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Analytical and innovative solutions for heat transfer problems involving phase change and interfaces

An improved lumped analysis for transient heat conduction in different geometries with heat generation

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ABSTRACT

This article deals with the analysis of transient one-dimensional heat conduction in both Cartesian and cylindrical geometry by employing the polynomial approximation method (PAM). Four different models such as specified heat flux for both slab and tube and heat generation in both slab and tube have been analyzed. The transient temperature is found to depend on various model parameters, namely, Biot number, heat source parameter and time. With the use of PAM, it has been possible to derive a unified relation for the transient thermal behavior of solid (slab and tube) with both internal generation and boundary heat flux. Present prediction is found to be in good agreement with other analytical results reported in the literature.

1. Introduction

When a hot solid comes in contact with a fluid of lower temperature, the removal of heat takes place between the surface of the solid and the neighboring fluid. The heat transfer usually takes place because of the heat conduction within the solid and heat convection between the hot surface and the fluid. In such a case, a distributed model can be used for the prediction of temperature both in spatial and temporal coordinates [1,2]. However, when the conductive thermal resistance in solid is sufficiently lower compared to the convective thermal resistance, one can consider a lumped model for the analysis [1,2]. In the recent years, the analysis of transient conduction by employing the lumped model is vastly used in numerous engineering applications, namely, the analysis of thermal hydraulics of nuclear reactors, the dynamics of chaotic instabilities in boiling water nuclear reactor and others.

It is observed that the classical lumped parameter approach or simple lumped models are usually used for solving conduction problems involving lower Biot numbers [3,4]. However, many engineering applications involve higher Biot number for the analysis. In such a case, the classical lumped models may not provide satisfactory results. In view of this, efforts have been made to improve the lumped model in order to predict the transient temperature at higher Biot numbers.

Several models have been reported in the literature to predict the transient behavior of temperature in solids. Earlier, Regis et al. [5] presented a lumped parameter model to solve the conduction equation by assuming a Hermite approximation for the temperature integrals and subsequently claimed a significant improvement over the classical lumped parameter formulation. Later on, the asymmetric cooling of slab geometry [6] was presented and the lumped model was shown to be the special case of the distributed model [7]. Cortes et al. [8] presented the unsteady heat conduction in various geometries, namely, slab, long cylinder and sphere and reported that lumped parameter model is a particular case of the general

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Nomenclature

A	parameter defined in Eq. (11)	$\bar{x}, \bar{r}$	length coordinates ... m
a	thermal diffusivity ... m^2/s	x, r	dimensionless length coordinates
B	parameter defined in Eq. (13)		
Bi	Biot number defined in Eq. (2)	*Greek symbols*	
C	specific heat ... J/kg °C	α, β, γ	constants defined in text
h	heat transfer coefficient ... W/m^2 °C	$\psi, \varphi, \bar{\varphi}$	parameters defined in Eq. (11)
K	thermal conductivity ... W/m °C	$\xi, \bar{\xi}$	parameter defined in Eq. (11)
L	length scale ... m	δ	wall thickness ... m
PAM	polynomial approximation method	θ	non-dimensional temperature defined in Eq. (2)
Q	dimensionless internal heat source parameter defined in Eq. (2)	$\bar{\theta}$	non-dimensional temperature integral defined in Eq. (6)
q	boundary heat flux ... W/m^2	θ_i	non-dimensional surface temperature
$\bar{q}$	internal heat generation ... W/m^3	ρ	density ... kg/m^3
R_i	radius $(i = 1, 2)$ ... m	ε	radius ratio defined in Eq. (2)
T	temperature ... °C	e	error between present model and that of CLSA model defined in Eq. (15)
T_0	initial temperature of the hot object ... °C		
t	time ... s		
U, V	parameter defined in Eq. (10)		

distributed model obtained from finite difference solution. Bairi and Laraqi [9] reported the analytical solutions for the unsteady heat conduction in spherical and cylindrical geometries. The transient analysis of long slab, long cylinder and sphere was reported by Sadat [10] by employing the singular perturbation method. The analysis of transient heat conduction in slab was reported by Bautisa et al. [11] by employing multiple scale analysis. Ostrogorsky [12] reported an analytical solution for transient heat conduction in spheres exposed to surroundings at a uniform temperature by employing Laplace transforms. In addition, numerous improved lumped parameter models [13] were presented to analyze transient heat conduction in slab geometry with temperature-dependent thermal conductivity by employing two point Hermite approximations for integrals. Recently, Tan et al. [14] reported an improved lumped model for the transient heat conduction of a wall involving both convective and radiative cooling by employing a two point Hermite type for integrals.

