C. R. Mecanique 340 (2012) 596–601

Contents lists available at SciVerse ScienceDirect

Comptes Rendus Mecanique

www.sciencedirect.com

Second order model in fluid film lubrication

Eduard Marušić-Paloka [a,*], Igor Pažanin [a], Sanja Marušić [b]

[a] Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia
[b] Faculty of Transport and Traffic Engineering, University of Zagreb, Vukelićeva 4, 10000 Zagreb, Croatia

ARTICLE INFO

Article history:
Received 19 February 2012
Accepted after revision 31 May 2012
Available online 23 June 2012

Keywords:
Lubrication
Fluid film lubrication
Asymptotic analysis
Second order model

ABSTRACT

The goal of this Note is to derive the second order model correcting the standard Reynolds equation for fluid film lubrication. Starting from microscopic model described by the Stokes system, we compute an asymptotic expansion for the solution. Instead of computing only the first term, as in the standard Reynolds approximation, we keep first two terms leading to the corrected model. We obtain equations similar to the Brinkman model for porous medium flow.

1. Introduction

Fluid film bearings are machine elements usually studied in the broader context of tribology — the science and technology of friction, lubrication and wear. Simply speaking, they consist of two surfaces in relative motion, separated by a thin fluid film, that lubricates the device and minimizes the friction and, consequently, the wear of the device. In our case, the fluid is an incompressible liquid and the two surfaces are rigid. Such elements are very important in mechanical engineering since they provide the reliability of the system and are crucial factor in limiting the dissipation of energy i.e. increasing the efficiency. If a fluid film bearing is well designed, the wear is not an issue, since two surfaces are completely separated by lubricant. It is therefore important to understand the behavior of the fluid film in the bearing.

The mathematical models for describing the motion of the lubricant usually result from simplifications of the basic equations of fluid motion. In our case the Newtonian fluid will be considered, but other models are also used in the literature. Simplification is based on the geometry of the lubricant film, i.e. its thickness. Using the thickness as a small parameter, a simple asymptotic approximation is easily derived and it gives a well-known Reynolds equation for the pressure of the fluid. Its derivation goes back to 19th century and the pioneering work of Reynolds [1]. The rigorous mathematical justification, i.e. the proof that it can be obtained as the limit of the Navier–Stokes (rather Stokes) system, as the thickness goes to zero, is due to Bayada and Chambat [2]. Error estimates and justifications in different norms and spaces have been given by different authors, see e.g. [3,5,13,14].

There is an analogy between flow through a porous medium and flow of a lubricant film. In both cases we have low permeable domain, and we use the domain permeability as the small parameter in the analysis. There are different models for description of the porous medium flow, and the analogous law for the Reynolds equation would be the Darcy law [6]. It gives the proportionality between the velocity and the pressure drop providing an elliptic PDE for the pressure. One of the problems with the Darcy law is that it cannot satisfy the no-slip boundary condition, natural for viscous fluids. It is due to the fact that the velocity enters the system only with first derivatives. In 1947 Brinkman [7] added a correction to the Darcy system, including the second derivatives of the velocity. That changes the nature of the system and allows to impose

* Corresponding author.
E-mail addresses: emarusic@math.hr (E. Marušić-Paloka), pazanin@math.hr (I. Pažanin), marusics@fpz.hr (S. Marušić).

1631-0721/$ – see front matter
http://dx.doi.org/10.1016/j.crme.2012.05.004

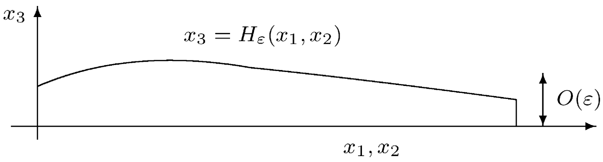

Fig. 1. The domain considered.

the Dirichlet boundary condition on the velocity. He assumed large permeability to compare his law with experimental data. Several papers can be found on that subject, both experimental as well as theoretical. We mention only theoretical papers that inspired our work. Sanchez-Palencia [4] and Levy [8] derived Brinkman law from Stokes system using formal homogenization and asymptotic analysis for flow through an array of small particles. Allaire [9] rigorously proved their result for periodic porous medium with large permeability. Auriault et al. [10] formally computed correctors for the Darcy law and derived the Brinkman's correction as a lower order term in an asymptotic expansion attributed to the flow. In case of flow through a thin fissure driven by the body force, the problem was recently investigated in [11] by the authors of this paper.

