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Hierarchical homogenization of fluid saturated porous solid with multiple porosity scales

Homogénéisation hiérarchique de solides poreux élastiques saturés par un fluide avec des échelles multiples de porosités

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ABSTRACT

In this Note, we investigate the macroscopic response of an elastic porous skeleton subjected to a mechanical loading. The pores of two different sizes are filled with a compressible fluid which can redistribute at both the microscopic and mesoscopic scales since the pores form one system of connected network. We apply the asymptotic homogenization method to upscale a microscopic fluid–structure interaction problem. The obtained poroelastic model describes the matrix behavior at the mesoscopic level. The homogenization procedure is repeated to give rise a double-porosity model relevant to the macroscopic scale. We discuss relationships obtained with the standard Biot poroelasticity theory. As an advantage, the approach reported here provides a direct procedure to calculate the effective properties for any 3D microstructure without any restrictions concerning the shape or topology of the pore network. Note that for the particular case of disconnected networks, i.e. with occluded pores, the model can be adapted easily.
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RÉSUMÉ

Dans cet article, nous étudions la réponse macroscopique d'un squelette poreux élastique soumis à des charges mécaniques. Les pores, présentant deux tailles différentes et formant un système de canaux connectés, sont saturés par un fluide compressible qui peut ainsi être redistribué à la fois aux échelles microscopique et mésoscopique. Dans un premier temps, nous appliquons une méthode d'homogénéisation asymptotique à l'échelle microscopique pour traiter le problème d'interaction entre les phases solide et fluide. Le modèle poroélastique obtenu décrit le comportement de la matrice au niveau mésoscopique. La procédure d'homogénéisation est répétée pour aboutir à un modèle à double porosité pertinent à l'échelle macroscopique. Les relations obtenues sont discutées en comparaison avec la théorie de la poroélasticité standard initiée par Biot. Un des avantages de la présente approche réside dans le développement d'une procédure directe de calcul des propriétés effectives pour toute microstructure tridimensionnelle sans aucune

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restriction concernant la forme ou la topologie du réseau de pores. Notons que dans le cas particulier d'un réseau de pores occlus, le modèle peut être adapté relativement facilement.
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Version française abrégée

Nous considérons l'état d'équilibre d'un milieu poreux déformable saturé par un fluide dont l'écoulement est contrôlé par le flux sur la frontière du domaine qu'il occupe. Le réseau poreux de ce milieu est caractérisé à deux niveaux puisqu'il offre deux tailles différentes de pores, les pores pouvant être interconnectés ou déconnectés par une phase imperméable solide.

Les deux niveaux de pores sont indexés par les exposants α et β respectivement associés aux échelles microscopique et mésoscopique. À l'échelle microscopique, nous considérons une phase solide élastique appelée « matrice » contenant des canaux saturés par un fluide qui peut être drainé vers l'extérieur du domaine ou vers le niveau de porosité plus élevée. Les lois de comportement sont supposées être linéairement élastique pour la matrice et newtonien compressible pour le fluide.

La procédure d'homogénéisation de ce milieu à deux phases permet d'obtenir un modèle décrivant le comportement poroélastique homogénéisé à l'échelle intermédiaire.

Ainsi, à l'échelle mésoscopique, ce modèle décrit le matériau occupé par la matrice de la mésostructure du niveau β. A ce niveau β, la phase fluide occupant les pores peut communiquer avec les canaux de la microporosité du niveau α, ou peut être complètement isolée. Ainsi, deux cas peuvent être traités :

- *Double porosité connectée* : les porosités aux différents niveaux sont connectées ; cela entraîne qu'une seule valeur de pression de fluide caractérise l'état d'équilibre.
- *Double porosité déconnectée* : les pores aux différents niveaux ne sont pas reliés entre eux et chaque niveau est caractérisé par une pression.

Dans ce travail, nous nous concentrons sur le premier cas pour lequel il existe de nombreuses applications, les nôtres étant motivées par la structure des pores du système canaliculo-lacunaire du tissu osseux compact.