It may be noted that during a postulated loss of coolant accident, the temperature of the fuel elements inside the reactor core may increase drastically due to the stored energy in the fuel element and also due to the continuing fission product decay heat. In view of this, it is essential to analyze the temporal behavior of temperature field in solids with internal heat source. Nevertheless, a few improved lumped models have been reported [3,15] considering heat generation inside the solid.

Polynomial approximation method is one of the many analytical methods used to solve heat conduction equations [16]. This is analogous to the classical integral technique used for fluid flow and convective heat transfer analysis [17]. Recently, the integral method has been employed to analyze a variety of rewetting problems [18] and phase change problems of a finite slab [19]. Keshavarz and Taheri [16] proposed an improved lumped model for the analysis of unsteady transient heat conduction in different geometries by employing a polynomial approximation method. Over the years, several improved lumped models have been reported to predict the transient behavior of temperature in solid. These include the solutions of the unsteady conduction by employing various techniques such as: finite difference method, perturbation method, Hermite approximation for integrals, polynomial approximation method, Laplace transforms technique and multiple scale analysis. However, a few efforts have been made to formulate a unified model irrespective of geometry involving both specified heat flux and heat generation in solid.

In the present study an attempt has been made to model both with boundary heat flux and heat generation for various geometries of slab and tube. It has been shown that all the different models can be analyzed by employing polynomial approximation method. The results obtained from the present analysis are compared with the published analytical results.

2. Theoretical analysis

Fig. 1 schematically depicts the symmetric cooling of a slab or a tube of infinite length. Fig. 1(a) schematically represents a tube either with boundary heat flux or heat generation; while the schematic representation of heat generation and boundary heat flux in slab is shown in Figs. 1(b) and 1(c), respectively. The following assumptions are made for the analysis:

1. Initially the temperature of hot solid (slab and tube) is maintained at constant temperature (say T_0), suddenly it is exposed to surrounding fluid of lower temperature (say T_∞);
2. The thermo-physical properties of solid, namely, thermal conductivity, specific heat and density are assumed to be constant;

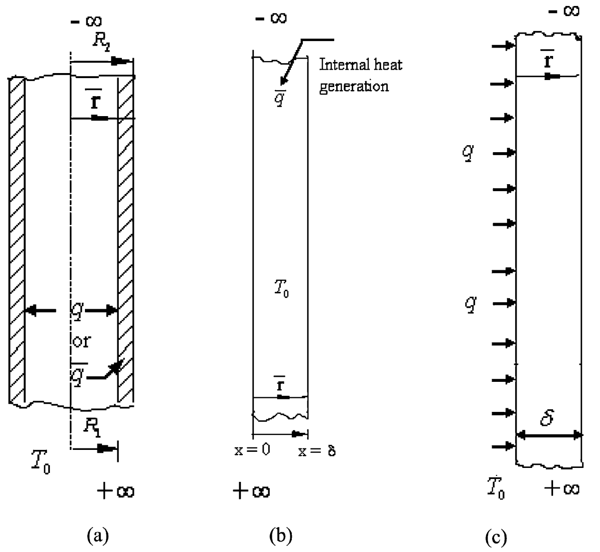

Fig. 1. Schematic of a hot object: (a) annular geometry with boundary heat flux or internal heat generation, (b) slab with internal heat generation, (c) slab with specified boundary heat flux.

3. To analyze the transient heat conduction, either a constant heat flux or a uniform heat generation in the solid (slab and tube) is considered. A constant heat transfer coefficient (h) is assumed at the surface between hot solid and neighboring ambient fluid.

