C. R. Mecanique 340 (2012) 629–640

Contents lists available at SciVerse ScienceDirect

Comptes Rendus Mecanique

www.sciencedirect.com

Mean velocity profiles of two-dimensional fully developed turbulent flows

Distribution de la vitesse moyenne dans un écoulement bidimensionnel turbulent

Carmine Di Nucci *, Aniello Russo Spena

Dipartimento di Ingegneria Civile, Edile-Architettura, Ambientale, University of L'Aquila, 67100 L'Aquila, Italy

ARTICLE INFO

Article history:
Received 14 May 2012
Accepted after revision 11 September 2012
Available online 29 September 2012

Keywords:
Fluid mechanics
Turbulent wall-bounded flows
Channel flows

Mots-clés :
Mécanique des fluides
Écoulement bidimensionnel turbulent

ABSTRACT

In the work presented in this Note, an Indirect Turbulence Model (ITM) is proposed to derive the mean velocity profiles in wall-bounded flows in hydraulically smooth channels having a very wide rectangular cross section. The analytical expression of the mean velocity distribution is given. The connection between the velocity distribution parameters and Reynolds' number is indicated. The thickness of the viscous sublayer is evaluated. The skin friction coefficient is computed, and the analytical expression of the turbulent viscosity coefficient is provided.
The validity of the proposed model is verified with reference to the velocity distributions, available in the literature, obtained with Direct Numerical Simulation (DNS) of Navier–Stokes' equations.
© 2012 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

RÉSUMÉ

Un modèle de turbulence a été développé pour représenter la distribution de la vitesse moyenne locale dans un écoulements bidimensionnel turbulent. Le modèle proposé permet de décrire la distribution de la vitesse moyenne (en fonction du nombre de Reynolds) dans un canal lisse à section rectangulaire très large. Le modèle permet aussi d'évaluer l'épaisseur de la sous-couche visqueuse, la viscosité turbulente et le coefficient de frottement local.
Le modèle proposé a été vérifié avec succès en utilisant les distributions de la vitesse disponibles dans la littérature et obtenues par intégration numérique directe des équations de Navier–Stokes (DNS).
© 2012 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

The mean velocity profile in wall-bounded turbulent uniform flows is usually divided into different regions: the layer close to the solid wall (viscous sublayer) is connected to the overlap layer through a buffer layer; the outer layer, located around the channel axis, finally characterizes the flow field in which the velocity distribution is mainly affected by the geometrical shape of the cross section (see, e.g., [1,2]).

* Corresponding author.
E-mail addresses: carmine.dinucci@univaq.it (C. Di Nucci), aniello.russospena@univaq.it (A. Russo Spena).

1631-0721/$ – see front matter © 2012 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
http://dx.doi.org/10.1016/j.crme.2012.09.004

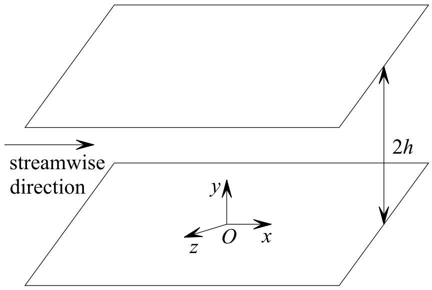

Fig. 1. Definition sketch of the flow configuration and coordinate system.

In the zones close to the solid walls of hydraulically smooth channels, some expressions have been proposed that are able to reproduce the mean velocity distribution for $0 \leqslant y^+ \leqslant 20$ [3–5], where $y^+ = yu_\tau\rho/\mu$ is the non-dimensional quantity introduced to represent the distance from the solid wall, which is defined by the distance from the wall y, the dynamic viscosity μ, the density ρ and the friction velocity u_τ; this latter variable defined by the relationship $u_\tau = (\tau_0/\rho)^{0.5}$ with τ_0 as the wall-shear stress.

The mean velocity distributions in the overlap layer are normally represented with Prandtl's logarithmic law $u^+ = (\ln y^+)/k + b$ (see, e.g., [6–8]), or with power laws such as $u^+ = cy^{+\alpha}$ (see, e.g., [9–11]), where $u^+ = u/u_\tau$ is the non-dimensional velocity, which is defined by the mean velocity u and the friction velocity u_τ; k is the von Kármán constant, and b, c and α are parametric quantities.

However, in the overlap layer, the mean velocity distribution shows a trend that is not fully in line with the logarithmic law, nor with the power law [12,13]. This observation has persuaded some authors to propose expressions for the velocity profile that are more complex than the logarithmic and power laws and based on asymptotic expansions of suitable turbulent quantities [14–18].

The velocity distribution in the buffer layer defined by $5 < y^+ < 70$ has been obtained by Gersten and Herwig [19] using an Indirect Turbulence Model (ITM) that assumes that the distribution of mean shear stresses (viscous or turbulent) in the cross section is known.

L'vov et al. [20] have proposed a turbulence model whose application makes it possible to deduce the mean velocity distribution in cross sections of smooth wall-bounded turbulent flows.

In the work presented in this Note, an Indirect Turbulence Model (ITM) is proposed to derive the mean velocity profiles in wall-bounded flows in hydraulically smooth channels having a very wide rectangular cross section. The analytical expression of the mean velocity distribution is given. The connection between the velocity distribution parameters and the Reynolds number is indicated. The thickness of the viscous sublayer is evaluated. The skin friction coefficient is computed, and the analytical expression of the turbulent viscosity coefficient is provided.

The validity of the proposed model is verified with reference to the velocity distributions, available in the literature, obtained with Direct Numerical Simulation (DNS) of Navier–Stokes' equations.

Comparisons with the turbulence model of L'vov et al. [20] and with the experimental correlations proposed by Dean [21] are finally performed.

2. Indirect turbulence model

Consider the uniform flow of a Newtonian viscous liquid in the space bounded by two horizontal and plane surfaces that are hydraulically smooth, indefinitely wide and located at a reciprocal distance $2h$. Take as the origin of an orthogonal Cartesian coordinate system x, y, z the generic point O of the surface, set at a conventional height zero. The horizontal axis x is turned in the streamwise direction, the vertical axis y is upwards, and the horizontal axis z is turned such that the triplet xyz is right-handed (Fig. 1).