On introduit le petit paramètre d'échelle ε, décrivant le rapport des longueurs caractéristiques des deux niveaux considérés, L^α et L^β, avec $\varepsilon = L^\alpha / L^\beta$. L'exposant $^\varepsilon$ sur une quantité (champ ou paramètre) indique une dépendance vis-à-vis de ce paramètre. La structure à deux niveaux est approchée quand $\varepsilon \to 0$ en utilisant la méthode d'homogénéisation des milieux périodiques. L'homogénéisation de la microstructure permet d'obtenir les propriétés effectives de la matrice poroélastique du niveau β. L'homogénéisation appliquée au niveau β à l'aide d'une procédure analogue avec $\varepsilon = L^\beta / L^{\text{macro}}$ conduit aux propriétés effectives du milieu poroélastique à l'échelle « macroscopique ».

L'introduction (section 1) présente le problème traité et introduit les deux échelles de porosité considérées. En section 2, l'homogénéisation du niveau α est présentée en précisant la configuration géométrique et les équations du problème. L'homogénéisation du niveau β est décrite dans la section 3. Les propriétés effectives obtenues sont mises en relation avec les propriétés physiques utilisées dans le modèle standard initié par Biot. En section 4, nous concluons qu'à la différence de l'approche classique de type Eshelby, l'homogénéisation présentée ici rend possible le traitement de géométries ou de topologies de pores quelconques. Ainsi, les tenseurs de localisation et de propriétés effectives peuvent être calculés comme la réponse à des problèmes indépendants aux échelles inférieures. Le modèle peut être adapté pour des porosités mutuellement déconnectées, ou pour une combinaison de pores connectés à un niveau et occlus à l'autre niveau.

1. Introduction

We consider the steady state of a deformable porous medium saturated by a fluid, whereby the fluid drainage is controlled by a boundary flux. The porosity of the medium is stretched over two levels, distinguishable by different sizes of pores, the primary and the secondary ones. They can be either interconnected or disconnected by an impermeable solid phase.

The two levels, further labeled by superscripts α and β, are associated with the microscopic and mesoscopic scales respectively. At the microscopic scale, we consider an elastic solid phase called "matrix", whereas the fluid fills the "canals" which can be drained, thus the fluid can be expelled from, or sucked in the higher level porosity. The constitutive relations are assumed to be linearly elastic for the matrix and Newtonian compressible for the fluid.

The homogenization procedure of this two-phase medium allows to obtain an α-model describing the poroelastic behavior at the intermediate scale.

Thus, at the mesoscopic scale, the above mentioned α-poroelasticity model describes the material occupying the matrix of the meso-structure β; at this "higher" level the canals can either communicate with the microscopic pores of the α level, or they can be completely insulated. The following two basic cases can be treated:

- *Connected double porosity*; the porosities at different hierarchies are connected, i.e. only one fluid pressure value characterizes the steady state.

– *Disconnected double porosity*; the pores of different hierarchies are not mutually connected, so that each level is characterized by a specific pressure.

We shall focus on the first case, being motivated by obvious existing applications; in particular, we have in mind the structure of pores in the canaliculo-lacunar system of compact bone tissue for which a poroelasticity model is adapted [1]. In this context, the assumption of incompressible constituents, while appropriate for soft tissues, is inaccurate for hard tissues. The absence of the assumption of incompressible constituents is a significant difference between the present paper and the theories of deformable porous media with double porosity developed in the literature and using a homogenization method (see for instance [2] and the pioneering work [3]). For completeness we recall [4], where a poroelastic medium with compressible inclusions forming a single porosity was studied. Although the second case, i.e. the disconnected double porosity, is beyond the scope of this paper, a brief information about the homogenization procedure is given in Section 4.

2. Homogenization of the microscopic scale problem – upscaling of the α level

2.1. The three spatial scales

Let us consider the scale parameter ε, describing the ratio of the characteristic sizes, L^α and L^β, of the two levels, i.e. $\varepsilon = L^\alpha/L^\beta$. By superscript $^\varepsilon$ we indicate dependence of functions and other parameters on ε. The hierarchical structure (double porosity) is studied for $\varepsilon \to 0$ using the periodic unfolding method of homogenization (other methods can be used); first we homogenize the microstructure, thus the effective poroelastic properties of the “matrix” at the β level are obtained. Then upscaling of the β level using analogous procedure with $\varepsilon = L^\beta/L^{\mathrm{macro}}$ leads to the effective poroelastic properties of the “macroscopic” level.

