the range of applicability of the result by making it somehow less perturbative.

2. A model statement

2.1. Let us now introduce the precise setting and state our result. Our system is defined by a Hamiltonian H as in (1). The integrable part h, together with its derivative up to order 3, is uniformly bounded on the domain $D = \mathbb{T}^n \times B$ by some constant $M > 0$. The norms used are the supremum norm for vectors and the induced norms. Moreover, h is quasi-convex, that is there exist positive constants η and m such that for all $I \in B$, $|\nabla h(I)| \geqslant \eta$ and for all $v \in \mathbb{R}^n$, using a dot for the Euclidean scalar product, if $\nabla h(I).v = 0$ then $\nabla^2 h(I)v.v \geqslant m|v|^2$. Finally, the system is analytic, that is for some $r, s > 0$, H has a bounded holomorphic extension to the complex domain $D_{r,s} = \{(\theta, I) \in (\mathbb{C}^n/(2\pi\mathbb{Z})^n) \times \mathbb{C}^n \mid \mathcal{I}(\theta)| < s,\ d(I, B) < r\}$, where $\mathcal{I}(\theta)$ is the imaginary part of θ and the distance d is associated to the supremum norm on $\mathbb{C}^n$. We assume that $|f|_{r,s} \leqslant 1$, where $|.|_{r,s}$ is the supremum norm for functions on the domain $D_{r,s}$. The constants n, R, M, η, m, r, s are fixed, and we take a free parameter $\rho > 0$.

Theorem 2.1. *Under the above assumptions, there exists a positive constant ρ_0 such that for any $\rho \leqslant \rho_0$, there exists a positive constant $\varepsilon_0(\rho)$ such that for any $\varepsilon \leqslant \varepsilon_0(\rho)$, there exists a positive constant $T(\rho, \varepsilon)$ such that for all initial data $(\theta_0, I_0) \in \mathbb{T}^n \times B_{R/2}$, $|I(t) - I_0| \leqslant \rho$ for $|t| \leqslant T(\rho, \varepsilon)$. Moreover, one can choose*

$$\rho_0 = \min\left\{\frac{\eta^2 m}{9M^2}, \frac{R}{2}\right\}$$

$$\varepsilon_0(\rho) = \min\left\{\left(\frac{\eta m \rho}{72M}\right)^{2n} \frac{m^{2n-1} r_0^2}{2^{10}(121 n^3 M^2)^{n-1} \coth^n(s_0/3)}, \left(\frac{\rho}{2}\right)^{2n} \frac{\eta^2 m^3}{2^{14} 9^3 M^2 n^{n-1} \coth^n(s_0/3)}\right\}$$

$$T(\rho, \varepsilon) = \frac{s_0}{3M} \exp\left(\frac{s_0}{198 M n^{3/2} \coth^{\frac{n}{2(n-1)}}(s_0/3)} \left(\frac{\eta m r_0}{2^8 9 M}\right)^{\frac{1}{n-1}} \left(\frac{\rho^2}{\varepsilon}\right)^{\frac{1}{2(n-1)}}\right)$$

We have decided to give explicit constants just to give an idea, needless to say that these are general estimates, they are far from being sharp and they can be greatly improved when focusing on an explicit system (see the comments at the end of this Note).

2.2. Let us now briefly sketch the proof of the above theorem. The strategy is very simple, and it is the same as the one used in [4]. We study motions "close" and "far away" from resonances. Our improvement, which is not accessible by classical methods, is that far way from resonances no analysis is required, just some easy geometric arguments are needed to obtain a result of confinement.

We take a parameter $K \geqslant 1$, which is free for the moment but it will be eventually chosen in terms of ρ, and we define the set of resonances of order K:

$$R_K = \left\{ I \in B_R \mid \exists k \in \mathbb{Z}^n \setminus \{0\}, |k| \leqslant K, k.\nabla h(I) = 0 \right\}$$

where for integer vectors, $|.|$ is also the supremum norm. Theorem 2.1 is easily deduced from the two lemmas below. The first one deals with solutions close to resonances.