In the present Note we derive and justify the Brinkman-type model for description of the thin fluid flow in lubrication theory. As far as we know, the same idea cannot be found in the existing literature in context of tribology. We start from the linearized Stokes equations describing the microscopic flow in a thin three-dimensional domain and employ the technique of two-scale asymptotic expansion in Section 3. Zero-order term in the expansion corresponds to the solution of Reynolds lubrication equation. We compute the successive terms in the asymptotic expansion of the solution leading to a higher-order correction of the standard Reynolds approximation. We do it in a way that those successive terms have zero mean value. That requirement forces us to correct the macroscopic equation. As a result, Brinkman-type system is obtained governing the two-dimensional macroscopic flow and that represents our main contribution. Rigorous justification of the formally derived asymptotic model is discussed in the concluding section.

2. The problem

Let $\mathcal{O} \subset \mathbf{R}^2$ be a bounded domain and $h : \overline{\mathcal{O}} \to \langle 0, +\infty\rangle$ a smooth positive function. For a small parameter $\varepsilon > 0$, we define our three-dimensional domain occupied by the fluid as

$$\Omega^\varepsilon = \left\{(x_1, x_2, x_3) \in \mathbf{R}^3 \colon x' = (x_1, x_2) \in \mathcal{O},\ 0 < x_3 < H_\varepsilon(x_1, x_2)\right\}, \quad H_\varepsilon(x_1, x_2) = \varepsilon h(x_1, x_2) \tag{1}$$

As we can see (Fig. 1), the lower surface is supposed to be plane, while the roughness of the upper surface is described by the given function h. We assume the flow to be governed by the stationary Stokes system

$$-\mu \Delta \mathbf{u}^\varepsilon + \nabla p^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon \tag{2}$$

$$\operatorname{div} \mathbf{u}^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon \tag{3}$$

The vector field $\mathbf{u}^\varepsilon$ denotes the fluid velocity whereas the pressure is given by the scalar field p^ε. The positive real number $\mu > 0$ corresponds to the viscosity of the fluid. The choice of boundary conditions highly depends on the devices to be considered. Here we want to study the lubrication process where two rigid surfaces are in relative motion and are separated by a thin layer of fluid. Thus, we impose the following boundary conditions:

$$\mathbf{u}^\varepsilon = 0 \quad \text{for } x_3 = \varepsilon h, \qquad \mathbf{u}^\varepsilon = \mathbf{w}_0 \quad \text{for } x_3 = 0 \tag{4}$$

$$\mathbf{u}^\varepsilon \times \mathbf{n} = 0 \quad \text{for } x' \in \partial\mathcal{O}, \qquad p^\varepsilon = q_\varepsilon \quad \text{for } x' \in \partial\mathcal{O} \tag{5}$$

for given outer pressure $q_\varepsilon = \varepsilon^{-2} q$ and constant velocity $\mathbf{w}_0$ of relative motion of two surfaces. Depending on q the bearing can be self-acting or externally pressurized. Obviously, $\mathbf{w}_0 \cdot \mathbf{k} = 0$ implying $u_3^\varepsilon|_{x_3=0} = 0$. Here and in the sequel $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ denotes the standard Cartesian basis. The boundary condition imposed along the lateral boundary (for $x' \in \partial\mathcal{O}$) involves pressure, so the existence and uniqueness result for the problem under consideration can be found in the referenced paper by Conca et al. [12]. The goal of this Note is to find an effective law of high order of accuracy describing the asymptotic behavior of the flow in Ω^ε.

3. Asymptotic expansion

We introduce the fast variable $y = \frac{x_3}{\varepsilon}$ and look for an asymptotic expansion of the unknowns $\mathbf{u}^\varepsilon$ and p^ε in the form[1]

[1] Note that $p^0 = p^0(x')$ since the lowest order approximation of the pressure requires $\frac{\partial p^0}{\partial y} = 0$.

$$\mathbf{u}^{\varepsilon} = \mathbf{u}^0(x', y) + \varepsilon \mathbf{u}^1(x', y) + \varepsilon^2 \mathbf{u}^2(x', y) + \cdots \tag{6}$$

$$p^{\varepsilon} = \frac{1}{\varepsilon^2} p^0(x') + \frac{1}{\varepsilon} p^1(x', y) + p^2(x', y) + \cdots \tag{7}$$