The transient conduction equation with either specified heat flux or with internal heat generation valid for both rectangular Cartesian coordinate and cylindrical polar coordinate system (Fig. 1) can be written in a generalized form:

$$\frac{1}{\bar{r}^n}\frac{\partial}{\partial \bar{r}}\left(\bar{r}^n\frac{\partial T}{\partial \bar{r}}\right)+m\left(\frac{\bar{q}}{K}\right)=\frac{\rho C}{K}\frac{\partial T}{\partial t}, \quad t>0$$
$$n=\begin{cases}0 & \text{for a Cartesian geometry, with } \bar{r}\equiv\bar{x},\ 0<\bar{r}<\delta\\ 1 & \text{for a cylindrical geometry, } R_1<\bar{r}<R_2\end{cases}$$
$$m=\begin{cases}0 & \text{boundary heat flux}\\ 1 & \text{uniform heat generation}\end{cases} \tag{1}$$

The following normalized variables are defined:

$$r=\frac{\bar{r}}{R_2}, \qquad Bi=\frac{hR_2}{K}, \qquad \theta=\frac{T-T_\infty}{T_0-T_\infty}, \qquad \varepsilon=\frac{R_1}{R_2}, \qquad \tau=\frac{at}{R_2^2}, \qquad Q=\frac{\bar{q}^m q^{1-m}R_2^{m+1}}{K(T_0-T_\infty)} \tag{2}$$

Utilizing Eq. (2), the energy equation (1) is transformed into the following form:

$$\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n\frac{\partial \theta}{\partial r}\right)+(mQ)=\frac{\partial \theta}{\partial \tau} \tag{3}$$

Subject to boundary conditions given by:

$$\left(\frac{\partial\theta}{\partial r}\right)_{r=1}=\begin{cases}-Bi\theta_i & (n=1,\ m=0)\\ -Bi\theta_i & (n=1, m=1)\\ -Bi\theta_i & (n=0,\ m=0)\\ -Bi\theta_i & (n=0, m=1)\end{cases} \tag{4a–d}$$

$$\left(\frac{\partial\theta}{\partial r}\right)_{r=\varepsilon}=\begin{cases}-Q & (n=1, m=0)\\ 0 & (n=1, m=1)\end{cases} \tag{4e–f}$$

$$\left(\frac{\partial\theta}{\partial r}\right)_{r=0}=\begin{cases}-Q & (n=0, m=0)\\ 0 & (n=0, m=1)\end{cases} \tag{4g–h}$$

$$\tau=0, \qquad \theta=1 \tag{4i}$$

$$r=1, \qquad \theta=\theta_i \tag{4j}$$

2.1. Solution procedure

In polynomial approximation method, it is customary to consider a guess profile for temperature variation. In the present conduction problem, it is assumed that the temperature variation with time is stronger compared to the heat conduction along the spatial direction. Therefore, it is decided to assume a temperature profile in the spatial direction. In the generalized form, the energy equation (3) can be integrated as follows:

$$\int_{n\varepsilon}^{1}\left[\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n\frac{\partial\theta}{\partial r}\right)+(mQ)-\frac{\partial\theta}{\partial\tau}\right]r^n\,\mathrm{d}r=0 \tag{5}$$

The average temperature $\bar{\theta}$ can be defined as

$$\bar{\theta}=\int_{n\varepsilon}^{1}\theta r^n\,\mathrm{d}r \tag{6}$$

This gives a single ordinary differential equation valid for both Cartesian and cylindrical geometry:

$$m\left(\int_{n\varepsilon}^{1}Qr^n\mathrm{d}r\right)+\left(r^n\frac{\partial\theta}{\partial r}\right)_{r=1}-\left(r^n\frac{\partial\theta}{\partial r}\right)_{r=n\varepsilon}-\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\tau}=0 \tag{7}$$

At this juncture it is necessary to assume a temperature profile as a function of spatial coordinate. A simple yet generalized profile valid for both Cartesian and cylindrical geometry has been chosen for the analysis and is shown below.