With reference to the schematic situation of Fig. 1, the mean quantities are introduced: velocity u, shear stress τ_v due to the dynamic viscosity μ, total shear stress τ, and shear stress τ_t connected to the phenomenon of turbulence. The following relationships exist:

$$\mu\frac{\partial u}{\partial y} = \tau_v \tag{1}$$

$$\frac{\partial u}{\partial x} = 0 \tag{2}$$

$$\tau_v = \tau - \tau_t \tag{3}$$

$$\tau = \left(1 - \frac{y}{h}\right)\tau_0 \tag{4}$$

where τ_0 is the shear stress in $y = 0$.

As a consequence, the velocity distribution is then expressed by

$$u(y)=\int \frac{1}{\mu}\left[\left(1-\frac{y}{h}\right)\tau_0-\tau_t\right]\mathrm{d}y+u_0 \tag{5}$$

where the integration constant u_0 is determined by imposing the no-slip condition

$$u(0)=0 \tag{6}$$

Eq. (5), the solution of the differential problem (1) and (2) completed with Eqs. (3) and (4), is physically legitimate if it makes it possible to reproduce the velocity mean values shown by experience or deduced through the numerical integration of fluid dynamics equations. This remark defines the posed problem in the inverse problems category and focuses it on the search for the turbulent shear stress distribution τ_t able to reproduce the velocity mean values that characterize the flow of a viscous fluid in a given physical domain. In defining the values to give to the distribution $\tau_t(y)$, it is necessary to consider that the symmetry of the system leads to stress values $\tau_t(h)=0$ for the axis ($y=h$). It is well known that for laminar flows ($Re<Re_c$, where Re is the Reynolds number and Re_c is the critical Reynolds number), $\tau_t=0$ in the domain $[0,h]$ and that for the flows characterized by $Re\to\infty$, $\tau_t=\tau$ for $y>0$. These two extreme distributions can be considered as degenerate forms of a second-order curve such as

$$\tau_t+2B\tau_t y+Cy^2+2D\tau_t+2Ey+F=0 \tag{7}$$

where the coefficients B, C, D, E, and F depend on the flow Reynolds number. Such a dependence defines the validity limits of Eq. (7) in the finite values field $Re>Re_c$.

From the condition $\tau_t(h)=0$, it follows that

$$F=-Ch^2-2Eh \tag{8}$$

3. System identification

The search for the turbulent shear stress distribution τ_t can be obtained with the identification theory methods. Introduced the usual non-dimensional quantities $u^+=u/u_\tau$ and $y^+=y\rho u_\tau/\mu=\tilde{y}Re_\tau$, where $\tilde{y}=y/h$ and $Re_\tau=y^+_{\max}=h\rho u_\tau/\mu$ is Reynolds' friction number, Eqs. (1), (3) and (7) become

$$\frac{\mathrm{d}u^+}{\mathrm{d}\tilde{y}}=\tilde{\tau}_v \tag{9}$$

$$\tilde{\tau}_v=Re_\tau(1-\tilde{y})-\tilde{\tau}_t \tag{10}$$

$$\tilde{\tau}_t^2+2\tilde{B}\tilde{\tau}_t\tilde{y}+\tilde{C}\tilde{y}^2+2\tilde{D}\tilde{\tau}_t+2\tilde{E}\tilde{y}+\tilde{F}=0 \tag{11}$$

where $\tilde{\tau}_v=\tau_v\frac{h}{\mu u_\tau}$, $\tilde{\tau}_t=\tau_t\frac{h}{\mu u_\tau}$, $\tilde{B}=B\frac{h^2}{\mu u_\tau}$, $\tilde{C}=Ch^2$, $\tilde{D}=D\frac{h}{\mu u_\tau}$, $\tilde{E}=Eh$, and $\tilde{F}=-\tilde{C}-2\tilde{E}$.

Put $\lambda_1=2\tilde{B}\tilde{y}+2\tilde{D}$ and $\lambda_2=\tilde{C}\tilde{y}^2+2\tilde{E}\tilde{y}+\tilde{F}$, solving Eq. (11) for $\tilde{\tau}_t$ (with the condition $0\leqslant\tilde{\tau}_t\leqslant Re_\tau$) yields the symbolic expression

$$\tilde{\tau}_t=-\frac{\lambda_1}{2}-\sqrt{\frac{\lambda_1^2}{4}-\lambda_2}=\Im(\tilde{B},\tilde{C},\tilde{D},\tilde{E},\tilde{y}) \tag{12}$$

With Eqs. (10) and (12), Eq. (9) becomes

$$\frac{\mathrm{d}u^+}{\mathrm{d}\tilde{y}}=\left[(1-\tilde{y})Re_\tau-\Im(\tilde{B},\tilde{C},\tilde{D},\tilde{E},\tilde{y})\right] \tag{13}$$

Integrating this equation with the no-slip condition (6) gives [22,23]

$$u^+=(\varphi_1+\psi_1/2)y^+/Re_\tau+\varphi_2\left(y^+/Re_\tau\right)^2+\varphi_3\psi_3+\varphi_4\ln(\psi_2) \tag{14}$$

where

$$\psi_1=\sqrt{P_1\left(y^+/Re_\tau\right)^2+P_2y^+/Re_\tau+P_3} \tag{15}$$

$$\psi_2=(P_2/\sqrt{P_1}+2\sqrt{P_3})/\left\{\left(2P_1y^+/Re_\tau+P_2\right)/\sqrt{P_1}+2\psi_1\right\} \tag{16}$$

$$\psi_3=\psi_1-\sqrt{P_3} \tag{17}$$

$$\varphi_1=Re_\tau+\tilde{D} \tag{18}$$

$$\varphi_2=(\tilde{B}-Re_\tau)/2 \tag{19}$$

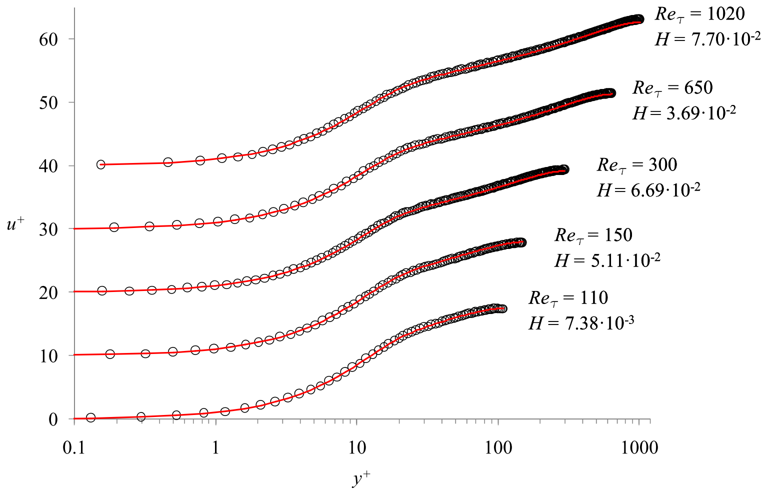

Fig. 2. Mean velocity profiles deduced with the proposed model (—); mean velocity obtained with the DNS technique (○) [25–27]. The plots are shifted vertically by 10 units. The symbol H indicates Hellinger's distance (see Section 4).