In the sequel we employ the following notation:

$$\nabla_{x'}\phi = \frac{\partial \phi}{\partial x_1}\mathbf{i} + \frac{\partial \phi}{\partial x_2}\mathbf{j}, \qquad \Delta_{x'}\mathbf{f} = \frac{\partial^2 \mathbf{f}}{\partial x_1^2} + \frac{\partial^2 \mathbf{f}}{\partial x_2^2}, \qquad \operatorname{div}_{x'}\mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \tag{8}$$

for a scalar function ϕ and a vector function $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$. Plugging the above expansions into Eq. (2) and collecting the terms with equal powers of ε yields

$$\varepsilon^{-2}\left[-\mu\frac{\partial^2 \mathbf{u}^0}{\partial y^2} + \nabla_{x'} p^0 + \frac{\partial p^1}{\partial y}\mathbf{k}\right] + \varepsilon^{-1}\left[-\mu\frac{\partial^2 \mathbf{u}^1}{\partial y^2} + \nabla_{x'} p^1 + \frac{\partial p^2}{\partial y}\mathbf{k}\right] + \left[-\mu\frac{\partial^2 \mathbf{u}^2}{\partial y^2} - \mu\Delta_{x'}\mathbf{u}^0 + \nabla_{x'} p^2 + \frac{\partial p^3}{\partial y}\mathbf{k}\right] + \cdots = 0 \tag{9}$$

If we keep only the main order term for the moment, we obtain

$$-\mu\frac{\partial^2 \mathbf{u}^0}{\partial y^2} + \nabla_{x'} p^0 + \frac{\partial p^1}{\partial y}\mathbf{k} = 0, \quad \mathbf{u}^0 = 0 \text{ for } y = h, \quad \mathbf{u}^0 = \mathbf{w}_0 \text{ for } y = 0 \tag{10}$$

leading to $p^1 = p^1(x')$ and

$$\mathbf{u}^0(x', y) = \frac{1}{2\mu} y\left(h(x') - y\right)\mathbf{v}(x') + \left(1 - \frac{y}{h(x')}\right)\mathbf{w}_0, \quad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} \tag{11}$$

Observe that

$$\mathbf{v} + \nabla_{x'} p^0 = 0 \tag{12}$$

From the divergence equation (3) we deduce

$$\operatorname{div}_{x'}\mathbf{u}^0 + \frac{\partial u_3^1}{\partial y} + \varepsilon\left(\operatorname{div}_{x'}\mathbf{u}^1 + \frac{\partial u_3^2}{\partial y}\right) + \cdots = 0 \tag{13}$$

Integrating from 0 to $h(x')$ with respect to y we get from the main order term

$$\operatorname{div}_{x'}\left(\int_0^h \mathbf{u}^0 \, \mathrm{d}y\right) = \operatorname{div}_{x'}\left(\frac{h^3}{12\mu}\mathbf{v} + \frac{h}{2}\mathbf{w}_0\right) = 0 \tag{14}$$

In view of (11)–(12), we obtain the classical Reynolds equation

$$\operatorname{div}_{x'}\left(h^3\nabla_{x'} p^0\right) = 6\mu\nabla_{x'} h \cdot \mathbf{w}_0 \quad \text{in } \mathcal{O} \tag{15}$$

Together with a boundary condition $p^0 = q$ on $\partial\mathcal{O}$ it forms a Dirichlet boundary value problem for linear elliptic equation of second order for the pressure. The velocity is then determined straightforward from a simple equation (12). From Eq. (15) we can compute an approximation for the mean pressure, i.e. it does not take into account pressure variations across the thin fluid film since they are small. Thus, in the Reynolds system, the pressure appears with second order derivatives and the velocity with no derivative. On the other hand, in the original Stokes system we had a velocity with second order derivatives, and a pressure with first order derivatives. Therefore, on the first glance, those two systems are of completely different types. Following the idea from Marušić-Paloka et al. [11], we continue the computation and keep the lower order terms. To be more precise, we compute the correctors in a way that they have zero mean value $\int_0^{h(x')} \cdot \mathrm{d}y$. That way the correctors do not contribute to the net flow rate. Such requirement will force us to change the leading order term $\mathbf{u}^0$ which now has to carry the whole flow rate. However those changes will be of the lower order. As a consequence, we are going to obtain the effective law very similar to the 2D Navier–Stokes system but with small *viscosity* term whose (effective) *viscosity* does not correspond to the physical viscosity of the liquid.