2.2. Governing equations

The geometrical configuration of the studied domain $\Omega^\alpha \subset \mathbb{R}^3$ is decomposed into the matrix and the canals occupying the domains $\Omega_m^{\alpha,\varepsilon}$ and $\Omega_c^{\alpha,\varepsilon}$, respectively, and their common boundary is $\Gamma^{\alpha,\varepsilon}$. More precisely, the following definitions are introduced

$$\Omega^\alpha = \Omega_m^{\alpha,\varepsilon} \cup \Omega_c^{\alpha,\varepsilon} \cup \Gamma^{\alpha,\varepsilon}, \qquad \Omega_m^{\alpha,\varepsilon} \cap \Omega_c^{\alpha,\varepsilon} = \emptyset, \quad \Gamma^{\alpha,\varepsilon} = \overline{\Omega_m^{\alpha,\varepsilon}} \cap \overline{\Omega_c^{\alpha,\varepsilon}} \tag{1}$$

In what follows, we will denote by ∇ and $\nabla\cdot$ the gradient and divergence operators, respectively. The symbol “$\cdot$” will denote the scalar product and the symbol “:” between tensors of any orders denotes their double contraction. Moreover, the notation ∇^S is associated with the symmetric part of the gradient operator. For instance, $\nabla^S \boldsymbol{u}$ is the linear strain tensor built from the displacement $\boldsymbol{u}$.

Assuming a static situation, the deformation of the matrix is governed by the equation system defined by the following problem: find $(\boldsymbol{u}^{\alpha,\varepsilon}, \bar{p}^{\alpha,\varepsilon})$ such that

$$\begin{aligned} &-\nabla \cdot \left(\mathbb{D}^{\alpha,\varepsilon}\nabla^S \boldsymbol{u}^{\alpha,\varepsilon}\right) = \boldsymbol{f}^{\alpha,\varepsilon}, \quad \text{in } \Omega_m^{\alpha,\varepsilon} \\ &\boldsymbol{n}^{[m]} \cdot \mathbb{D}^{\alpha,\varepsilon}\nabla^S \boldsymbol{u}^{\alpha,\varepsilon} = \boldsymbol{g}^{\alpha,\varepsilon}, \quad \text{on } \partial_{\mathrm{ext}}\Omega_m^{\alpha,\varepsilon} \\ &\boldsymbol{n}^{[m]} \cdot \mathbb{D}^{\alpha,\varepsilon}\nabla^S \boldsymbol{u}^{\alpha,\varepsilon} = -\bar{p}^{\alpha,\varepsilon}\boldsymbol{n}^{[m]}, \quad \text{on } \Gamma^{\alpha,\varepsilon} \end{aligned} \tag{2}$$

and (by $\widetilde{\ }$ we denote a “matrix-to-canal” extension, see below)

$$\int_{\partial\Omega_c^{\alpha,\varepsilon}} \widetilde{\boldsymbol{u}^{\alpha,\varepsilon}} \cdot \boldsymbol{n}^{[c]}\, \mathrm{dS}_x + \gamma^\alpha \bar{p}^{\alpha,\varepsilon}\left|\Omega_c^{\alpha,\varepsilon}\right| = -J^{\alpha,\varepsilon} \tag{3}$$

