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A new paraxial asymptotic model for the relativistic Vlasov–Maxwell equations

Un nouveau modèle paraxial asymptotique pour approcher le système d'équations de Vlasov–Maxwell

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ABSTRACT

We introduce a new paraxial asymptotic model to approximate the Vlasov–Maxwell equations. This formulation is fourth order accurate in a parameter η which denotes the ratio between the transverse characteristic velocity of the beam and the speed of light. The model is interesting because it is simpler than the complete Vlasov–Maxwell equations, and is an accurate approximation of them. This model should give an accurate, fast and easy to implement numerical method of solution.

RÉSUMÉ

On propose un nouveau modèle paraxial asymptotique pour approcher le système d'équations de Vlasov–Maxwell. Cette formulation est précise à l'ordre quatre par rapport à un paramètre η désignant le quotient de la vitesse transverse caractéristique du faisceau par rapport à celle de la lumière. L'intéret de ce modèle est qu'il est plus simple que le système complet des équations de Vlasov–Maxwell, tout en étant une approximation suffisamment précise. Ce nouveau modèle devrait conduire à une méthode numérique fiable, rapide et facile à implementer.

Version française abrégée

On considère un faisceau de particules chargées non colisionnelles se déplacant dans un champ électromagnétique. Ce système peut être modélisé par le système d'équations couplées de Vlasov–Maxwell. Cependant, sa résolution numérique s'avère souvent lourde et coûteuse. Il est donc important de pouvoir construire, pour des hypothèses physiques données, des modèles d'équations plus simples, qui approchent cependant le modèle initial avec la meilleure précision possible.

Dans cette Note, nous nous intéressons au cas d'un faisceau de particules de grande énergie. L'exemple typique que nous considérons ici est le cas d'un faisceau court de particules chargées, se déplacant avec un facteur relativiste γ élevé (voir Éq. (1)) dans un tube cylindrique parfaitement conducteur. Nous proposons alors un nouveau modèle paraxial asymptotique pour approcher le système d'équations de Vlasov–Maxwell.

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Pour dériver ce modèle, on commence par effectuer un changement de variables dans le système d'équations de Vlasov–Maxwell (1)–(4), en transformant la variable longitudinale z en la variable $\zeta = ct - z$, où c désigne la vitesse de la lumière, z l'axe du faisceau et t le temps. En décomposant les composantes de toutes les quantités (position, vitesse, champs, ...) en parties transverses et longitudinales, nous obtenons alors le système (6)–(14).

La seconde étape consiste à introduire une mise à l'échelle des équations, adaptée à la physique considérée. Pour ce faire, on suppose que le faisceau est court, que sa vitesse v_z est proche de la vitesse de la lumière c et que sa vitesse transverse $\mathbf{v}_\perp$ est petite devant c. On introduit alors un petit paramètre η défini par (15) et on déduit le système d'équations adimensionnées (18)–(25).

Ensuite, on considère les développements asymptotiques des quantités f, n, $\mathbf{j}$ et $\mathbf{E}$, $\mathbf{B}$, $\boldsymbol{\mathcal{E}}_\perp$, $\mathbf{F}$ en fonction de ce petit paramètre η. On montre ainsi que pour obtenir une approximation de la solution $f(\mathbf{x}, \mathbf{p}, t)$ de l'équation de Vlasov en $\mathcal{O}(\eta^4)$, il suffit de connaître le développement asymptotique de la force électromagnétique transverse $\mathbf{F}_\perp$ à l'ordre 2, et de la force électromagnétique longitudinale F_z à l'ordre 1. En utilisant l'expression de ces forces (26), on détermine quels sont les termes dans le développement asymptotique des champs électromagnetiques nécessaires pour fermer le système.

La dernière étape consiste alors à déterminer des équations caractéristiques pour ces champs électromagnétiques, c'est-à-dire des termes intervenant dans le développement asymptotique. C'est l'objet des lemmes 4.2–4.3 où sont caracterisés les termes d'ordre 0, 1 ou 2 des champs suivants :

$$\mathbf{E}_\perp^0,\ \mathbf{B}_\perp^0,\ \boldsymbol{\mathcal{E}}_\perp^0,\ E_z^0,\ B_z^0,\ \boldsymbol{\mathcal{E}}_\perp^1,\ E_z^1,\ B_z^1,\ \boldsymbol{\mathcal{E}}_\perp^2$$

On dérive ensuite le nouveau modèle paraxial en revenant aux variables physiques initiales (cf. Éqs. (29)–(32)). Ce nouveau modèle, après discrétisation, devrait conduire à une méthode numérique précise, rapide et facile à implémenter.

1. Introduction

Charged particle beams problems are often used in science and technology (see for instance [1]). Considering non-collisional beams, one of the most complete mathematical models is the time-dependent Vlasov–Maxwell system of equations (cf. [2]). However, the numerical solution of this model, which is unavoidable in many situations [3,4], requires a large computational effort. Therefore, whenever possible, it is worthwhile to take into account the particularities of the physical problem to derive approximate models leading to cheaper simulations (see [5–8]).

In this Note, we are interested in the case of high energy short beams. The typical example we considered here is the transport of a bunch of relativistic charged particles in the interior of a perfectly conducting hollow tube.

Following the principle exposed in [8], we derive a new paraxial asymptotic model to approximate these equations. The formulation first introduces a change of variables from z to $\zeta = ct - z$, where c denotes the speed of light, z denotes the axis of the beam and t the time. Then, one considers a scaling of the equations which reflects the characteristics of the high energy short beam. Finally, we introduce a small parameter η and we use an asymptotic expansion technique to obtain a paraxial model which is fourth order accurate in η, i.e. such that the asymptotic expansions of f in the Vlasov–Maxwell and in the paraxial model coincide up the third order in η.