We return to (9) and (13). We deduce

$$u_3^1 = -\int_0^y \operatorname{div}_{x'} \mathbf{u}^0(x',\xi)\,\mathrm{d}\xi = \left(\frac{y^3}{6\mu} - \frac{y^2 h}{4\mu}\right)\operatorname{div}_{x'} \mathbf{v} - \frac{y^2}{4\mu}\nabla_{x'} h \cdot \mathbf{v} + \frac{y^2}{2}\mathbf{w}_0 \cdot \nabla_{x'}\left(\frac{1}{h}\right) \tag{16}$$

$$u_\beta^1 = 0, \quad \beta = 1,2, \qquad p^1 = 0 \tag{17}$$

$$p^2 = \mu\frac{\partial u_3^1}{\partial y} + Q^2(x') = -\frac{y}{2}\operatorname{div}_{x'}(h\mathbf{v}) + \frac{y^2}{2}\operatorname{div}_{x'}\mathbf{v} + \mu y \mathbf{w}_0 \cdot \nabla_{x'}\left(\frac{1}{h}\right) + Q^2(x') \tag{18}$$

In order to get optimal error estimate for the pressure, we keep the mean value of p^2 equal to zero as well. In view of that, from (18) we obtain $Q^2 = 0$. Now we construct the corrector $\mathbf{u}^2$. In view of (9), it is given by

$$-\mu\frac{\partial^2 \mathbf{u}^2}{\partial y^2} = \mu\Delta_{x'}\mathbf{u}^0 - \nabla_{x'} p^2 - \frac{\partial p^3}{\partial y}\mathbf{k}, \qquad \mathbf{u}^2 = 0 \quad \text{for } y = 0, h \tag{19}$$

To keep the divergence equation (13) satisfied as well, we put $p^3 = 0$ implying $u_3^2 = 0$. For given $\mathbf{u}^0$, p^2, p^3 and unknown $\mathbf{u}^2$, (19) is a boundary value problem for an ordinary differential equation, with respect to y. It has a unique solution, with non-zero mean value. That does not suit our purpose as we do not want the corrector $\mathbf{u}^2$ to contribute to the net flow rate. To correct that, we add an additional term to Eq. (19), denoted by $\mathbf{A}(x')$. Thus, we consider the following *modified* problem for $\mathbf{u}^2$:

$$\begin{cases} -\mu\dfrac{\partial^2 \mathbf{u}^2}{\partial y^2} = \mu\Delta_{x'}\mathbf{u}^0 - \nabla_{x'} p^2 + \mathbf{A}(x') \\ \mathbf{u}^2 = 0, \quad \text{for } y = 0, h \\ \displaystyle\int_0^h \mathbf{u}^2\,\mathrm{d}y = 0 \end{cases} \tag{20}$$

Note that $\mathbf{A}(x')$ is an unknown in the system and it is to be determined in order to satisfy $\int_0^h \mathbf{u}^2\,\mathrm{d}y = 0$. As the consequence the zero order term in (9) will not lead to the macroscopic equation of the Reynolds type (12) and (15). Instead, an additional term caused by $\mathbf{A}(x')$ will appear bringing in a second order (*viscous*) term in the effective law. However, such term will be of the lower order ε^2. Now we compute $\mathbf{u}^2$. Employing the decomposition

$$\Delta_{x'}(h\mathbf{v}) = (\Delta_{x'}h)\mathbf{v} + 2\nabla_{x'}h \cdot (\nabla_{x'}\mathbf{v})^\tau + h\Delta_{x'}\mathbf{v} \tag{21}$$

from (11) we deduce

$$\mu\Delta_{x'}\mathbf{u}^0 = \frac{y}{2}(\Delta_{x'}h)\mathbf{v} + y\nabla_{x'}h \cdot (\nabla_{x'}\mathbf{v})^\tau + \frac{1}{2}(hy - y^2)\Delta_{x'}\mathbf{v} - \mu y \Delta_{x'}\left(\frac{1}{h}\right)\mathbf{w}_0 \tag{22}$$