where $\boldsymbol{u}^{\alpha,\varepsilon}$ is the displacement vector of the solid matrix, $\bar{p}^{\alpha,\varepsilon}$ is the fluid pressure, $\mathbb{D}^{\alpha,\varepsilon}$ is the elasticity fourth-order tensor of the solid matrix and γ^α is the fluid compressibility. The applied surface-force and volume-force fields are denoted respectively by $\boldsymbol{g}^{\alpha,\varepsilon}$ and $\boldsymbol{f}^{\alpha,\varepsilon}$. The outer unit normal vector of the boundary $\Omega_m^{\alpha,\varepsilon}$ is denoted by $\boldsymbol{n}^{[m]}$. Condition (3) means that the fluid volume $-J^{\alpha,\varepsilon}$ injected from outside through $\partial_{\mathrm{ext}}\Omega_c^{\alpha,\varepsilon}$ into $\Omega_c^{\alpha,\varepsilon}$, i.e. the right-hand side term, is balanced by the increase of the pore volume and by the fluid compression resulting in an increased pressure $\bar{p}$. Note that the solvability condition yields $\int_{\partial_{\mathrm{ext}}\Omega_m^{\alpha,\varepsilon}} \boldsymbol{g}^{\alpha,\varepsilon}\, \mathrm{dS}_x + \int_{\Omega_m^{\alpha,\varepsilon}} \boldsymbol{f}^{\alpha,\varepsilon}\, \mathrm{dV}_x = \boldsymbol{0}$ where dS_x and dV_x are the differential elements of surface and volume, respectively. In (3) we use the extension $\widetilde{z} \in \mathscr{C}(\Omega^\alpha)$ defined for any $z \in \mathscr{C}(\Omega_m^{\alpha,\varepsilon})$, where $\mathscr{C}(\Omega_m^{\alpha,\varepsilon})$ is the admissible function space of the problem constituted by the sufficiently differentiable real-valued functions defined in $\Omega_m^{\alpha,\varepsilon}$ (due to the boundary conditions in (2), the displacements are determined up to rigid body motions, typically $\mathscr{C}(\Omega_m^{\alpha,\varepsilon}) \subset W^{1,2}(\Omega_m^{\alpha,\varepsilon})$ where $W^{1,2}(\Omega_m^{\alpha,\varepsilon})$ is the classical Sobolev space of square integrable functions including their first derivatives).

The boundary value problem given by (2) and (3) can be rewritten in the terms of the weak formulation: find $(\boldsymbol{u}^{\alpha,\varepsilon}, \bar{p}^{\alpha,\varepsilon}) \in \mathscr{C}(\Omega_m^{\alpha,\varepsilon}) \times \mathbb{R}$ such that

$$\int_{\Omega_m^{\alpha,\varepsilon}} \left(\mathbb{D}^{\alpha,\varepsilon}\nabla^S \boldsymbol{u}^{\alpha,\varepsilon}\right) : \nabla^S \boldsymbol{v}\, \mathrm{dV}_x + \bar{p}^{\alpha,\varepsilon} \int_{\Gamma_m^{\alpha,\varepsilon}} \boldsymbol{n}^{[m]} \cdot \boldsymbol{v}\, \mathrm{dS}_x$$
$$= \int_{\partial_{\mathrm{ext}}\Omega_m^{\alpha,\varepsilon}} \boldsymbol{g}^{\alpha,\varepsilon} \cdot \boldsymbol{v}\, \mathrm{dS}_x + \int_{\Omega_m^{\alpha,\varepsilon}} \boldsymbol{f}^{\alpha,\varepsilon} \cdot \boldsymbol{v}\, \mathrm{dV}_x, \quad \text{for all } \boldsymbol{v} \in \mathscr{C}\left(\Omega_m^{\alpha,\varepsilon}\right)$$
$$\int_{\partial\Omega_c^{\alpha,\varepsilon}} \widetilde{\boldsymbol{u}^{\alpha,\varepsilon}} \cdot \boldsymbol{n}^{[c]}\, \mathrm{dS}_x + \gamma^\alpha \bar{p}^{\alpha,\varepsilon} \left|\Omega_c^{\alpha,\varepsilon}\right| = -J^{\alpha,\varepsilon} \tag{4}$$

2.3. Homogenization process

We assume that the domain Ω^α is obtained from a periodic microstructure generated by the representative unit cell Y^α decomposed as follows

$$Y^\alpha = Y_m^\alpha \cup Y_c^\alpha \cup \Gamma_Y^\alpha, \qquad Y_c^\alpha = Y^\alpha \setminus \overline{Y_m^\alpha}, \qquad \Gamma_Y^\alpha = \overline{Y_m^\alpha} \cap \overline{Y_c^\alpha} \tag{5}$$

As a result, the domain Ω^α is defined by $\bigcup_{k\in\mathbb{K}^\varepsilon} \varepsilon(Y^\alpha + k)$ with $\mathbb{K}^\varepsilon = \{k \in \mathbb{Z}^3,\ \varepsilon(Y^\alpha + k) \subset \Omega^\alpha\}$. The upscaling procedure of the heterogeneous continuum consists in the limit analysis with respect to $\varepsilon \to 0$. For this we use the periodic unfolding method [5,6] based on the coordinate decomposition $x = \xi + \varepsilon y$, where $\xi = \varepsilon[\frac{x}{\varepsilon}]_Y$ is the lattice coordinate at the mesoscopic scale, thus given by the brackets, so that $y \in Y$ is the local coordinate of the microscopic scale. The analogous notation is employed when upscaling from the mesoscopic-to-macroscopic scale.