The model is interesting as it is simpler than the complete Vlasov–Maxwell equations, and is an accurate approximation of them. This model should give an accurate, fast and easy to implement numerical method of solutions.

2. The Vlasov–Maxwell model

Consider a beam of charged particles with a mass m and a charge q moving in a perfectly conducting cylindrical tube, whose axis is constituted by the z-axis. We denote by Ω the transverse section of boundary Γ, $\nu = (\nu_x, \nu_y, 0)$ denoting the unit exterior normal to the tube. We suppose that an external magnetic field $\mathbf{B}^e$ confines the beam in a neighborhood of the z-axis which may be therefore chosen as the optical axis of the beam. Let $\mathbf{x} = (x, y, z)$ be the position of the particle, $\mathbf{p} = (p_x, p_y, p_z)$ its momentum and $\mathbf{v} = (v_x, v_y, v_z)$ its velocity. We assume that the beam is relativistic and non-collisional so that its distribution function $f = f(\mathbf{x}, \mathbf{p}, t)$ in the phase space $(\mathbf{x}, \mathbf{p})$ is a solution to the Vlasov equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}.\nabla_x f + \mathbf{F}.\nabla_p f = 0, \quad \text{where } \mathbf{p} = \gamma m \mathbf{v},\ \gamma = \left(1 - \frac{|\mathbf{v}|^2}{c^2}\right)^{-1/2} \tag{1}$$

which can be expressed in the position–velocity phase space by

$$\frac{\partial f}{\partial t} + \mathbf{v}.\nabla_x f + \nabla_{\mathbf{v}} \cdot \left(\frac{1}{\gamma m}\left(\mathbf{I} - \frac{1}{c^2}\mathbf{v} \otimes \mathbf{v}\right)\mathbf{F} f\right) = 0 \tag{2}$$

where $\mathbf{I}$ denotes the unit tensor.

Above, $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ denotes the electromagnetic force acting on the particles. The electric field $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$ and the magnetic field $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$ are solutions to Maxwell's equations:

$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mu_0 \mathbf{J}, & \dfrac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \\ \nabla.\mathbf{E} = \dfrac{1}{\varepsilon_0}\rho, & \nabla.\mathbf{B} = 0 \end{cases} \tag{3}$$

where the charge and the current density ρ and $\mathbf{J}$ are obtained from the distribution function f with

$$\rho = q\int f\,\mathrm{d}\mathbf{p}, \qquad \mathbf{J} = q\int \mathbf{v} f\,\mathrm{d}\mathbf{p} \tag{4}$$

We assume now that the beam is highly relativistic, i.e., satisfies $\gamma \gg 1$. Since $v_z \simeq c$ for any particle in the beam, we rewrite the Vlasov–Maxwell equations in a frame which moves along z-axis with the light velocity c. For this purpose, we set $\zeta = ct - z$, $v_\zeta = c - v_z$ and we perform the change of variables $(x, y, z, v_x, v_y, v_z, t) \to (x, y, \zeta, v_x, v_y, v_\zeta, t)$. It is also convenient to introduce the transverse quantities:

$$\mathbf{x}_\perp = (x, y), \qquad \mathbf{v}_\perp = (v_x, v_y)$$

and to define the transverse operators:

$$\mathbf{grad}_\perp \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right), \qquad \mathbf{curl}_\perp \varphi = \left(\frac{\partial \varphi}{\partial y}, -\frac{\partial \varphi}{\partial x}\right), \qquad \Delta_\perp \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}$$

where $\varphi = \varphi(x, y)$ is a scalar function. Similarly, for $\mathbf{A}_\perp = (A_x, A_y)$ denoting a transverse vector field, we set

$$div_\perp \mathbf{A}_\perp = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}, \qquad curl_\perp \mathbf{A}_\perp = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}, \qquad div_{\mathbf{v}_\perp} \mathbf{A}_\perp = \frac{\partial A_x}{\partial v_x} + \frac{\partial A_y}{\partial v_y}$$

We introduce $\mathbf{A}_\perp$ defined by $\mathbf{A}_\perp \times \mathbf{e}_z = (A_y, -A_x)$ and we readily get the following identities:

$$div_\perp(\mathbf{A}_\perp \times \mathbf{e}_z) = curl_\perp \mathbf{A}_\perp, \qquad curl_\perp(\mathbf{A}_\perp \times \mathbf{e}_z) = -div_\perp \mathbf{A}_\perp, \qquad curl_\perp \mathbf{curl}_\perp \varphi = -\Delta_\perp \varphi \tag{5}$$

Moreover, denoting by $\tau = (-\nu_y, \nu_x)$ the unit tangent along Γ, we have the relation $\mathbf{curl}_\perp \varphi \cdot \tau = -\frac{\partial \varphi}{\partial \nu}$.