$$\theta=\alpha(\tau)+\beta(\tau)f(x)+\gamma(\tau)f(x) \tag{8}$$

The temperature profile contains three unknown parameters namely, α, β and γ which are function of time and $f(x)$ is a function of space coordinate. By using Eqs. (6)–(8) and boundary conditions (4a)–(4j) one may get,

$$\theta(\tau)=\frac{\exp(-U\tau)+V}{U} \tag{9}$$

where the values of U and V are defined as below:

$$U=\begin{cases}\begin{cases}Bi/\psi & \text{Tube with heat flux}\\ Bi/\varphi & \text{Tube with heat generation}\\ Bi/\xi & \text{Cylinder with heat generation}\end{cases}\\ \begin{cases}Bi/A & \text{Slab with heat flux}\\ Bi/A & \text{Slab with heat generation}\end{cases}\end{cases}$$

$$V=\begin{cases}\begin{cases}Q/\psi & \text{Tube with heat flux}\\ Q/\bar{\varphi} & \text{Tube with heat generation}\\ Q/\bar{\xi} & \text{Cylinder with heat generation}\end{cases}\\ \begin{cases}Q/A & \text{Slab with heat flux}\\ Q/A & \text{Slab with heat generation}\end{cases}\end{cases} \tag{10}$$

where,

$$\begin{aligned}&\psi\equiv 0.5\big[1+Bi-0.5Bi/(1-\varepsilon)\big]\big(1-\varepsilon^2\big)+\big[Bi\varepsilon\big(1+\varepsilon+\varepsilon^2\big)/3\big]-(Bi/8)(1+\varepsilon)\big(1+\varepsilon^2\big)\\ &\varphi\equiv 0.5\big[1+0.5Bi(1-2\varepsilon)/(1-\varepsilon)\big]\big(1-\varepsilon^2\big)+\big[Bi\varepsilon\big(1+\varepsilon+\varepsilon^2\big)/3\big]-(Bi/8)(1+\varepsilon)\big(1+\varepsilon^2\big)\\ &\bar{\varphi}\equiv 2\varphi/\big(1-\varepsilon^2\big),\qquad A\equiv 1+Bi/3,\qquad \xi\equiv 0.5+(Bi/8),\qquad \bar{\xi}\equiv 1+(Bi/4)\end{aligned} \tag{11}$$

It may be noted that under the framework of present analysis, the transient behavior of temperature in hot solid with both heat generation and specified heat flux can be expressed through Eq. (9) as a four parameter relationship irrespective of geometry (slab/tube). It is of interest to examine the solution of Eq. (9) for the case of simple slab without heat generation. For slab geometry, with $Q=0$, Eq. (9) reduces to the simple expression of Keshavarz and Taheri [16]:

$$\theta(\tau)=\exp(-B\tau) \tag{12}$$

where,

$$B=Bi/(1+Bi/3) \tag{13}$$

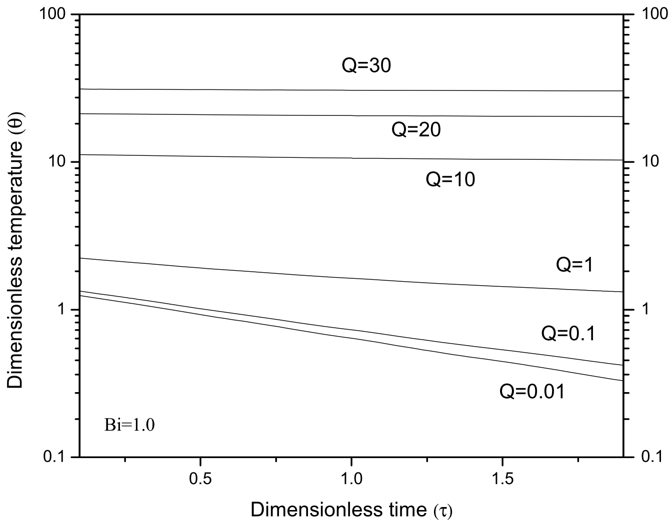

Fig. 2. Transient behavior of average temperature for a slab geometry with various heat source parameters.