2.3.1. Response at the microscopic scale

We assume weak convergence of the external forces; denoting by χ_m^ε the characteristic function of the matrix, $\chi_m^\varepsilon \boldsymbol{f}^{\alpha,\varepsilon}$ converge towards $(1-\phi^\alpha)\boldsymbol{f}^\alpha$ where $\boldsymbol{f}^\alpha \in \mathscr{C}(\Omega^\alpha)$ is a local averaged volume-force acting on the matrix. The volume fraction of pores is defined by $\phi^\alpha = |Y_c^\alpha|/|Y^\alpha|$. Denoting by ϕ_S^α the *exterior surface porosity* on $\partial\Omega^\alpha$, we assume existence of surface-force $\boldsymbol{g}^\alpha$ such that $\int_{\partial_{\mathrm{ext}}\Omega_m^{\alpha,\varepsilon}} \boldsymbol{g}^{\alpha,\varepsilon} \cdot \boldsymbol{v}\, \mathrm{dS}_x \to \int_{\partial\Omega^\alpha} (1-\phi_S^\alpha)\boldsymbol{g}^\alpha \cdot \boldsymbol{v}\, \mathrm{dS}_x$ for any $\boldsymbol{v} \in \mathscr{C}(\Omega^\alpha)$. (Note that for statistically distributed pores we can choose $\phi = \phi_S$ according to the Delessian law.)

It can be shown that $\boldsymbol{u}^{\alpha,\varepsilon}$ extended to whole Ω^α converges strongly to field $\boldsymbol{u}(x) \in \mathscr{C}(\Omega^\alpha)$ which is defined at the mesoscopic scale. When $\varepsilon \to 0$, the strain is a two-scale function defined from its macroscopic part $\nabla_x^S \boldsymbol{u}(x)$ and its fluctuating part $\nabla_y^S \boldsymbol{u}^1(x, y)$, whereby fluctuations are proportional to macroscopic strains. There are so-called characteristic displacements $\boldsymbol{\omega}^{ij}(y)$ and $\boldsymbol{\omega}^P(y)$ such that $\boldsymbol{u}^1(x, y) = \boldsymbol{\omega}^{ij}(y)\partial_j u_i(x) - \boldsymbol{\omega}^P(y)\bar{p}$, where $\bar{p}$ is the constant fluid pressure in Ω^α. Functions $\boldsymbol{\omega}^{ij}(y)$ and $\boldsymbol{\omega}^P(y)$ are obtained as solutions of the following problems: find $(\boldsymbol{\omega}^{ij}, \boldsymbol{\omega}^P) \in \mathscr{C}_\#(Y_m) \times \mathscr{C}_\#(Y_m)$ satisfying

$$a_Y^m\left(\boldsymbol{\omega}^{ij} + \boldsymbol{\Pi}^{ij}, \boldsymbol{v}\right) = 0, \quad \forall \boldsymbol{v} \in \mathscr{C}_\#(Y_m)$$
$$a_Y^m\left(\boldsymbol{\omega}^P, \boldsymbol{v}\right) = \fint_{\Gamma_Y} \boldsymbol{v} \cdot \boldsymbol{n}^{[m]}\, \mathrm{dS}_y, \quad \forall \boldsymbol{v} \in \mathscr{C}_\#(Y_m) \tag{6}$$

where $a_Y^m(\boldsymbol{w}, \boldsymbol{v}) = \fint_{Y_m} (\mathbb{D}\nabla_y^S \boldsymbol{w}) : \nabla_y^S \boldsymbol{v}$ and $\boldsymbol{\Pi}^{ij} = (\Pi_k^{ij})$, $i, j, k = 1, 2, 3$ with $\Pi_k^{ij} = y_j\delta_{ik}$. Above we employed $\mathscr{C}_\#(Y_m)$, a subspace of $\mathscr{C}(Y_m)$ containing Y-periodic functions only. For the sake of brevity we use the notation $Y := Y^\alpha$, thus dropping out the superscript α.