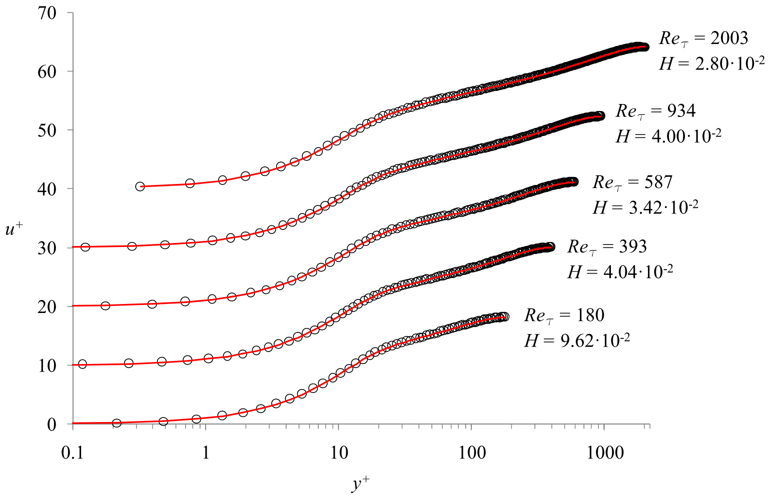

Fig. 3. Mean velocity profiles deduced with the proposed model (—); mean velocity obtained with the DNS technique (○) [28–30]. The plots are shifted vertically by 10 units.

$$\varphi_3 = (\sqrt{P_1}P_2)/(4P_1^{3/2}) \tag{20}$$

$$\varphi_4 = (P_2^2 - 4P_1P_3)/(8P_1^{3/2}) \tag{21}$$

$$P_1 = \tilde{B}^2 - \tilde{C} \tag{22}$$

$$P_2 = 2(\tilde{B}\tilde{D} - \tilde{E}) \tag{23}$$

$$P_3 = \tilde{D}^2 + \tilde{C} + 2\tilde{E} \tag{24}$$

The relationships between the coefficients that define the velocity distribution (14) and Reynolds' friction number are expressed as

$$\tilde{B} = Re_\tau(1 - f_1) \tag{25}$$

$$\tilde{C} = \frac{Re_\tau^2 - \tilde{B}Re_\tau}{f_2} - Re_\tau^2 + 2\tilde{B}Re_\tau \tag{26}$$

$$\tilde{D} = -\frac{1}{Re_\tau}\left(\frac{Re_\tau^2 - \tilde{B}Re_\tau}{f_3} + Re_\tau^2\right) \tag{27}$$

$$\tilde{E} = \frac{Re_\tau^2 - \tilde{B}Re_\tau}{f_4} + Re_\tau^2 - \tilde{B}Re_\tau + \tilde{D}Re_\tau \tag{28}$$

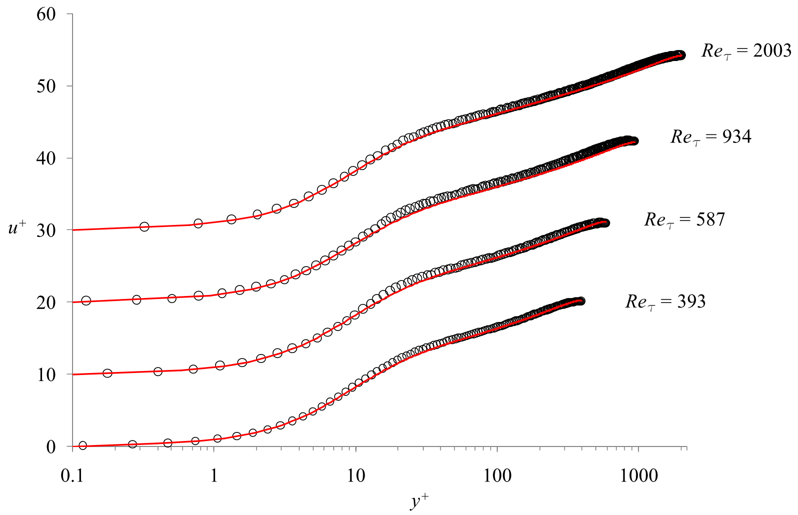

Fig. 4. Mean velocity profiles deduced with the proposed model (○); mean velocity obtained with the turbulence model of L'vov et al. [20] (—). The plots are shifted vertically by 10 units.

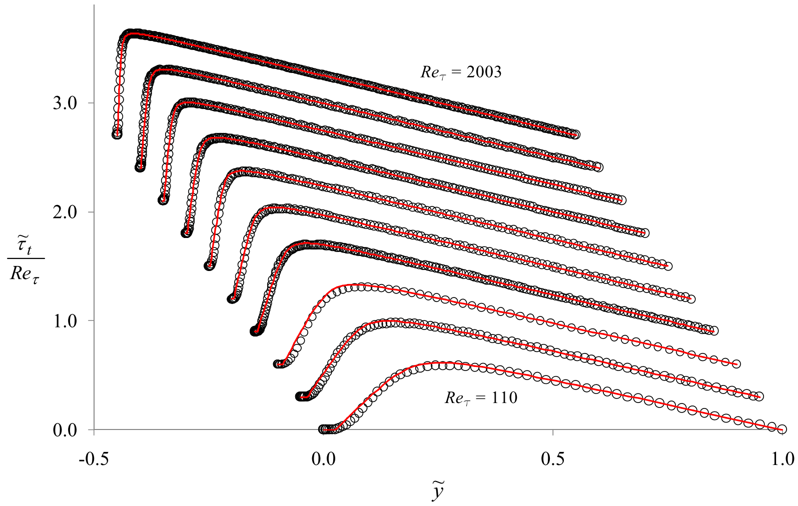

Fig. 5. Turbulent shear stresses deduced with the proposed model (—); turbulent shear stresses obtained using the DNS technique (○). The plots are shifted vertically by 0.3 units and horizontally by −0.05 units. $Re_\tau = 110, 150, 180, 300, 393, 587, 650, 934, 1020, 2003$.

with

$$f_1 = \left(3.655Re_\tau^2 + 25\,704.994Re_\tau - 55\,013.808\right) \cdot 10^{-6} \tag{29}$$

$$f_2 = \left(6.991Re_\tau^2 + 39\,476.172Re_\tau + 2\,873\,405.419\right) \cdot 10^{-6} \tag{30}$$

$$f_3 = \left(-7.490Re_\tau^2 - 49\,231.626Re_\tau + 556\,178.423\right) \cdot 10^{-6} \tag{31}$$

$$f_4 = \left(-23.766Re_\tau^2 - 82\,908.798Re_\tau - 4\,325\,049.776\right) \cdot 10^{-6} \tag{32}$$

These results have been obtained with methods based on the theory of optimal control [24], using the velocity distribution available in the literature that has been deduced with the numerical procedure DNS for $Re_\tau = 110, 150, 300, 650$, and 1020 [25–27] (Fig. 2).