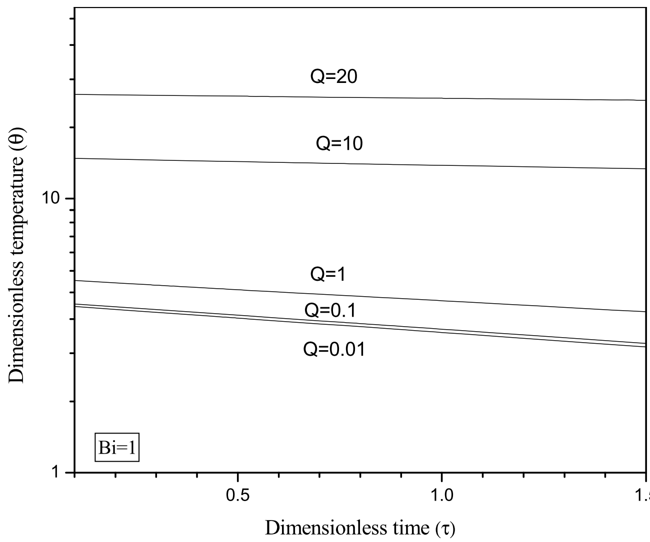

Fig. 3. Transient behavior of average temperature for a cylindrical geometry with various heat source parameters.

Additionally, the present solution simulates the transient analysis of solid rod with symmetric cooling (by assigning $Q = 0$ and $\varepsilon = 0$) in Eq. (9); this reduces to a three parameter relationship involving non-dimensional time, Biot number and temperature expressed as $\theta(\tau) = \exp[(-2\tau Bi)/(1 + (Bi/4))]$. The above expression is exactly the same as that of Keshavarz and Taheri [16] which they obtained through polynomial approximation method. Correa and Cotta [3] reported the solution of conduction equation for plain slab geometry with internal heat generation considering a lumped model. The solution is represented as

$$\theta(\tau) = \exp(-Bi\tau) + Q/Bi\big(1 - \exp(-Bi\tau)\big) \tag{14}$$

The reported classical lumped analysis is only applicable to problems with lower Biot numbers. It may be noted that the above expression (Eq. (14)) is valid only if $Q/Bi \leqslant 1.0$. However, several engineering applications involve a higher Q/Bi for their analysis. In such a case, the above model may not provide satisfactory results for the situations that involve higher Q/Bi ratio. Additionally, Ozisik [2] presented the exact solution for the spatial variation of the average temperature in a slab with internal heat generation and expressed as

$$\theta(\tau) = Q\left(\frac{1}{3} + \frac{1}{Bi}\right) + F(Q, Bi, X) \tag{15}$$

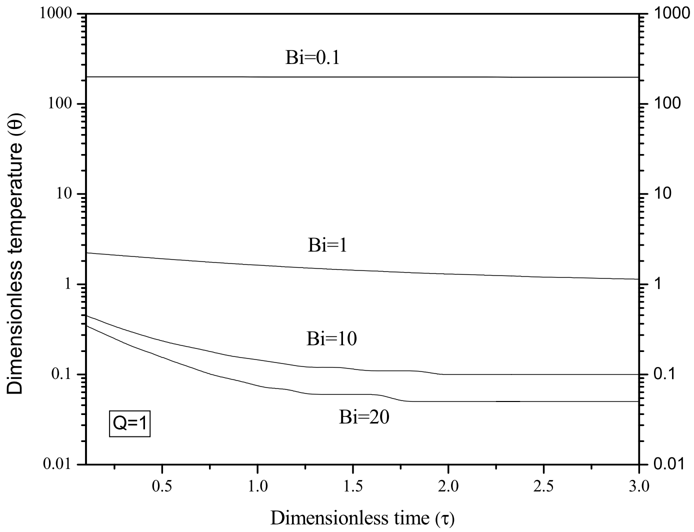

Fig. 4. Transient behavior of average temperature for a slab with various Biot numbers.