2.3.2. Model obtained by homogenization

The effective properties of the deformable porous medium are introduced using the characteristic responses obtained at the microscopic scale

$$A_{ijkl} = a_Y^m\left(\boldsymbol{\omega}^{ij} + \boldsymbol{\Pi}^{ij}, \boldsymbol{\omega}^{kl} + \boldsymbol{\Pi}^{kl}\right), \qquad B_{ij} = -\fint_{Y_m} \mathrm{div}_y\, \boldsymbol{\omega}^{ij}, \qquad M = a_Y^m\left(\boldsymbol{\omega}^P, \boldsymbol{\omega}^P\right) \tag{7}$$

Obviously, the tensors $\mathbb{A} = (A_{ijkl})$ and $\boldsymbol{B} = (B_{ij})$ are symmetric; moreover $\mathbb{A}$ is positive definite and $M > 0$.

Model of poroelasticity. At this first level of the homogenization process, we obtain the model of poroelasticity governing the skeleton displacements $\boldsymbol{u} \in \mathscr{C}(\Omega)$ and the fluid pressure $\bar{p}$ which verify the following equations

$$\int_\Omega \left(\mathbb{A}\nabla_x^S \boldsymbol{u} - \bar{p}\hat{\boldsymbol{B}}\right) : \nabla_x^S \boldsymbol{v} = \int_\Omega (1-\phi)\boldsymbol{f} \cdot \boldsymbol{v} + \int_{\partial\Omega} \bar{\boldsymbol{g}} \cdot \boldsymbol{v}\, \mathrm{dS}_x, \quad \forall \boldsymbol{v} \in \mathscr{C}(\Omega)$$
$$\int_\Omega \hat{\boldsymbol{B}} : \nabla_x^S \boldsymbol{u} + \bar{p}(M + \bar{\phi}\gamma)|\Omega| = -J, \quad \text{with } \hat{\boldsymbol{B}} := \boldsymbol{B} + \phi\mathbf{I} \tag{8}$$

where J is the limit of the total flux $J^{\alpha,\varepsilon}$ (outwards Ω^α), $\bar{\boldsymbol{g}} := (1-\phi_S)\boldsymbol{g} + \phi_S(-\bar{p})\boldsymbol{n}$ is the mean surface stress (traction) and $\bar{\phi}$ is the mean porosity, i.e. $\bar{\phi} = |\Omega|^{-1}\int_\Omega \phi$. Note that all $\mathbb{A}$, $\boldsymbol{B}$, M, ϕ, ϕ_S and J are associated with upscaling from the α level to the β level, therefore they will be further labeled by superscript α.

3. Homogenization of the mesoscopic scale problem – upscaling of the β level

At the second level, i.e. at the mesoscopic scale, the geometrical configuration of the meso-structure consists of two compartments: 1) the *matrix* $\Omega_m^{\beta,\varepsilon}$ which is formed by the porous medium associated with the upscaled microstructure of the α level, 2) the *canals* $\Omega_c^{\beta,\varepsilon}$ which are filled with fluid and connected with pores of the α level through the interface $\Gamma_m^{\beta,\varepsilon}$. Without risk of ambiguity, the dimensionless parameter ε now designates the ratio between the mesoscopic and the macroscopic scales.

3.1. Description at the mesoscopic scale level

In analogy with the α level, the domain Ω^β at the mesoscopic scale is split as follows: $\Omega^\beta = \Omega_m^{\beta,\varepsilon} \cup \Omega_c^{\beta,\varepsilon} \cup \Gamma^{\beta,\varepsilon}$, where $\Gamma^{\beta,\varepsilon}$ denotes the interface between the subdomains. The structure is loaded on $\partial_{\text{ext}}\Omega_m^{\beta,\varepsilon} = \partial\Omega^\beta \cap \partial\Omega_m^{\beta,\varepsilon}$ by mean surface stresses $\bar{\boldsymbol{g}}^\alpha$, see (8), and by a volume-force field $\hat{\boldsymbol{f}}^\alpha = (1-\phi^\alpha)\boldsymbol{f}^\alpha$ acting on the matrix. The mesoscopic pores $\Omega_c^{\beta,\varepsilon}$ are drained-out on $\partial_{\text{ext}}\Omega_c^{\beta,\varepsilon}$; the total outflow from Ω^β through $\partial\Omega^\beta$ is denoted as $J^{\beta,\varepsilon}$, it incorporates also the flux from the micro-porosity α through $\partial_{\text{ext}}\Omega_m^{\beta,\varepsilon}$ and from the mesoscopic canals through $\partial_{\text{ext}}\Omega_c^{\beta,\varepsilon}$. Note that on the interior part of $\partial\Omega_m^{\beta,\varepsilon}$ surface-force $\bar{\boldsymbol{g}}$, as considered on the left-hand side of (8), is represented by the interstitial pressure in the β-porosity. So, the displacement $\boldsymbol{u}^{\beta,\varepsilon} \in \mathscr{C}(\Omega_m^{\beta,\varepsilon})$ and the pressure $\bar{p}^\varepsilon \in \mathbb{R}$ must satisfy the equation