Using the above notations, the Vlasov equation in the new variables can be written as

$$\begin{aligned} &\frac{\partial f}{\partial t} + \mathbf{v}_\perp \cdot \mathbf{grad}_\perp f + v_\zeta \frac{\partial f}{\partial \zeta} + div_{\mathbf{v}_\perp}\left[\frac{1}{\gamma m}\left(\left(\mathbf{I} - \frac{1}{c^2}\mathbf{v}_\perp \otimes \mathbf{v}_\perp\right)\cdot \mathbf{F}_\perp - \frac{1}{c}\left(1 - \frac{v_\zeta}{c}\right)\mathbf{v}_\perp F_z\right) f\right] \\ &\quad + \frac{\partial}{\partial v_\zeta}\left[\frac{1}{\gamma m c}\left(\left(1 - \frac{v_\zeta}{c}\right)\mathbf{v}_\perp \cdot \mathbf{F}_\perp + \left(2 - \frac{v_\zeta}{c}\right) v_\zeta F_z\right) f\right] = 0 \end{aligned} \tag{6}$$

where

$$\gamma = \left(2\frac{v_\zeta}{c} - \frac{1}{c^2}\left(|\mathbf{v}_\perp|^2 + {v_\zeta}^2\right)\right)^{-1/2} \tag{7}$$

Setting $\boldsymbol{\mathcal{E}}_\perp = (\mathcal{E}_x = E_x - cB_y, \mathcal{E}_y = E_y + cB_x)$ and $J_\zeta = \rho c - J_z = q\int v_\zeta f\,\mathrm{d}\mathbf{v}$, Ampere's law and the Poisson equation of (3) give

$$\frac{1}{c^2}\frac{\partial \mathbf{E}_\perp}{\partial t} - \mathbf{curl}_\perp B_z + \frac{1}{c}\frac{\partial \boldsymbol{\mathcal{E}}_\perp}{\partial \zeta} = -\mu_0 \mathbf{J}_\perp \tag{8}$$

$$\frac{1}{c^2}\frac{\partial E_z}{\partial t} + \frac{1}{c} div_\perp \boldsymbol{\mathcal{E}}_\perp = -\mu_0 J_\zeta \tag{9}$$

$$div_\perp \mathbf{E}_\perp - \frac{\partial E_z}{\partial \zeta} = \frac{1}{\epsilon_0}\rho \tag{10}$$

Similarly, Faraday's law and the absence of magnetic monopoles of (3), both written in the new variables become

$$\frac{\partial B_\perp}{\partial t} + \mathbf{curl}_\perp E_z + \frac{\partial}{\partial \zeta}(\boldsymbol{\mathcal{E}}_\perp \times \mathbf{e}_z) = 0 \tag{11}$$

$$\frac{\partial B_z}{\partial t} + curl_\perp \boldsymbol{\mathcal{E}}_\perp = 0 \tag{12}$$

$$div_\perp \mathbf{B}_\perp - \frac{\partial B_z}{\partial \zeta} = 0 \tag{13}$$

Finally, the electromagnetic force becomes

$$\mathbf{F}_\perp = g\big(\boldsymbol{\mathcal{E}}_\perp + (\mathbf{v}_\perp \times \mathbf{e}_z)B_z + v_\zeta(\mathbf{B}_\perp \times \mathbf{e}_z)\big), \qquad F_z = g\big(E_z + \mathbf{v}_\perp \cdot (\mathbf{B}_\perp \times \mathbf{e}_z)\big) \tag{14}$$

Let us write down the boundary conditions for the electromagnetic fields, which will be useful for the rest of the paper. We refer the reader to [8,9] for more details. On the surface on the tube, that is for $x_\perp \in \Gamma$, $\zeta \in]0, Z[$, since the tube is assumed to be a perfectly conducting one, we impose the perfect conductor boundary conditions which expressed that the tangential electric field vanished, namely

$$\mathbf{E}\cdot\tau = 0, \qquad E_z = 0$$

On the artificial boundary at $\zeta = 0$, we notice that the static external electromagnetic fields which exist ahead of the beam cannot be modified by the electromagnetic waves generated by the beam since they travel to the left in the variable $\zeta = ct - z$. Assuming that there is no external electric field, these boundary conditions are written, for $\mathbf{x}_\perp \in \Omega$, $\zeta = 0$

$$\mathbf{E} = 0, \qquad \mathbf{B} = \mathbf{B}^e$$

Similarly, on the artificial boundary $\zeta = Z$, we have to allow the outgoing waves to leave the domain. As we will see in the future derivations, this type of condition will have no influence on the paraxial model.

3. A scaling of the equations

The second step to derive a paraxial model is to introduce an *ad hoc* scaling of the equations. Assuming that we deal with a short beam, we introduce a scaling of the equations, by handling the following properties of the beam:

(i) The beam dimensions are small compared to the longitudinal length of the device, denoted L;
(ii) The longitudinal velocities v_z of the particles are close to the light velocity c;
(iii) The transverse velocities $\mathbf{v}_\perp$ of the particles are small compared to c.

Thus, we introduce the two characteristic quantities:

(i) l denotes the characteristic dimension of the beam,
(ii) $\bar{v}$ is the transverse characteristic velocity of the particles.