Lemma 2.2. *Assume that ε, K and ρ satisfy*

$$\varepsilon K^{2n} \leqslant \frac{m^{2n-1} r_0^2}{2^{10}(121 n^3 M^2)^{n-1} \coth^n(s_0/3)}, \qquad \sqrt{n}\left(2^5 K\right)^{\frac{1}{n-1}} \left(m^{-1} \coth^n(s_0/3)\varepsilon\right)^{\frac{1}{2(n-1)}} \leqslant \rho/2 \tag{2}$$

and set

$$T(\varepsilon, K) = \frac{s_0}{3M} \exp\left(\frac{s_0}{198 M n^{3/2} \coth^{\frac{n}{2(n-1)}}(s_0/3)} \left(\frac{r_0}{2^5 K}\right)^{\frac{1}{n-1}} \left(\frac{1}{\varepsilon}\right)^{\frac{1}{2(n-1)}} \right)$$

Then any solution $(\theta(t), I(t))$ with $I_0 \in B_{R/2} \cap R_K$ satisfy $|I(t) - I_0| \leqslant \rho/2$ for $|t| \leqslant T(\rho, K)$.

This lemma is just Theorem 3 from [5] in the special case where the lattice is one-dimensional, and where we have adapted the constants to our context (in this Note we use different norms for vectors and for analytic functions). The second part of (2) is just to ensure the confinement in terms of ρ. Away from resonances, we have the following lemma:

Lemma 2.3. *Assume that ε and K satisfy*

$$K^{-1} \leqslant \min\left\{ \frac{\eta^3 m^2}{648 M^3}, \frac{R\eta m}{72 M} \right\}, \quad \varepsilon K \leqslant 3 \tag{3}$$

and let $(\theta(t), I(t))$ be a solution with $I_0 \in B_{R/2}$ and defined for $|t| \leqslant T^*$, for some $0 < T^* \leqslant +\infty$. If $I(t) \notin R_K$ for $|t| \leqslant T^*$, then $|I(t) - I_0| \leqslant 36M(\eta m K)^{-1}$ for $|t| \leqslant T^*$.

This lemma uses the iso-energetic non-degeneracy of h, a quantitative version of the implicit function theorem and an elementary result on rational approximations (see [4] for a non-quantitative version).

If a solution stays always far from resonances, Lemma 2.3 applies and gives infinite stability. If not, one can also apply Lemma 2.2 to obtain the desired result. As for the constants, one eventually chooses $K = 72M(\eta m \rho)^{-1}$ to ensure that the confinement in Lemma 2.3 is no more than $\rho/2$. The first condition in (3) defines ρ_0 and gives the condition $\rho \leqslant \rho_0$. Plugging this value of K in Lemma 2.2, condition (2) defines $\varepsilon_0(\rho)$, gives the condition $\varepsilon \leqslant \varepsilon_0(\rho)$ and finally this determines the time of stability $T(\varepsilon, \rho)$.

3. Towards a possible application

Here we shall briefly explain a work in progress with Jean-Pierre Marco and Laurent Niederman [6], where we try to apply this approach to a planetary three body problem (Sun–Jupiter–Saturn). Our aim is to obtain a statement similar to [3], Theorem 5.3, but with a ratio of mass closer to a realistic value.

What would appear as a complication is that the system here is properly degenerate (the non-degenerate action variables are the semi-major axes, the degenerate ones are the eccentricities and the inclinations) but this is quite easy to handle (see [3]) and in fact one can take advantage of that. Indeed, from the quantitative point of view, a great loss of accuracy in the result we presented in this Note comes from the analysis (Lemma 2.2) close to simple resonances. In this model three body problem, simple resonances are just periodic orbits, and one can obtain a much more precise normal form in this case, using the period τ of the periodic orbit as a free parameter instead of the rather artificial parameter K we used here. Also, the integrable Hamiltonian here is just a sum of squares, hence the geometry (Lemma 2.3) is trivial and much better constants can be obtained. Finally, if the threshold $\varepsilon_0 = \varepsilon_0(\rho)$ cannot be adjusted in a satisfactory way, it is very easy to introduce yet another parameter $k \in \mathbb{N}^*$ to weaken even more the result, but making it even more applicable: basically, $\varepsilon_0(\rho, k)$ increases when k decreases, the radius of confinement is not affected but instead of being an exponential, the time of stability $T(\varepsilon, \rho, k)$ is now just a polynomial (with a degree depending on k) in ε^{-1}.

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