In view of (15) we can obtain

$$\begin{aligned} \nabla_{x'} p^2 = {} & \left(y - \frac{3y^2}{2h}\right)\nabla_{x'}(\nabla_{x'}h \cdot \mathbf{v}) - \frac{3y^2}{2}\nabla_{x'}\left(\frac{1}{h}\right)(\nabla_{x'}h \cdot \mathbf{v}) \\ & - 2\mu y \nabla_{x'}\left(\nabla_{x'}\left(\frac{1}{h}\right) \cdot \mathbf{w}_0\right) - 3\mu y^2 \nabla_{x'}\left(\frac{1}{h^3}\nabla_{x'}h \cdot \mathbf{w}_0\right) \end{aligned} \tag{23}$$

Taking into account (22) and (23), we solve (20) by putting

$$\begin{aligned} \mathbf{u}^2 = {} & \frac{1}{24\mu}(y^4 - 2hy^3)\Delta_{x'}\mathbf{v} - \frac{y^3}{12\mu}(\Delta_{x'}h)\mathbf{v} - \frac{y^3}{6\mu}\nabla_{x'}h \cdot (\nabla_{x'}\mathbf{v})^\tau - \frac{y^4}{8\mu}\nabla_{x'}\left(\frac{1}{h}\right)(\nabla_{x'}h \cdot \mathbf{v}) \\ & + \frac{1}{\mu}\left(\frac{y^3}{6} - \frac{y^4}{8h}\right)\nabla_{x'}(\nabla_{x'}h \cdot \mathbf{v}) + \frac{y^3}{6}\Delta_{x'}\left(\frac{1}{h}\right)\mathbf{w}_0 - \frac{y^3}{3}\nabla_{x'}\left(\nabla_{x'}\left(\frac{1}{h}\right) \cdot \mathbf{w}_0\right) \\ & - \frac{y^4}{4}\nabla_{x'}\left(\frac{1}{h^3}\nabla_{x'}h \cdot \mathbf{w}_0\right) - \frac{y^2}{2\mu}\mathbf{A}(x') - \frac{y}{\mu}\mathbf{B}(x') \end{aligned} \tag{24}$$

where

$$\mathbf{A}(x') = -\frac{h}{4}(\Delta_{x'}h)\mathbf{v} - \frac{h}{2}\nabla_{x'}h\cdot(\nabla_{x'}\mathbf{v})^{\tau} - \frac{h^2}{10}\Delta_{x'}\mathbf{v} + \frac{h}{20}\nabla_{x'}(\nabla_{x'}h\cdot\mathbf{v}) - \frac{9h^2}{20}\nabla_{x'}\left(\frac{1}{h}\right)(\nabla_{x'}h\cdot\mathbf{v})$$
$$+\frac{\mu h}{2}\Delta_{x'}\left(\frac{1}{h}\right)\mathbf{w}_0 - \mu h\nabla_{x'}\left(\nabla_{x'}\left(\frac{1}{h}\right)\cdot\mathbf{w}_0\right) - \frac{9\mu h^2}{10}\nabla_{x'}\left(\frac{1}{h^3}\nabla_{x'}h\cdot\mathbf{w}_0\right) \tag{25}$$

$$\mathbf{B}(x') = \frac{h^2}{24}(\Delta_{x'}h)\mathbf{v} + \frac{h^2}{12}\nabla_{x'}h\cdot(\nabla_{x'}\mathbf{v})^{\tau} + \frac{h^3}{120}\Delta_{x'}\mathbf{v} + \frac{h^2}{60}\nabla_{x'}(\nabla_{x'}h\cdot\mathbf{v}) + \frac{h^3}{10}\nabla_{x'}\left(\frac{1}{h}\right)(\nabla_{x'}h\cdot\mathbf{v})$$
$$-\frac{\mu h^2}{12}\Delta_{x'}\left(\frac{1}{h}\right)\mathbf{w}_0 + \frac{\mu h^2}{6}\nabla_{x'}\left(\nabla_{x'}\left(\frac{1}{h}\right)\cdot\mathbf{w}_0\right) + \frac{\mu h^3}{5}\nabla_{x'}\left(\frac{1}{h^3}\nabla_{x'}h\cdot\mathbf{w}_0\right) \tag{26}$$