Now, we define a small parameter η with

$$\eta \equiv \frac{\bar{v}}{c} \ll 1 \tag{15}$$

Since the particle velocities are close to c, we have $|\mathbf{v}|^2 = |\mathbf{v}_\perp|^2 + v_z^2 = |\mathbf{v}_\perp|^2 + c^2 - 2cv_\zeta + v_\zeta^2 \simeq c^2$. Moreover, even if $|\mathbf{v}_\perp|^2$, cv_ζ and v_ζ^2 are $O(c^2)$, v_ζ^2 is negligible in comparison with cv_ζ, since $v_\zeta \ll c$, so that $2cv_\zeta \simeq |\mathbf{v}_\perp|^2$. Hence, v_ζ appears to be of the order $\frac{\bar{v}^2}{c}$ and we choose $\overline{w} = \eta^2 c$ as a characteristic longitudinal velocity of the particles in the variable ζ and consequently $\overline{w} = \eta\bar{v}$. This gives a longitudinal characteristic velocity $\overline{w}$ related to the transverse characteristic velocity $\bar{v}$ by

$$\overline{w} = \eta\bar{v}$$

In the same way, we introduce a longitudinal characteristic dimension $\overline{l_\parallel}$ different from the transverse characteristic dimension l, and satisfying the same relation, that is

$$\overline{l_\parallel} = \eta l$$

Finally, one also takes $T = \frac{l}{\bar{v}}$ as a characteristic time. Introducing now the dimensionless-independent variables $\mathbf{x}'_\perp$, ζ', t', $\mathbf{v}'_\perp$, v'_ζ defined by

$$x = lx', \qquad y = ly', \qquad \zeta = \overline{l_\parallel}\zeta', \qquad t = Tt', \qquad v_x = \bar{v}v'_x, \qquad v_y = \bar{v}v'_y, \qquad v_\zeta = \overline{w}v'_\zeta \tag{16}$$

and choosing the scaling factors $\bar{f}$, $\bar{\rho}$, $\bar{J}$, $\bar{E}$, $\bar{B}$, $\bar{F}$ as

$$\begin{cases} \bar{f} = \dfrac{\varepsilon_0 m}{q^2 l^2 \overline{\omega}}, \qquad \bar{\rho} = q\bar{f}\bar{v}^2\overline{\omega}, \qquad \bar{J} = \bar{\rho}c \\ \bar{E} = \dfrac{m\overline{v}^2}{ql}, \qquad \bar{B} = \dfrac{\bar{E}}{c}, \qquad \bar{F} = q\bar{E} \end{cases} \tag{17}$$

we look for the dependent variables f, $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{F}$ as functions of the following forms: $f(\mathbf{x}_\perp, \zeta, \mathbf{v}_\perp, v_\zeta, t) = \bar{f} f'(\mathbf{x}'_\perp, \zeta', \mathbf{v}'_\perp, v'_\zeta, t)$, $\mathbf{E}(\mathbf{x}_\perp, \zeta, t) = \bar{E}\mathbf{E}'(\mathbf{x}'_\perp, \zeta', t')'$, $\mathbf{B}(\mathbf{x}_\perp, \zeta, t) = \bar{B}\mathbf{B}'(\mathbf{x}'_\perp, \zeta', t')'$, $\mathbf{F}(\mathbf{x}_\perp, \zeta, \mathbf{v}_\perp, v_\zeta, t) = \bar{F}\mathbf{F}'(\mathbf{x}'_\perp, \zeta', \mathbf{v}'_\perp, v'_\zeta, t)$. Note that $\rho = \bar{\rho}n'$, with $n' = \int f' \,\mathrm{d}v'$, $v' = (\mathbf{v}'_\perp, v_\zeta)$ and $\mathbf{J}_\perp = \eta\bar{J}\mathbf{j}'_\perp$, $J_\zeta = \eta^2\bar{J}j'_\zeta$.

We are able now to write down the Vlasov–Maxwell equations using these dimensionless variables. Dropping the primes for simplicity, the Vlasov equation in dimensionless variables is

$$\frac{\partial f}{\partial t} + \mathbf{v}_\perp \cdot \mathbf{grad}_\perp f + v_\zeta \frac{\partial f}{\partial \zeta} + div_{\mathbf{v}_\perp}\left[\frac{1}{\gamma}\left(\left(\mathbf{I} - \eta^2 \mathbf{v}_\perp \otimes \mathbf{v}_\zeta\right) \cdot \mathbf{F}_\perp - \eta\left(1 - \eta^2 v_\zeta\right)\mathbf{v}_\perp F_z\right) f\right]$$
$$+ \frac{\partial}{\partial v_\zeta}\left[\frac{1}{\gamma}\left(\left(1 - \eta^2 v_\zeta\right)\mathbf{v}_\perp \cdot \mathbf{F}_\perp + \eta\left(2 - \eta^2 v_\zeta\right) v_\zeta F_z\right) f\right] = 0 \tag{18}$$

where, with (7), we have $\gamma = \frac{1}{\eta}(2v_\zeta - (v_\perp^2 + \eta^2 v_\zeta^2))^{-\frac{1}{2}}$.

On the other hand, Ampere's law and the Poisson equation (8)–(10) give

$$\eta \frac{\partial \mathbf{E}_\perp}{\partial t} - \mathbf{curl}_\perp B_z + \frac{1}{\eta}\frac{\partial \boldsymbol{\mathcal{E}}_\perp}{\partial \zeta} = -\eta \mathbf{j}_\perp \tag{19}$$

$$\eta \frac{\partial E_z}{\partial t} + div_\perp \boldsymbol{\mathcal{E}}_\perp = \eta^2 j_\zeta \tag{20}$$

$$div_\perp \mathbf{E}_\perp - \frac{1}{\eta}\frac{\partial E_z}{\partial \zeta} = n \tag{21}$$

whereas Faraday's law and the absence of monopoles equations (10)–(11) are written

$$\eta \frac{\partial \mathbf{B}_\perp}{\partial t} + \mathbf{curl}_\perp E_z + \frac{1}{\eta}\frac{\partial}{\partial \zeta}(\boldsymbol{\mathcal{E}}_\perp \times \mathbf{e}_z) = 0 \tag{22}$$

$$\eta \frac{\partial B_z}{\partial t} + curl_\perp \boldsymbol{\mathcal{E}}_\perp = 0 \tag{23}$$

$$div_\perp \mathbf{B}_\perp - \frac{1}{\eta}\frac{\partial B_z}{\partial \zeta} = 0 \tag{24}$$