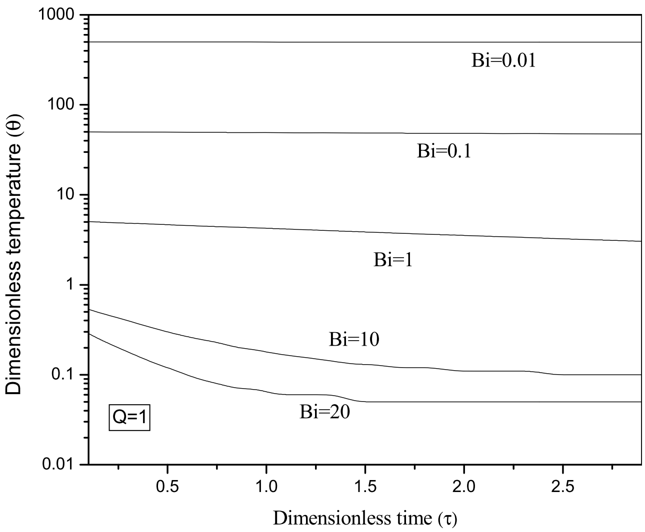

Fig. 5. Transient behavior of average temperature for a cylinder with various Biot numbers.

3. Results and discussion

An effort has been made to analyze the transient heat conduction in various geometries (slab/tube) by considering either heat flux at the surface or uniform heat generation inside the hot object. Polynomial approximation method has been employed to solve the conduction equation in the hot solid. It may be noted that the present analysis yields a single solution (Eq. (9)) valid for all the different cases. In rest of the work, we have tried to depict the variation of temperature with non-dimensional time.

The transient behavior of temperature for both Cartesian and cylindrical geometry with various heat source parameters is depicted in Figs. 2 and 3, respectively. With the decrease in heat source parameter (Q), the temperature of slab exhibits a sharp variation with time. However, with the increase of heat source parameter ($Q = 30$), the temperature of slab does not vary significantly with time. This may be explained by the fact that with the increase in heat source parameter, the convection cooling at the surface of hot solid due to the surrounding fluid is significantly lower compared to the heating caused by the internal heat source leading to a negligible change in the transient response of temperature in the solid.

Fig. 4 depicts the average temperature as a function of dimensionless time for different values of Biot number for a slab. A sharp variation in temperature with time is observed at higher Biot number ($Bi = 20$, 10). This may be due to the fact that higher Biot number represents a higher heat removal from the hot surface leading to sudden decrease in temperature. For higher Biot number ($Bi = 20$, 10), a sharp drop in temperature is observed for a dimensional time ranging from 0.01

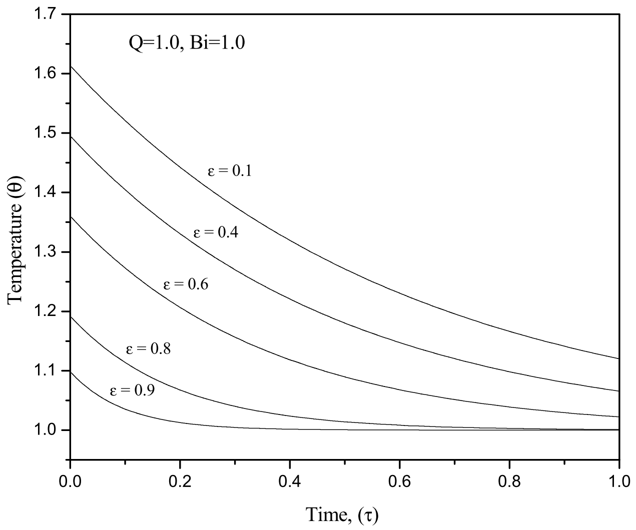

Fig. 6. Transient behavior of average temperature for a tube with specified heat flux and varying wall thickness ($Q = 1.0$, $Bi = 1.0$).