Now we integrate (9) with respect to y from 0 to $h(x')$ leading to

$$\mathbf{v} + \nabla_{x'}p^0 - \varepsilon^2\left[\frac{h^2}{10}\Delta_{x'}\mathbf{v} + \frac{h}{2}\nabla_{x'}h\cdot(\nabla_{x'}\mathbf{v})^{\tau} - \frac{h}{20}\nabla_{x'}(\nabla_{x'}h\cdot\mathbf{v}) + \frac{h}{4}(\Delta_{x'}h)\mathbf{v} + \frac{9h^2}{20}\nabla_{x'}\left(\frac{1}{h}\right)(\nabla_{x'}h\cdot\mathbf{v})\right]$$
$$= \mu h\varepsilon^2\left[\nabla_{x'}\left(\nabla_{x'}\left(\frac{1}{h}\right)\cdot\mathbf{w}_0\right) + \frac{9h}{10}\nabla_{x'}\left(\frac{1}{h^3}\nabla_{x'}h\cdot\mathbf{w}_0\right) - \frac{1}{2}\Delta_{x'}\left(\frac{1}{h}\right)\mathbf{w}_0\right] \quad \text{in } \mathcal{O}$$
$$\mathrm{div}_{x'}\left(h^3\mathbf{v}\right) = -6\mu\nabla_{x'}h\cdot\mathbf{w}_0 \quad \text{in } \mathcal{O}$$
$$\mathbf{v}\times\mathbf{n} = 0 \quad \text{and} \quad p^0 = q \quad \text{on } \partial\mathcal{O} \tag{27}$$

This is the new, Brinkman-type, effective law satisfied by $(\mathbf{v}, p^0)$ and describing the macroscopic flow. Unlike Reynolds system (12), (15), at first glance it is very similar to 2D Navier–Stokes system (notice the *viscous* term $\frac{H_\varepsilon^2}{10}\Delta\mathbf{v}$). Solvability of (27) can be established using the techniques from [12] employed for the governing system as well. Note that (27) is, in fact, linear system so the main difficulty is the divergence equation, not allowing us to eliminate the pressure in the variational formulation. That can be elegantly fixed by introducing the new unknown $\mathbf{z} = h^3\mathbf{v} + 6\mu h\mathbf{w}_0$ (being, evidently, divergence-free) and deriving the momentum equation for $\mathbf{z}$ instead of $\mathbf{v}$. In addition the third component of the velocity is determined by (16). It is important to notice that the viscosity $\frac{H_\varepsilon^2}{10}$ is not equal to the viscosity μ of the lubricant.

Remark 1. To illustrate the difference between the standard Reynolds model, given by (12), (15), and derived effective law (27), let us consider a simple case of rectangular domain, namely $h = 1$ and $\mathcal{O} =]0,1[^2$. We prescribe the pressure drop $q_0 - q_1$ between sides $x_1 = 0$ and $x_1 = 1$ and take $\mu = 1$ for notational simplicity. In this case, we can solve those two problems and obtain the explicit solutions in the form[2]:

$$\mathbf{v}_R = (q_0 - q_1)\mathbf{i} \tag{28}$$
$$\mathbf{v}_B = (q_0 - q_1)\left[1 + \frac{1 - e^{-\frac{\sqrt{10}}{\varepsilon}}}{e^{\frac{-2\sqrt{10}}{\varepsilon}} - 1}\left(e^{-\frac{\sqrt{10}x_2}{\varepsilon}} + e^{\frac{\sqrt{10}(x_2-1)}{\varepsilon}}\right)\right]\mathbf{i} \tag{29}$$

Even in such simplified setting, we observe an additional term in the solution $\mathbf{v}_B$ of the Brinkman-type equations representing the correction of the classical Reynolds solution $\mathbf{v}_R$. Moreover, note that $\mathbf{v}_B = 0$ for $x_2 = 0, 1$.