In the above equations, the right-hand sides n, j fulfill the charge conservation equation

$$\frac{\partial n}{\partial t} + div_\perp \mathbf{j}_\perp + \frac{\partial j_\zeta}{\partial \zeta} = 0$$

The electromagnetic force $\mathbf{F}$ that we obtain has the same expression as in [8] or [9]:

$$\begin{cases} \mathbf{F}_\perp = \boldsymbol{\mathcal{E}}_\perp + \eta(\mathbf{v}_\perp \times \mathbf{e}_z)B_z + \eta^2 v_\zeta(\mathbf{B}_\perp \times \mathbf{e}_z) \\ F_z = E_z + \eta\left(\mathbf{v}_\perp \cdot (\mathbf{B}_\perp \times \mathbf{e}_z)\right) \end{cases} \tag{25}$$

We turn to the boundary conditions. The scaled electric field $\mathbf{E}$ on the perfectly conducting boundary of the tube, i.e., for $x_\perp \in \Gamma$, $\zeta \in]0, Z[$ obeys

$$\mathbf{E}_\perp \cdot \tau = 0, \qquad E_z = 0, \qquad \boldsymbol{\mathcal{E}}_\perp \cdot \tau = \mathbf{B}_\perp \cdot \nu, \qquad \left(\eta\frac{\partial}{\partial t} + \frac{1}{\eta}\frac{\partial}{\partial \zeta}\right)\mathbf{B}_\perp \cdot \nu = 0$$
$$\eta \int_\Omega \frac{\partial B_z}{\partial t}\, \mathrm{d}\mathbf{x}_\perp + \int_\Gamma B_\perp \cdot \nu \, \mathrm{d}l = 0, \qquad \int_\Omega \left(\eta\frac{\partial}{\partial t} + \frac{1}{\eta}\frac{\partial}{\partial \zeta}\right) B_z \, \mathrm{d}\mathbf{x}_\perp = 0$$

whereas, for $\mathbf{x}_\perp \in \Omega$, $\zeta = 0$, we get $\mathbf{E} = \mathbf{0}$, $\mathbf{B} = \mathbf{B}^e$ and for $\mathbf{x}_\perp \in \Omega$, $\zeta = z$, we obtain $\boldsymbol{\mathcal{E}}_\perp = 0$.

4. An asymptotic expansion

The third step to derive a paraxial model is to derive, from the scaled Vlasov–Maxwell equations, asymptotic expansions of the quantities f, n, $\mathbf{j}$ and $\mathbf{E}$, $\mathbf{B}$, $\boldsymbol{\mathcal{E}}_\perp$, $\mathbf{F}$ in powers of the small parameter η, like:

$$f = f^0 + \eta f^1 + \eta^2 f^2 + \cdots, \qquad n = n^0 + \eta n^1 + \eta^2 n^2 + \cdots, \qquad \mathbf{j} = \mathbf{j}^0 + \eta \mathbf{j}^1 + \eta^2 \mathbf{j}^2 + \cdots$$
$$\mathbf{E} = \mathbf{E}^0 + \eta \mathbf{E}^1 + \eta^2 \mathbf{E}^2 + \cdots, \qquad \mathbf{B} = \mathbf{B}^0 + \eta \mathbf{B}^1 + \eta^2 \mathbf{B}^2 + \cdots, \qquad \boldsymbol{\mathcal{E}}_\perp = \boldsymbol{\mathcal{E}}_\perp^0 + \eta \boldsymbol{\mathcal{E}}_\perp^1 + \eta^2 \boldsymbol{\mathcal{E}}_\perp^2 + \cdots$$
$$\mathbf{F} = \mathbf{F}^0 + \eta \mathbf{F}^1 + \eta^2 \mathbf{F}^2 + \cdots$$

Then, we replace *formally* in the scaled Vlasov–Maxwell equations the functions by their asymptotic expansions, and we identify the coefficients of η^0, η^1, etc. For instance for the Vlasov equation (18), we get at the zeroth order:

$$\frac{\partial f^0}{\partial t} + \mathbf{v}_\perp \cdot \mathbf{grad}_\perp f^0 + v_\zeta \frac{\partial f^0}{\partial \zeta} = 0$$

or at the first order:

$$\frac{\partial f^1}{\partial t} + \mathbf{v}_\perp \cdot \mathbf{grad}_\perp f^1 + v_\zeta \frac{\partial f^1}{\partial \zeta} + div_{\mathbf{v}_\perp}\left(\left(2v_\zeta - \mathbf{v}_\perp^2\right)^{1/2}\mathbf{F}_\perp^0 f^0\right) + \frac{\partial}{\partial v_\zeta}\left(\left(2v_\zeta - \mathbf{v}_\perp^2\right)^{1/2}\mathbf{v}_\perp \cdot \mathbf{F}_\perp^0 f^0\right) = 0$$