4. Model validation and results analysis

The validity of the proposed indirect turbulence model has been verified with reference to the numerical results deduced with the DNS technique for $Re_\tau = 180, 393, 587, 934$, and 2003 [28–30] (Fig. 3). The good agreement between the velocity

Table 1
Hellinger's distance $H(\tilde{\tau}_{tDNS}||\tilde{\tau}_{tITM})$ between the DNS and theoretical turbulent shear data.

Re_τ	H
110	5.50×10^{-2}
150	8.91×10^{-2}
180	1.35×10^{-1}
300	1.50×10^{-1}
393	1.33×10^{-1}
587	1.35×10^{-1}
650	1.23×10^{-1}
934	1.08×10^{-1}
1020	4.69×10^{-2}
2003	1.21×10^{-1}

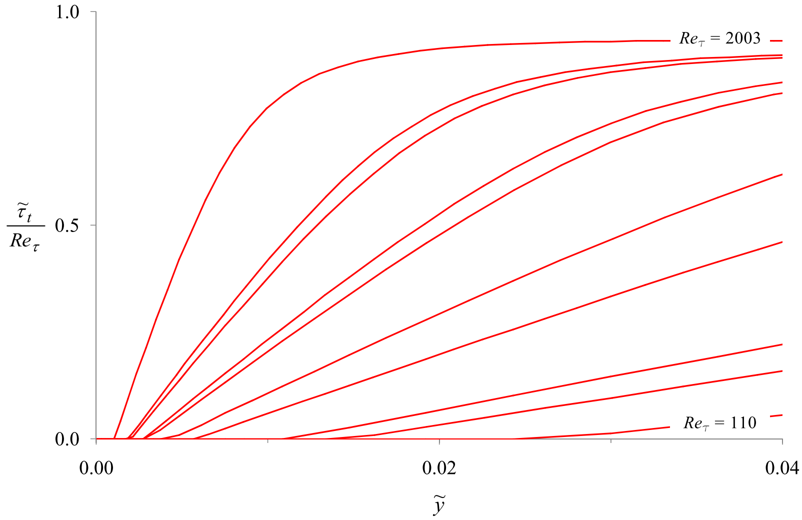

Fig. 6. Turbulent shear stresses deduced with the proposed model for $Re_\tau = 110, 150, 180, 300, 393, 587, 650, 934, 1020, 2003$.

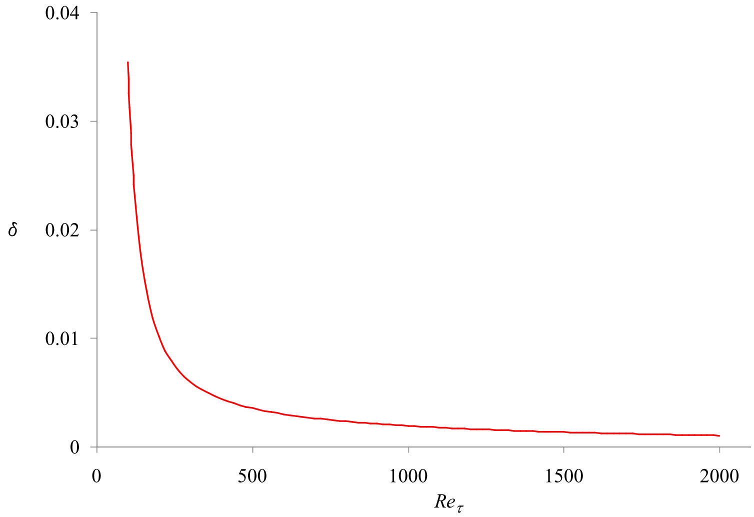

Fig. 7. Thickness of the viscous sublayer $\tilde{y} = \delta(Re_\tau)$ deduced with the proposed model.

distribution deduced with the adopted approach and the one obtained with the DNS technique is testified to by the values taken from Hellinger's distance, defined as

$$H(u^+_{DNS}||u^+_{ITM}) = \sum_{y^+=0}^{Re_\tau} \{[u^+_{DNS}(y^+)]^{0.5} - [u^+_{ITM}(y^+)]^{0.5}\}^2 \tag{33}$$

Table 2
The average and the maximum error between the DNS and theoretical mean velocity data in the range $\delta < \tilde{y} \leqslant 1$.

Re_τ	δ	Maximum error	Average error
110	2.91×10^{-2}	0.80%	0.49%
150	1.60×10^{-2}	4.87%	1.14%
180	1.21×10^{-2}	3.52%	0.46%
300	6.05×10^{-3}	3.82%	0.88%
393	4.51×10^{-3}	4.09%	0.83%
587	3.08×10^{-3}	2.32%	0.67%
650	2.84×10^{-3}	4.02%	0.56%
934	2.05×10^{-3}	5.21%	0.58%
1020	1.91×10^{-3}	4.79%	0.70%
2003	1.04×10^{-3}	4.68%	0.33%

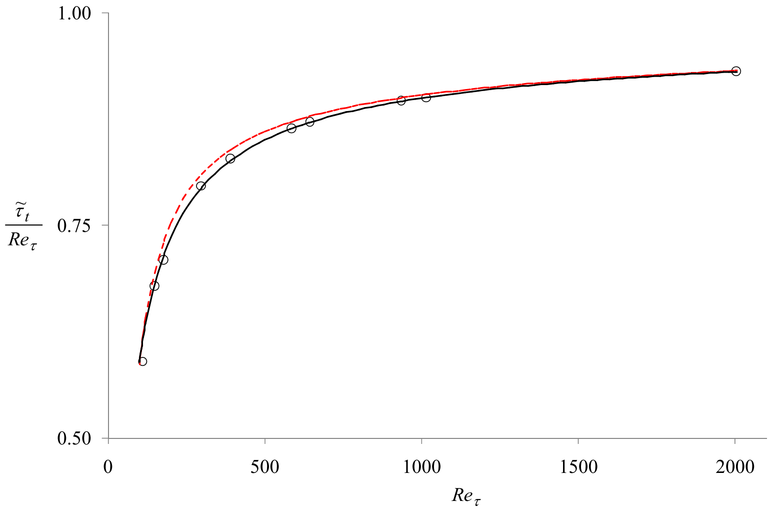

Fig. 8. Maximum turbulent shear stress deduced with: proposed model (– –); turbulence model of L'vov et al. [20] (—); DNS technique (○) for $Re_\tau = 110, 150, 180, 300, 393, 587, 650, 934, 1020, 2003$.