4. Rigorous justification

In this concluding section, we discuss justification by error estimate of the formally derived asymptotic model. It is well known that, for the Reynolds solution the error of approximation, expressed in the rescaled L^2 norm, is not better than $O(\varepsilon)$, depending on the boundary conditions under consideration (see e.g. [13]). We need to get better estimates for the effective law (27) in order to justify its usage. For the original boundary conditions (4)–(5) describing real physical situation, a satisfactory L^2 or H^1 error estimate seams to be unfeasible. That is due to the boundary layer effects polluting those estimates.[3] To avoid the boundary layer, we prove the error estimate in case of periodic boundary condition. We take $\mathcal{O}$ to be a rectangle, h is assumed to be periodic, and we impose on $\partial\mathcal{O}$ periodicity condition:

$$\left(\mathbf{u}^\varepsilon, p^\varepsilon\right) \quad \text{is } \mathcal{O} \text{ periodic in } x' \text{ variable} \tag{30}$$

Using standard techniques (see e.g. [11, Proof of Theorem 1]), in such setting we can prove:

[2] In both cases, the pressure is given by $p^0 = q_0 + (q_1 - q_0)x_1$.

[3] Indeed, in (29) the function $e^{-\frac{\sqrt{10}x_2}{\varepsilon}} + e^{\frac{\sqrt{10}(x_2-1)}{\varepsilon}}$ is of the boundary layer type.

Theorem 4.1. *The following estimates hold*:

$$\left| \int_0^h \mathbf{u}^\varepsilon \, dy - \frac{h^3}{12\mu}\mathbf{v} - \frac{h}{2}\mathbf{w}_0 \right|_{L^2(\mathcal{O})} \leqslant C\varepsilon^3, \qquad \left| \varepsilon^2 \int_0^h p^\varepsilon \, dy - p^0 \right|_{L^2(\mathcal{O})} \leqslant C\varepsilon^3 \tag{31}$$

where $(\mathbf{v}, p^0)$ *is given by* (27).

In case of Dirichlet condition, however, H^{-1} and H^{-2} error estimates can be proved, but the proof is tedious and technical.

Acknowledgements

This research was supported by the Ministry of Science, Education and Sports, Republic of Croatia, Grants: 037-0372787-2797 and 135-0372787-2819. The authors would like to thank the referees for their valuable remarks and comments.

References

[1] O. Reynolds, On the theory of lubrication and its applications to Mr. Beauchamp Towers experiments, including an experimental determination of the viscosity of olive oil, Philos. Trans. Roy. Soc. London 177 (1886) 157–234.
[2] G. Bayada, M. Chambat, The transition between the Stokes equations and the Reynolds equation: A mathematical proof, Appl. Math. Opt. 14 (1986) 73–93.
[3] S.A. Nazarov, Asymptotic solution of the Navier–Stokes problem on the flow of a thin layer of fluid, Siberian Math. J. 31 (1990) 296–307.
[4] E. Sanchez-Palencia, On the asymptotics of the fluid flow past an array of fixed obstacles, Int. J. Eng. Sci. 20 (1982) 1291–1301.
[5] J. Wilkening, Practical error estimates for Reynolds' lubrication approximation and its higher order corrections, SIAM J. Math. Anal. 41 (2009) 588–630.
[6] H. Darcy, Les fontaines publiques de la ville de Dijon, Victor Darmon, Paris, 1856.
[7] H. Brinkman, A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles, Appl. Sci. Res. A 1 (1947) 27–34.
[8] T. Levy, Fluid flow through an array of fixed particles, Int. J. Eng. Sci. 21 (1983) 11–23.
[9] G. Allaire, Homogenization of the Navier–Stokes equations in open sets perforated with tiny holes I. Abstract framework, a volume distribution of holes, Arch. Rational. Mech. Anal. 113 (1991) 209–259.
[10] J.L. Auriault, C. Geindreau, C. Boutin, Filtration law in porous media with poor separation of scales, Transp. Porous Med. 60 (2005) 89–108.
[11] E. Marušić-Paloka, I. Pažanin, S. Marušić, Comparison between Darcy and Brinkman laws in a fracture, Appl. Math. Comput. 218 (2012) 7538–7545.
[12] C. Conca, F. Murat, O. Pironneau, The Stokes and Navier–Stokes equations with boundary conditions involving the pressure, Japan J. Math. 20 (1994) 263–318.
[13] A. Duvnjak, E. Marušić-Paloka, Derivation of the Reynolds equation for lubrication of a rotating shaft, Archivum Mat. (Brno) 36 (2000) 239–253.
[14] S. Marušić, E. Marušić-Paloka, Two-scale convergence for thin domains and its applications to some lower-dimensional models in fluid mechanics, Asymptotic Anal. 23 (2000) 23–58.