$$\int_{\Omega_m^{\beta,\varepsilon}} \left(\mathbb{A}^\alpha \nabla^S \boldsymbol{u}^{\beta,\varepsilon} - \bar{p}^\varepsilon \hat{\boldsymbol{B}}^\alpha\right) : \nabla^S \boldsymbol{v} + \bar{p}^\varepsilon \int_{\Gamma^{\beta,\varepsilon}} \boldsymbol{v}\cdot\boldsymbol{n}^{[m]}\,\mathrm{dS}_x = \int_{\partial_{\text{ext}}\Omega_m^{\beta,\varepsilon}} \bar{\boldsymbol{g}}^\alpha \cdot \boldsymbol{v}\,\mathrm{dS}_x + \int_{\Omega_m^{\beta,\varepsilon}} \hat{\boldsymbol{f}}^\alpha \cdot \boldsymbol{v}, \quad \forall \boldsymbol{v} \in \mathscr{C}\left(\Omega_m^{\beta,\varepsilon}\right) \tag{9}$$

and the volume conservation

$$\int_{\Omega_m^{\beta,\varepsilon}} \hat{\boldsymbol{B}}^\alpha : \nabla^S \boldsymbol{u}^{\beta,\varepsilon} + \int_{\partial\Omega_c^{\beta,\varepsilon}} \widetilde{\boldsymbol{u}^{\beta,\varepsilon}} \cdot \boldsymbol{n}^{[c]}\,\mathrm{dS}_x + \bar{p}^\varepsilon\left[\left(M^\alpha + \gamma\bar{\phi}^\alpha\right)\left|\Omega_m^{\beta,\varepsilon}\right| + \gamma\left|\Omega_c^{\beta,\varepsilon}\right|\right] = -J^{\beta,\varepsilon} \tag{10}$$

where we used the displacement extension $\widetilde{\boldsymbol{u}^{\beta,\varepsilon}}$ to canals $\Omega_c^{\beta,\varepsilon}$.

3.2. Homogenized problem at the β level

Analysis of problem (9)–(10) when $\varepsilon \to 0$ leads to equations involving effective poroelastic properties of the second level which are evaluated using the characteristic responses $\boldsymbol{\omega}^{ij}$ and $\boldsymbol{\omega}^P$.

In analogy with the α level, let $Y^\beta = Y_m^\beta \cup Y_c^\beta \cup \Gamma_Y^\beta$ be the reference periodic cell. The following local problems must be solved: find $\boldsymbol{\omega}^{ij}$ and $\boldsymbol{\omega}^P \in \mathscr{C}_\#(Y_m^\beta)$, $i,j = 1,2,3$, such that

$$\begin{aligned}
&\int_{Y_m^\beta} \left[\mathbb{A}^\alpha \nabla_y^S\left(\boldsymbol{\omega}^{ij} + \boldsymbol{\Pi}^{ij}\right)\right] : \nabla_y^S \boldsymbol{v} = 0, \quad \forall \boldsymbol{v} \in \mathscr{C}_\#\left(Y_m^\beta\right)\\
&\int_{Y_m^\beta} \left[\mathbb{A}^\alpha \nabla_y^S \boldsymbol{\omega}^P\right] : \nabla_y^S \boldsymbol{v} = -\int_{Y_m^\beta} \hat{\boldsymbol{B}}^\alpha : \nabla_y^S \boldsymbol{v} + \fint_{\Gamma_Y^\beta} \boldsymbol{v}\cdot\boldsymbol{n}^{[m]}\,\mathrm{dS}_y, \quad \forall \boldsymbol{v} \in \mathscr{C}_\#\left(Y_m^\beta\right)
\end{aligned} \tag{11}$$

In what follows by $(\boldsymbol{\omega}^{ij}, \boldsymbol{\omega}^P)$ we mean the solutions of (11) and not those of (6). While the solutions of $(11)_1$ express fluctuations with respect to the unit strain of the macroscopic scale, the solutions of $(11)_2$ interpret the local response with respect to the unit pressure.