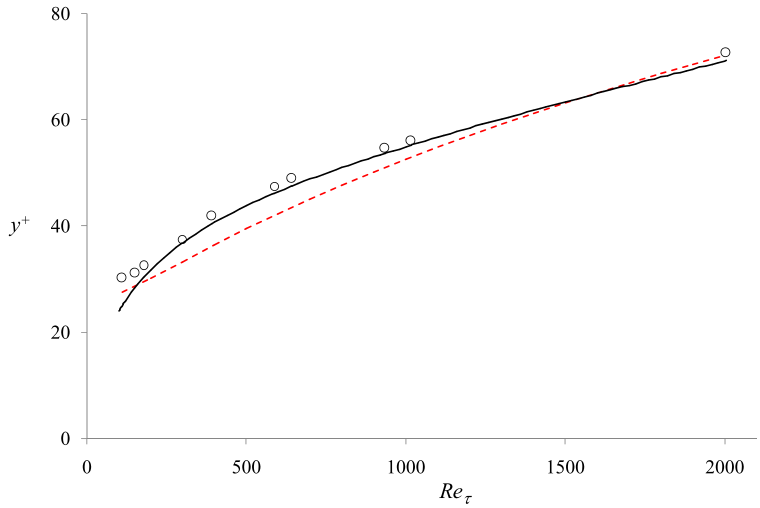

Fig. 9. Location of the maximum turbulent shear stress deduced with: proposed model (– –); turbulence model of L'vov et al. [20] (—); DNS technique (○) for $Re_\tau = 110, 150, 180, 300, 393, 587, 650, 934, 1020, 2003$.

where $u^+_{DNS}(y^+)$ is the velocity distribution computed with the DNS technique and $u^+_{ITM}(y^+)$ is the velocity distribution obtained with the proposed indirect turbulence model. Hellinger's distance is a good indicator to evaluate how the two velocity distributions resemble each other [31–33].

Fig. 4 shows the comparison of the mean velocity profiles deduced with the adopted approach with the one obtained with the turbulence model of L'vov et al. [20] for $Re_\tau = 393, 587, 934$, and 2003.

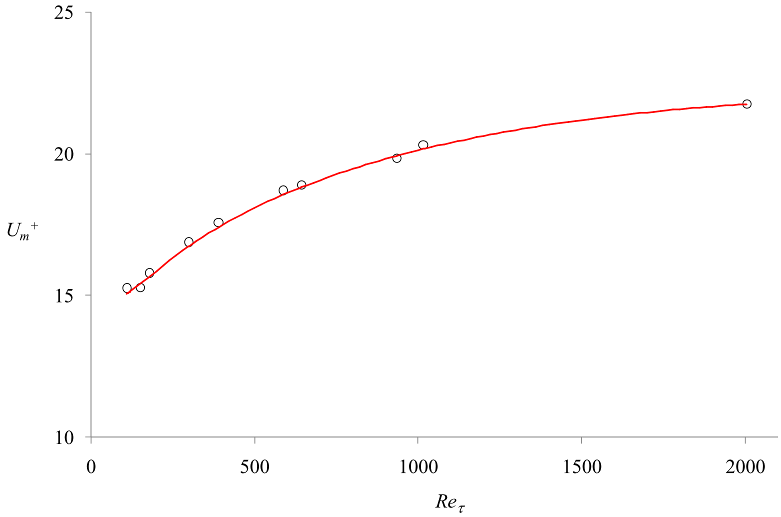

Fig. 10. Mean bulk velocity (normalized by friction velocity) U_m^+ deduced with the proposed model (—) and with the DNS technique (○) for $Re_\tau = 110, 150, 180, 300, 393, 587, 650, 934, 1020, 2003$.

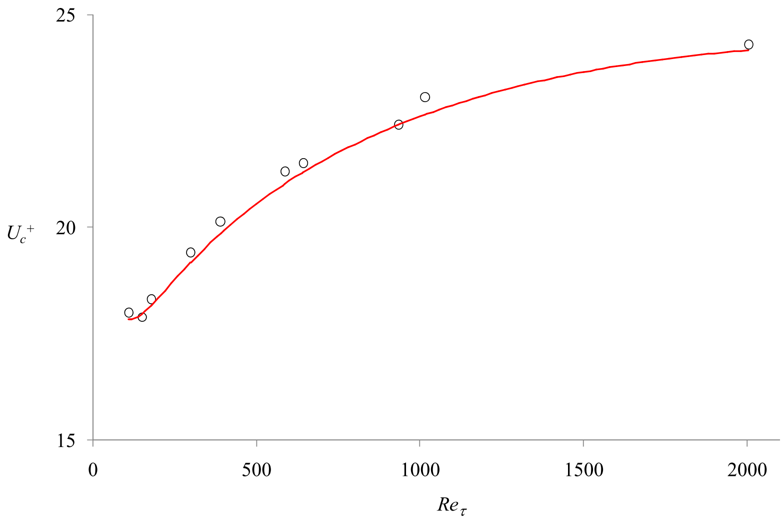

Fig. 11. Mean centerline velocity (normalized by friction velocity) U_c^+ deduced with the proposed model (—) and with the DNS technique (○) for $Re_\tau = 110, 150, 180, 300, 393, 587, 650, 934, 1020, 2003$.

Fig. 5 shows the distribution of the stresses $\tilde{\tau}_t(\tilde{y})$ deduced with the adopted approach and the one obtained with the DNS technique. The agreement between the DNS and theoretical turbulent shear data is testified to by the values taken from Hellinger's distance $H(\tilde{\tau}_{tDNS}||\tilde{\tau}_{tITM})$ (Table 1). The analysis of Fig. 6 highlights the fact that the turbulent shear stress computed with the proposed model takes nil values near the wall in a region referred to as the viscous sublayer, whose thickness $\tilde{y} = \delta$ is a function of Re_τ (Fig. 7).