Writing in the same way the second and the third order (details can be found in [10]), one can prove that for determining the asymptotic expansion $f^0 + \eta f^1 + \eta^2 f^2 + \eta^3 f^3$ of the distribution function f up to the order 3 in η, it is enough to know the expansion $\mathbf{F}_\perp^0 + \eta\mathbf{F}_\perp^1 + \eta^2\mathbf{F}_\perp^2$ of the transverse electromagnetic force $\mathbf{F}_\perp$ up to the order 2 and the expansion $F_z^0 + \eta F_z^1$ of the longitudinal electromagnetic force F_z up to the order 1 only. Then, using the expressions (25) of the forces, we get:

$$\begin{cases} \mathbf{F}_\perp^0 = \boldsymbol{\mathcal{E}}_\perp^0, \\ F_z^0 = E_z^0, \end{cases} \qquad \begin{cases} \mathbf{F}_\perp^1 = \boldsymbol{\mathcal{E}}_\perp^1 + (\mathbf{v}_\perp \times \mathbf{e}_z)B_z^0, \\ F_z^1 = E_z^1 + \left(\mathbf{v}_\perp \cdot (\mathbf{B}_\perp^0 \times \mathbf{e}_z)\right), \end{cases} \qquad \mathbf{F}_\perp^2 = \boldsymbol{\mathcal{E}}_\perp^2 + (\mathbf{v}_\perp \times \mathbf{e}_z)B_z^1 + v_\zeta\left(\mathbf{B}_\perp^0 \times \mathbf{e}_z\right) \tag{26}$$

Hence, the asymptotic expressions of the forces (26) are entirely determined as soon as we know

- the principal part $\mathbf{B}_\perp^0$ of $\mathbf{B}_\perp$,
- the expansions of E_z and B_z up to the order 1, which requires E_z^0, B_z^0 and E_z^1, B_z^1,
- the expansion of $\boldsymbol{\mathcal{E}}_\perp$ up to the order 2, which requires $\boldsymbol{\mathcal{E}}_\perp^0$, $\boldsymbol{\mathcal{E}}_\perp^1$, $\boldsymbol{\mathcal{E}}_\perp^2$.

Our aim is now to determine equations that characterize these "required" electromagnetic asymptotic fields. In this Note, we will restrict ourselves to give examples of such characterizations for $\mathbf{B}_\perp^0$, $\boldsymbol{\mathcal{E}}_\perp^0$ and E_z^1, B_z^1.

Remark 4.1. We assume the external fields $(\mathbf{E}^e, \mathbf{B}^e)$ to be of the order $(\frac{mc^2}{qL}, \frac{mc}{qL})$. Setting that the characteristic length of the device L satisfies $L = cT = c\frac{l}{v} = \frac{l}{\eta}$, we note that because $\frac{mc^2}{qL} = \frac{m\bar{v}^2}{qL}\frac{l}{L}\frac{c^2}{v^2} = \frac{1}{\eta}\frac{m\bar{v}^2}{qL}$, the asymptotic expansion of $\mathbf{E}$ and $\mathbf{B}$ will formally begin with $\eta^{-1}\mathbf{E}$ and $\eta^{-1}\mathbf{B}$.

We obtain for the zeroth order parts the following lemma:

Lemma 4.2. *The transverse component $\mathbf{B}_\perp^0$ of $\mathbf{B}^0$ is the unique solution to*

$$\begin{cases} div_\perp \mathbf{B}_\perp^0 = \dfrac{\partial B_z^1}{\partial \zeta} & \text{in } \Omega \\ curl_\perp \mathbf{B}_\perp^0 = n^0 + \dfrac{\partial E_z^1}{\partial \zeta} & \text{in } \Omega \\ \mathbf{B}_\perp^0 \cdot \nu = \mathbf{B}_\perp^0 \cdot \nu_{|\zeta=0} & \text{on } \Gamma \end{cases} \tag{27}$$

On the other hand, the transverse component $\boldsymbol{\mathcal{E}}_\perp^0$ of $\boldsymbol{\mathcal{E}}^0$ is the unique solution to

$$\begin{cases} div_\perp \boldsymbol{\mathcal{E}}_\perp^0 = 0 & \text{in } \Omega \\ curl_\perp \boldsymbol{\mathcal{E}}_\perp^0 = 0 & \text{in } \Omega \\ \boldsymbol{\mathcal{E}}_\perp^0 \cdot \tau = \mathbf{B}_\perp^0 \cdot \nu_{|\zeta=0} & \text{on } \Gamma \end{cases} \tag{28}$$

Proof. Considering the zeroth order (i.e. for η^{-1}), we easily get that:

$$\frac{\partial \boldsymbol{\mathcal{E}}_\perp^0}{\partial \zeta} = 0, \qquad \frac{\partial E_z^0}{\partial \zeta} = 0, \qquad \frac{\partial}{\partial \zeta}\left(\boldsymbol{\mathcal{E}}_\perp^0 \times \mathbf{e}_z\right) = 0, \qquad \frac{\partial B_z^0}{\partial \zeta} = 0$$

whereas the boundary conditions yield:

$$\frac{\partial}{\partial \zeta}\left(\mathbf{B}_\perp^0 \cdot \nu\right) = 0, \qquad \int_\Gamma \mathbf{B}_\perp^0 \cdot \nu \, \mathrm{d}l = 0$$

which can be summarized with $\boldsymbol{\mathcal{E}}_\perp^0$, E_z^0, B_z^0 are independent of ζ, and $\frac{\partial}{\partial\zeta}(\mathbf{B}_\perp^0 \cdot \nu) = 0$ on the boundary Γ. Considering then the first order (i.e. for η^0), we obtain that:

$$-\mathbf{curl}_\perp B_z^0 + \frac{\partial \boldsymbol{\mathcal{E}}_\perp^1}{\partial \zeta} = 0, \qquad div_\perp \boldsymbol{\mathcal{E}}_\perp^0 = 0, \qquad div_\perp \mathbf{E}_\perp^0 - \frac{\partial E_z^1}{\partial \zeta} = n^0$$

and

$$\mathbf{curl}_\perp E_z^0 + \frac{\partial}{\partial \zeta}\big(\boldsymbol{\mathcal{E}}_\perp^1 \times \mathbf{e}_z\big) = 0, \qquad curl_\perp \boldsymbol{\mathcal{E}}_\perp^0 = 0, \qquad div_\perp \mathbf{B}_\perp^0 - \frac{\partial B_z^1}{\partial \zeta} = 0$$

together with the boundary conditions:

$$\mathbf{E}_\perp^0 \cdot \tau = 0, \qquad E_z^0 = 0, \qquad \boldsymbol{\mathcal{E}}_\perp^0 \cdot \tau = \mathbf{B}_\perp^0 \cdot \nu, \qquad \frac{\partial}{\partial \zeta}\mathbf{B}_\perp^1 \cdot \nu = 0, \qquad \int_\Gamma \mathbf{B}_\perp^0 \cdot \nu \, dl = 0, \qquad \int_\Omega \frac{\partial}{\partial \zeta} B_z^1 \cdot d\mathbf{x}_\perp = 0$$

Finally, one obtains the results by noticing that $div_\perp \boldsymbol{\mathcal{E}}_\perp^0 = div_\perp \mathbf{E}_\perp^0 - curl_\perp \mathbf{B}_\perp^0 = 0$ and that $curl_\perp \boldsymbol{\mathcal{E}}_\perp^0 = curl_\perp \mathbf{E}_\perp^0 - div_\perp \mathbf{B}_\perp^0 = 0$. □

These characterizations have been obtained by considering the terms in η^{-1} and η^0. By using now the second order terms (i.e. formally in η^1), we can get with similar arguments the characterizations of the longitudinal components E_z^0, B_z^0, and $\boldsymbol{\mathcal{E}}_\perp^0$. To determine now the components E_z^1, B_z^1, $\boldsymbol{\mathcal{E}}_\perp^2$, we proceed in the same way by considering the third order terms (i.e. formally in η^2). We get, for instance for E_z^1 and B_z^1

Lemma 4.3. *The first order longitudinal electric component E_z^1 is the unique solution to*

$$\begin{cases} 2\dfrac{\partial^2 E_z^1}{\partial \zeta \, \partial t} - \Delta_\perp E_z^1 = -\dfrac{\partial n^0}{\partial t} + \dfrac{\partial j_\zeta^0}{\partial \zeta} & \text{in } \Omega \\ E_z^1 = 0 & \text{on } \Gamma \\ E_z^1(\zeta = 0) = 0 \end{cases}$$

whereas the first order longitudinal magnetic component B_z^1 is the unique solution to

$$\begin{cases} 2\dfrac{\partial^2 B_z^1}{\partial \zeta \, \partial t} - \Delta_\perp B_z^1 = -curl_\perp \mathbf{j}_\perp^0 & \text{in } \Omega \\ \dfrac{\partial B_z^1}{\partial \nu} = \mathbf{j}_\perp^0 \cdot \tau + \dfrac{\partial \mathbf{B}_\perp^0}{\partial t} \cdot \nu_{|\zeta=0} \\ B_z^1(\zeta = 0) & \text{being a given function} \end{cases}$$

In this lemma, the longitudinal electric and magnetic components both satisfy an initial condition, namely $E_z^1(t=0)$ and $B_z^1(t=0)$ being equal respectively to a given function.

In order to construct the paraxial model, we need to distinguish the self-consistent fields (denoted by ^{s}) from the external ones (denoted by ^{e}). Recall briefly that, as introduced by Vlasov, a self-consistent field is a field created by all the charged particles, contrary to an external field which is chosen freely and generally imposed by an external device (magnets, accelerators, ...). From the above characterizations, one can deduce the systems of equations satisfied by the self-consistent parts of the fields. Using a decomposition of the fields into their external and self-consistent parts:

$$\mathbf{E} = \mathbf{E}^e + \mathbf{E}^s, \qquad \mathbf{B} = \mathbf{B}^e + \mathbf{B}^s, \quad \text{and consequently,} \quad \boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^e + \boldsymbol{\mathcal{E}}^s$$

one easily obtains characterizations for the self-consistent fields. Consider for example the system of Eqs. (28) in which we set to zero all the external quantities and keep only the self-consistent parts of the fields, we get the same equations except for the boundary condition where $\mathbf{B}_\perp^0 \cdot \nu_{|\zeta=0}$ is replaced by 0 so that $\boldsymbol{\mathcal{E}}_\perp^{0,s} \equiv 0$. Similarly, one can prove that $E_z^{0,s} = B_z^{0,s} \equiv 0$.

As another example, consider the case of $E_z^{1,s}$ and $B_z^{1,s}$. From Lemma 4.3, one obtains that $E_z^{1,s}$ solves the same system of equations, whereas for $B_z^{1,s}$, one has to suppress the term $\frac{\partial \mathbf{B}_\perp^0}{\partial t} \cdot \nu_{|\zeta=0}$ from the right-hand side.