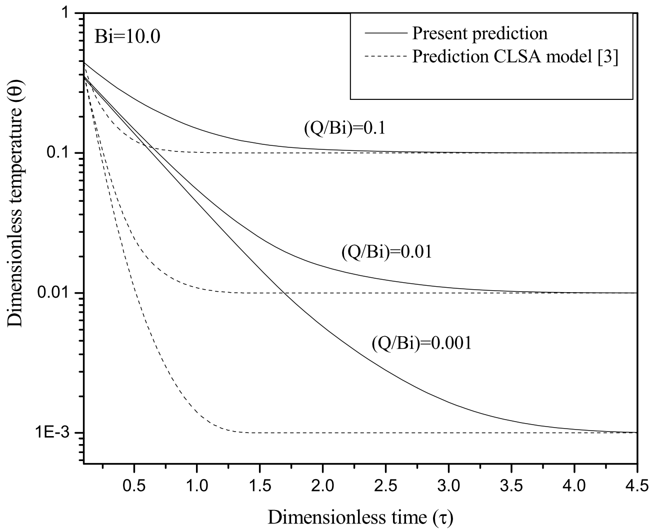

Fig. 7. Comparison of present prediction with CLSA for a slab with varying heat source parameter.

to 1.5; while the temperature attains an asymptotic value after 1.5. Additionally, the symmetric cooling of a long cylinder has been analyzed and the results are shown in Fig. 5. It is observed that the transient behavior of temperature in cylinder exhibits similar trend as that of slab geometry.

Fig. 6 depicts the transient behavior of temperature of an annular geometry for various radius ratio (ε). For a given set of model parameters, namely, Bi and Q, the initial temperature of tube decreases with increase in ε. This may be attributed to the fact that with increase of ε, both the thickness and heat capacity of the tube decrease, which is conducive to the removal of heat from the solid. The transient thermal behavior of tube exhibits the general trend as is observed in slab geometry.

Present prediction for slab geometry with heat generation is compared with that obtained from the classical lumped model (CLSA) of Correa and Cotta [3] and is shown in Fig. 7. The transient behavior of temperature of slab predicted by classical lumped model is lower compared to the present prediction. The deviation of transient temperature obtained by present prediction and CLSA was found to depend on the ratio of internal heat source parameter and Biot number. It is observed that for a higher Q/Bi, the deviation of temperature is higher between present prediction and CLSA. However, after a certain time period, both the models predict the same value of temperature for the solid.

It may be noted that the normalized error or dimensionless deviation of the average temperature is defined as follows:

$$e = \left(\left|\theta_{present}(\tau) - \theta_{lumped}(\tau)\right|\right)/\theta_{present}(\tau) \tag{16}$$

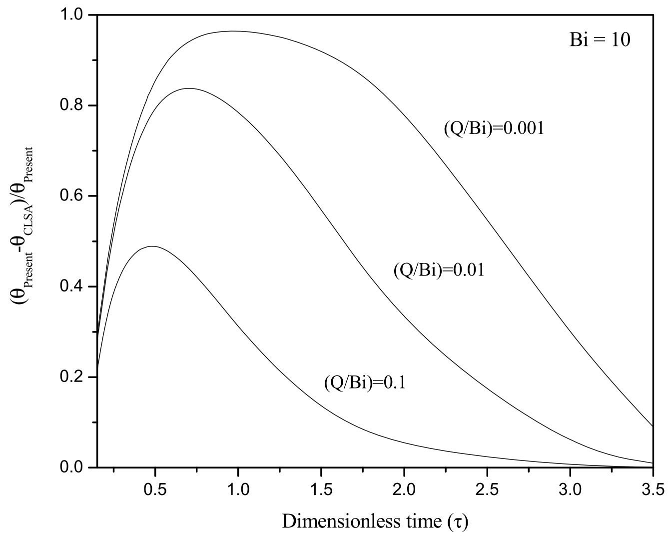

Fig. 8. Distribution of error as a function of time for the lumped models.

The distribution of error between present prediction and that obtained by CLSA model as a function of time is depicted in Fig. 8. It is observed that the error rises to maximum in early times and decays to zero as the time increases. The largest error at any time is less than 90%. However, the typical error is found to vary within 20% to 40% between the present prediction and CLSA model.

4. Conclusions

The polynomial approximation method has been applied for the comprehensive analysis of transient heat conduction in solid (slab and tube) considering a specified heat flux and a uniform heat generation. A simple closed form solution is obtained for the temperature distribution in all the cases. It has been shown that all the different models can be analyzed by employing polynomial approximation method and the generalization of the analysis is also possible with the use of PAM.

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