The average and the maximum error between the DNS and theoretical mean velocity data in the range $\delta < \tilde{y} \leqslant 1$ are reported in Table 2.

The maximum turbulent shear stress, as well as the location of the maximum turbulent shear stress, deduced with the proposed model are in agreement with the one obtained using the DNS technique and the turbulence model of L'vov et al. [20] (Figs. 8 and 9).

The agreement between the DNS and theoretical data is also verified with reference to: mean bulk velocity, normalized by the friction velocity, defined as $U_m^+ = \int_0^1 u^+ \, \mathrm{d}\tilde{y}$ (Fig. 10); mean centerline velocity, normalized by the friction velocity, U_c^+ (Fig. 11); skin friction coefficient based on mean bulk velocity U_m, $C_f(U_m) = \tau_0 / \frac{1}{2}\rho U_m^2$ (Fig. 12); skin friction coefficient based on the centerline velocity U_c, $C_f(U_c) = \tau_0 / \frac{1}{2}\rho U_c^2$ (Fig. 13); Reynolds' number based on the mean bulk velocity U_m and the full channel width, $Re_m = 2hU_m\rho/\mu$ (Fig. 14).

The skin friction coefficient $C_f(U_m)$ computed with the proposed model is also in agreement with Dean's suggested correlation of $C_f = 0.073 Re_m^{-0.25}$ [21] (Fig. 15).

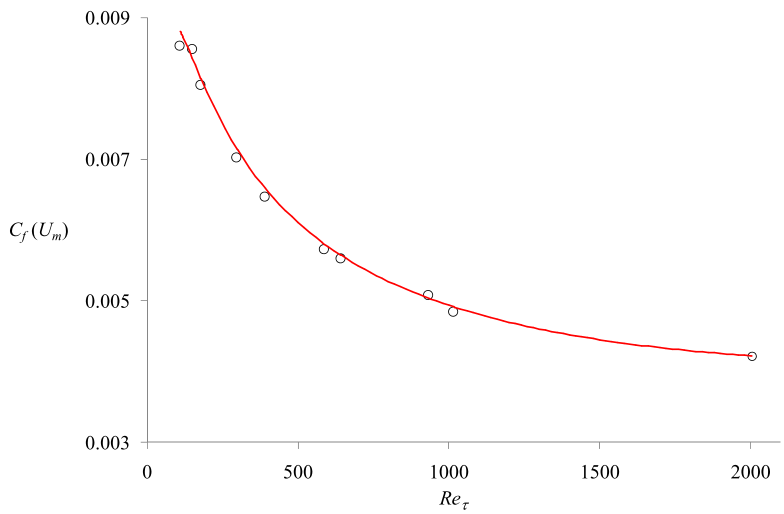

Fig. 12. Skin friction coefficient $C_f(U_m)$ deduced with the proposed model (—) and with the DNS technique (○) for $Re_\tau = 110, 150, 180, 300, 393, 587, 650, 934, 1020, 2003$.

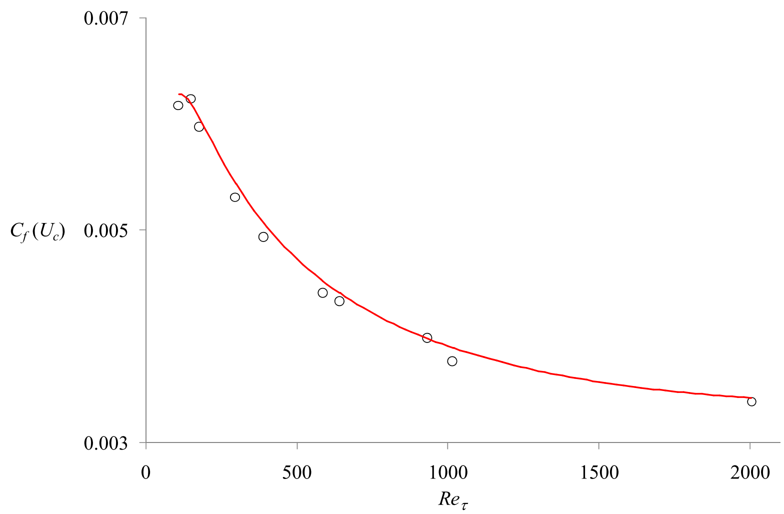

Fig. 13. Skin friction coefficient $C_f(U_c)$ deduced with the proposed model (—) and with the DNS technique (○) for $Re_\tau = 110, 150, 180, 300, 393, 587, 650, 934, 1020, 2003$.

The application of the proposed model makes it possible to obtain the analytical expression of the turbulent viscosity coefficient $\tilde{\mu}$ defined by the relationship

$$\tilde{\mu}\frac{\mathrm{d}u^+}{\mathrm{d}\tilde{y}} = \tilde{\tau}_t \qquad (34)$$

From Eqs. (12) and (13), it follows that

$$\tilde{\mu} = \frac{\Im(\tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{y})}{(1-\tilde{y})Re_\tau - \Im(\tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{y})} \qquad (35)$$

Fig. 16 shows the viscosity coefficient distributions $\tilde{\mu}$ deduced with the proposed model.

The mean velocity distribution computed using the parametric equation (14) shows a trend that is not in line with the Prandtl's logarithmic law, and consequently the proposed model does not allow to evaluate the von Kármán constant. In fact, contrary to the logarithmic law, the diagnostic function $g(y^+) = y^+\,\mathrm{d}u^+/\mathrm{d}y^+$ [34] deduced using Eq. (13) does not assume constant values in significant portions of the flow field (Fig. 17). This result is in conformity with the observations of Zanoun et al. [35], according to which the logarithmic law is a good representation of the mean velocity distribution in the overlap layer for $Re_\tau > 2000$.