Same characterizations may be easily obtained for the other components. In particular, one gets that $E_z^{0,s} = B_z^{0,s} = 0$ and $\boldsymbol{\mathcal{E}}_\perp^{0,s} = \boldsymbol{\mathcal{E}}_\perp^{1,s} = 0$. For this reason, since there is no ambiguity, we will drop the superscript of the order in the final model below, for example, $\boldsymbol{\mathcal{E}}_\perp^s$ instead of $\boldsymbol{\mathcal{E}}_\perp^{2,s}$ or E_z^s instead of $E_z^{1,s}$.

5. The paraxial model

We are now ready to introduce the paraxial model, which provides an approximation of the distribution function f which is formally fourth order accurate in η: this means that the asymptotic expansions of f in the Vlasov–Maxwell and in the paraxial model coincide up the third order in η. We will derive this model coming back to the physical variables. First note that

$$\boldsymbol{\mathcal{E}}_\perp^s \ll \mathbf{E}_\perp^s, c\mathbf{B}_\perp^s$$

Indeed, $\boldsymbol{\mathcal{E}}_\perp^{1,s}$, $\boldsymbol{\mathcal{E}}_\perp^{2,s}$ are equal to zero, whereas $\boldsymbol{\mathcal{E}}_\perp^{2,s}$ is the first non-vanishing pseudo-field. At the contrary, the magnetic field $\mathbf{B}_\perp^{0,s}$ is different from zero and depends on the charge density ρ (or n^0 in dimensionless variables).

To begin with, let us derive the equations satisfied by $\mathbf{B}_\perp^s$. From Eqs. (27), we get

$$\begin{cases} curl_\perp \mathbf{B}_\perp^s = \dfrac{1}{c\varepsilon_0}\rho + \dfrac{1}{c}\dfrac{\partial E_z^s}{\partial \zeta} & \text{in } \Omega \\ div_\perp \mathbf{B}_\perp^s = \dfrac{\partial B_z^s}{\partial \zeta} & \text{in } \Omega \\ \mathbf{B}_\perp^s \cdot \nu = 0 & \text{on } \Gamma \end{cases} \tag{29}$$

Hence, we have obtained a quasi-static model for the transverse self-consistent field $\mathbf{B}_\perp^s$. Practically, this field will be computed after that the longitudinal components E_z^s, B_z^s will be determined.

Let us now deal with these longitudinal self-consistent fields. In our paraxial model, E_z^s is a solution to a second order wave-like equation:

$$\begin{cases} 2\dfrac{\partial^2 E_z^s}{\partial\zeta\,\partial t} - c\Delta_\perp E_z^s = \dfrac{1}{\varepsilon_0}\left(-\dfrac{\partial\rho}{\partial t} + \dfrac{\partial J_\zeta}{\partial \zeta}\right) & \text{in } \Omega \\ E_z^s = 0 & \text{on } \Gamma \\ E^s_{z|\zeta=0} & \text{being a data} \end{cases} \tag{30}$$

with a given initial condition $E^s_{z|t=0}$. Similarly, the longitudinal magnetic part B_z^s solved the following system:

$$\begin{cases} 2\dfrac{\partial^2 B_z^s}{\partial\zeta\,\partial t} - c\Delta_\perp B_z^s = \mu_0 c\, curl_\perp \mathbf{J}_\perp & \text{in } \Omega \\ \dfrac{\partial B_z^s}{\partial \nu} = \mu_0 \mathbf{J}_\perp \cdot \tau & \text{on } \Gamma \\ B^s_{z|\zeta=0} = 0 & \end{cases} \tag{31}$$

with a given initial condition $B^s_{z|t=0}$. This allows to compute first the longitudinal fields E_z^s, B_z^s, then the transverse ones $\mathbf{E}_\perp^s$, $\mathbf{B}_\perp^s$. From these quantities, one can also determine the transverse pseudo-field $\boldsymbol{\mathcal{E}}_\perp^s$, by solving the quasi-static system of equations:

$$\begin{cases} div_\perp \boldsymbol{\mathcal{E}}_\perp^s = \mu_0 c J_\zeta - \dfrac{1}{c}\dfrac{\partial E_z^s}{\partial t} & \text{in } \Omega \\ curl_\perp \boldsymbol{\mathcal{E}}_\perp^s = -\dfrac{\partial B_z^s}{\partial t} & \text{in } \Omega \\ \boldsymbol{\mathcal{E}}_\perp^s \cdot \tau = 0 & \text{on } \Gamma \end{cases} \tag{32}$$

We can summarize our main result in the following theorem:

Theorem 5.1. *Eqs.* (29)–(32) *determine the triple* $(\mathbf{E}^s, \mathbf{B}^s, \boldsymbol{\mathcal{E}}_\perp^s)$ *from* $(\rho, \mathbf{J})$ *in a unique way. Moreover, the paraxial model provides an approximation of the distribution function* f *which is formally fourth order accurate in* η*, i.e., the asymptotic expansions of* f *in the Vlasov–Maxwell and in the paraxial model coincide up the third order in* η*.*

6. Conclusion

In this Note, we proposed a new asymptotic model for the relativistic Vlasov–Maxwell equations. It has been derived from a paraxial approximation of the system of equations, and is fourth order accurate in the small parameter η. The simplicity of the obtained formulation compared to the Vlasov–Maxwell equations allows us to use very simple numerical schemes, like finite-difference discretization and particle-in-cell technique. This approach would be very powerful in its ability to get an accurate, fast and easy to implement algorithm. In a more general way, we suggest that our approach could be applied to other complex models in order to derive simple but accurate models. Hence, an interesting perspective could be to compare this model to similar approximate ones, as for example introduced in [8]. A relevant method of comparison of these asymptotic models could be to use data mining techniques, as introduced in [11].

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