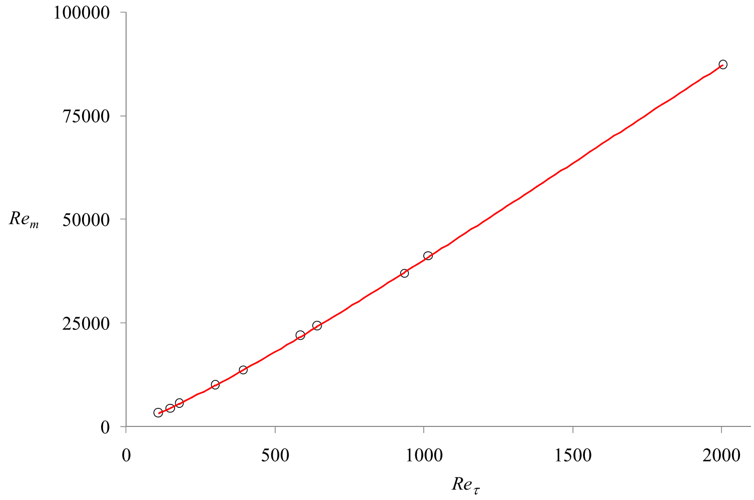

Fig. 14. Reynolds' number (based on the mean bulk velocity) Re_m deduced with the proposed model (—) and with the DNS technique (○) for $Re_\tau =$ 110, 150, 180, 300, 393, 587, 650, 934, 1020, 2003.

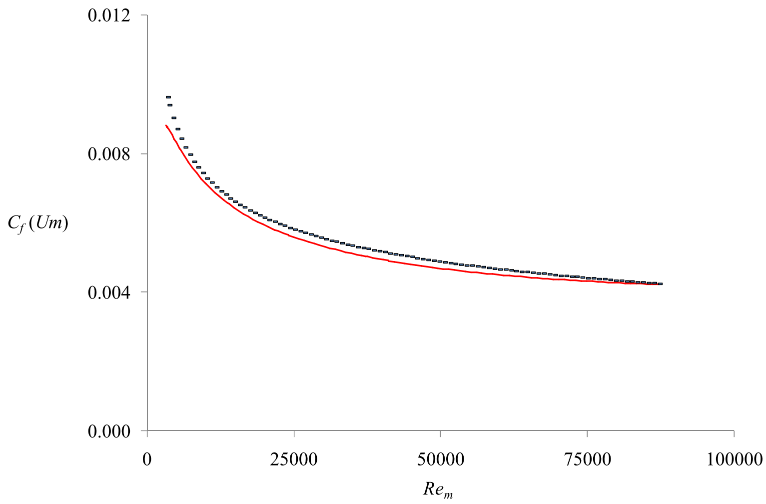

Fig. 15. Skin friction coefficient $C_f(U_m)$ deduced with the proposed model (—) and with the Dean's correlation (– –) [21].

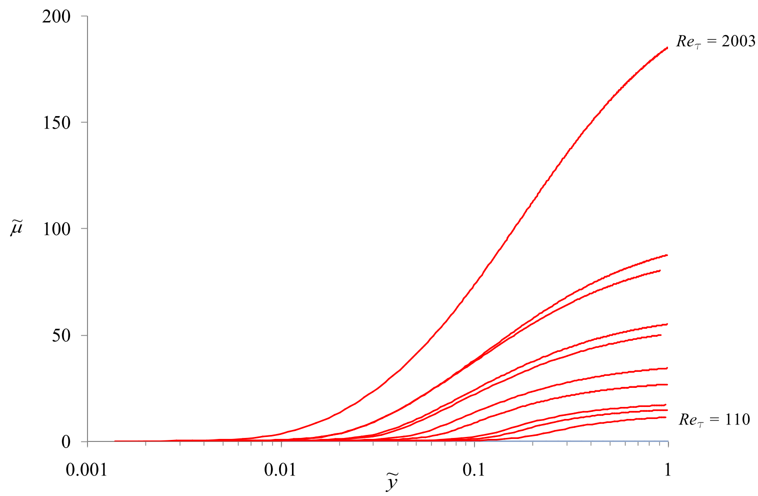

Fig. 16. Turbulent viscosity coefficient $\tilde{\mu}$ deduced with the proposed model for $Re_\tau = 110, 150, 180, 300, 393, 587, 650, 934, 1020, 2003$.

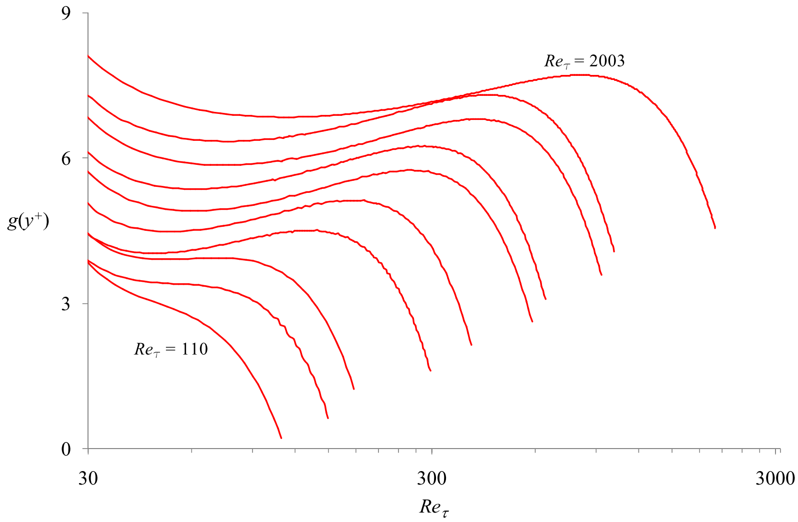

Fig. 17. Diagnostic function $g(y^+) = y' \mathrm{d}u^+ \mathrm{d}y^+$ [34] deduced with the proposed model for $Re_\tau = 110, 150, 180, 300, 393, 587, 650, 934, 1020, 2003$. The plots are shifted vertically by 0.5 units.

5. Conclusions

This short Note is inspired by the works of Di Nucci et al. [22,23]. In these works, the parametric equation (14) has been proposed to represent the mean velocity distribution in wall-bounded flows in hydraulically smooth channels having a very wide rectangular cross section.

After giving a predictive character to Eq. (14) by specifying the connection between the velocity distribution parameters and the Reynolds number, this work evaluates the thickness of the viscous sublayer, provides the analytical expression of the turbulent viscosity coefficient, gives the skin friction coefficient.

In the range $110 \leqslant Re_\tau \leqslant 2003$, the validity of the proposed model has been verified with reference to the velocity distributions, available in the literature, obtained with Direct Numerical Simulation (DNS) of Navier–Stokes' equations.

Acknowledgements

The authors desire to thank R.D. Moser, J. Kim, N.N. Mansour, K. Iwamoto, Y. Suzuki, N. Kasagi, J.C. del Álamo, J. Jiménez, P. Zandonade, S. Hoya, H. Abe, H. Kawamura, Y. Matsuo for providing the DNS data.

References

[1] F.M. White, Fluid Mechanics, fourth edition, McGraw–Hill, New York, 2010.
[2] J.H. Spurk, N. Aksel, Fluid Mechanics, second edition, Springer-Verlag, Berlin, 2008.
[3] G. Bucci, E. Fiorucci, A. Russo Spena, C. Di Nucci, Analysis of turbulent flow speed profiles in pressure pipes using the dissimilar similitude technique applied to an electrolytic tank: implementation and experimental characterization, IEEE Trans. Instr. Meas. 57 (2008) 1547–1553.
[4] R. Absi, A simple eddy viscosity formulation for turbulent boundary layers near smooth walls, C. R. Mecanique 337 (2009) 158–165.
[5] C. Di Nucci, E. Fiorucci, Mean velocity profiles of fully-developed turbulent flows near smooth walls, C. R. Mecanique 339 (2011) 388–395.
[6] B.J. McKeon, F.L. Morrison, J. Li, W. Jiang, A.J. Smits, Further observations on the mean velocity in fully-developed pipe flow, J. Fluid Mech. 501 (2004) 135–147.
[7] E.-S. Zanoun, Answers to some open questions in wall-bounded laminar and turbulent shear flows, PhD thesis, Lehrstuhl für Strömungsmechanik, Universität Erlangen–Nürnberg, 2003.
[8] O.A.B. Saleh, Fully developed turbulent smooth and rough channel and pipe flows, PhD thesis, Lehrstuhl für Strömungsmechanik, Universität Erlangen–Nürnberg, 2005.
[9] G.I. Barenblatt, Scaling laws for fully developed turbulent shear flows. Part 1: Basic hypotheses and analysis, J. Fluid Mech. 248 (1993) 513–520.
[10] G.I. Barenblatt, V.M. Prostokishin, Scaling laws for fully developed turbulent shear flows. Part 2: Processing of experimental data, J. Fluid Mech. 248 (1993) 521–529.
[11] N. Afzal, A. Seena, A. Bushra, Power law velocity profile in fully developed turbulent pipe and channel flows, J. Hydraulic Eng. 133 (2007) 1080–1086.
[12] M.H. Buschmann, M. Gad-el-Hak, Evidence of nonlogarithmic behavior of turbulent channel and pipe flow, AIAA J. 47 (2009) 535–541.
[13] M.V. Zagarola, A.J. Smits, Scaling of the mean velocity profile of turbulent pipe flow, Phys. Rev. Lett. 78 (1997) 239–242.
[14] M.H. Buschmann, M. Gad-el-Hak, Generalized logarithmic law and its consequences, AIAA J. 41 (2003) 40–48.
[15] M.H. Buschmann, M. Gad-el-Hak, Recent developments in scaling of wall-bounded flows, Prog. Aerospace Sci. 42 (2007) 419–467.
[16] W.K. George, Is there a universal log law for turbulent wall-bounded flows?, Phil. Trans. R. Soc. A 365 (2007) 789–806.
[17] R.L. Panton, Composite asymptotic expansions and scaling wall turbulence, Phil. Trans. R. Soc. A 365 (2007) 733–754.
[18] H.M. Nagib, K.A. Chauhan, Variations of von Kármán coefficient in canonical flows, Phys. Fluids 20 (2008) 101518.
[19] K. Gersten, H. Herwig, Strömungsmechanik, Fundamentals and Advances in the Engineering Science, Verlag Vieweg, Wiesbaden, 1992.
[20] V.S. L'vov, I. Procaccia, O. Rudenko, Universal model of finite Reynolds number turbulent flow in channels and pipes, Phys. Rev. Lett. 100 (2008) 054504.

[21] R.B. Dean, Reynolds number dependence of skin friction and other bulk flow variables in two-dimensional rectangular duct flow, J. Fluids Eng. 100 (1978) 215–223.
[22] C. Di Nucci, A. Russo Spena, Un modello indiretto di turbolenza per la deduzione della distribuzione di velocità in condotti idraulicamente lisci. Parte I – Presupposto ad indagini numeriche, in: Proc. 31st Convegno di Idraulica e Costruzioni Idrauliche, Perugia, Italy, 2008, p. 395 (CD-ROM).
[23] C. Di Nucci, M. Petrilli, Un modello indiretto di turbolenza per la deduzione della distribuzione di velocità in condotti idraulicamente lisci. Parte II – Determinazione dei parametri della distribuzione di velocità, in: Proc. 31st Convegno di Idraulica e Costruzioni Idrauliche, Perugia, Italy, 2008, p. 396 (CD-ROM).
[24] A. Quarteroni, Modellistica Numerica per Problemi Differenziali, fourth edition, Springer, Milano, 2008.
[25] K. Iwamoto, Y. Suzuki, N. Kasagi, Reynolds number effect on wall turbulence: toward effective feedback control, Int. J. Heat Fluid Flow 23 (2002) 678–689.
[26] K. Iwamoto, Database of fully developed channel flow, THTLAB Internal Report No. ILR-0201, Dept. Mech. Eng., Univ. Tokyo, 2002.
[27] H. Abe, H. Kawamura, Y. Matsuo, Surface heat-flux fluctuations in a turbulent channel flow up to $Re_\tau = 1020$ with $Pr = 0.025$ and 0.71, Int. J. Heat Fluid Flow 25 (2004) 404–419.
[28] R.D. Moser, J. Kim, N.N. Mansour, Direct numerical simulation of turbulent channel flow up to $Re_\tau = 590$, Phys. Fluids 11 (1999) 943–945.
[29] J.C. del Álamo, J. Jiménez, P. Zandonade, R.D. Moser, Scaling of the energy spectra of turbulent channels, J. Fluid Mech. 500 (2004) 135–144.
[30] S. Hoyas, J. Jiménez, Scaling of the velocity fluctuations in turbulent channels up to $Re_\tau = 2003$, Phys. Fluids 18 (2006) 011702.
[31] A.N. Shiryaev, Probability, second edition, Springer-Verlag, New York, 1996.
[32] H. Heyer, Theory of Statistical Experiments, Springer-Verlag, New York, 1982.
[33] Y. Tsuji, I. Nakamura, Probability density function in the log-law region of low Reynolds number turbulent boundary layer, Phys. Fluids 11 (1999) 647–658.
[34] J.M. Österlund, A.V. Johansson, H.M. Nagib, M.H. Hites, A note on the overlap region in turbulent boundary layers, Phys. Fluids 12 (2000) 1–4.
[35] E.-S. Zanoun, F. Durst, H. Nagib, Evaluating the law of the wall in two-dimensional fully developed turbulent channel flows, Phys. Fluids 15 (2003) 3079–3089.