The effective poroelasticity properties of the upscaled mesoscale are represented by $\mathbb{A}^\beta = (A_{ijkl}^\beta)$, $\boldsymbol{B}^\beta = (B_{ij}^\beta)$ and M^β, given as follows

$$\begin{aligned}
A_{ijkl}^\beta &= \int_{Y_m^\beta} \left[\mathbb{A}^\alpha \nabla_y^S\left(\boldsymbol{\omega}^{kl} + \boldsymbol{\Pi}^{kl}\right)\right] : \nabla_y^S\left(\boldsymbol{\omega}^{ij} + \boldsymbol{\Pi}^{ij}\right)\\
B_{ij}^\beta &= \int_{Y_m^\beta} \hat{\boldsymbol{B}}^\alpha : \nabla_y^S\left(\boldsymbol{\omega}^{ij} + \boldsymbol{\Pi}^{ij}\right) - \fint_{Y_m^\beta} \mathrm{div}_y\, \boldsymbol{\omega}^{ij}
\end{aligned}$$

$$M^\beta = \int_{Y_m^\beta} \left[\mathbb{A}^\alpha \nabla_y^S \boldsymbol{\omega}^P\right] : \nabla_y^S \boldsymbol{\omega}^P \tag{12}$$

The response of the homogenized medium at macroscopic scale, i.e. upscaled mesoscale for $\varepsilon \to 0$, is represented by the displacement field $\boldsymbol{u} \in \mathscr{C}(\Omega^\beta)$ and by the constant pressure $\bar{p} \in \mathbb{R}$ satisfying

$$\begin{aligned}
&\int_{\Omega^\beta} \left(\mathbb{A}^\beta \nabla_x^S \boldsymbol{u} - \bar{p}\hat{\boldsymbol{B}}^\beta\right) : \nabla_x^S \boldsymbol{v} = \int_{\partial\Omega^\beta} \bar{\boldsymbol{g}}^\beta \cdot \boldsymbol{v}\, \mathrm{dS}_x + \int_{\Omega^\beta} \left(1 - \phi^\beta\right)\hat{\boldsymbol{f}}^\alpha \cdot \boldsymbol{v} \\
&\int_{\Omega^\beta} \hat{\boldsymbol{B}}^\beta : \nabla_x^S \boldsymbol{u} + \bar{p}\left|\Omega^\beta\right|\hat{M}^\beta = -J^\beta
\end{aligned} \tag{13}$$

for all $\boldsymbol{v} \in \mathscr{C}(\Omega^\beta)$ where $\bar{\boldsymbol{g}}^\beta := (1 - \phi^\beta)\bar{\boldsymbol{g}}^\alpha + \phi^\beta(-\bar{p})\boldsymbol{n}$ is the mean surface stress (the traction force density) and (denoting by $\bar{\phi}^\beta$ the mean β level porosity)

$$\hat{\boldsymbol{B}}^\beta := \phi^\beta \mathbf{I} + \boldsymbol{B}^\beta, \qquad \hat{M}^\beta := M^\beta + \bar{\phi}^\beta\left[\left(M^\alpha + \gamma\bar{\phi}^\alpha\right)\left(1 - \bar{\phi}^\beta\right) + \gamma\bar{\phi}^\beta\right] \tag{14}$$

3.3. Relationship with the poroelasticity model of Biot

Using the upscaling procedure involving the two sub-scales, i.e. the α and β levels, we can associate the coefficients involved in the macroscopic formulation with the Biot poroelasticity model: on the left side, the standard notation is proposed whereas, on the right side, we put the equations of the present two-level homogenized poroelasticity model

$$\begin{aligned}
\boldsymbol{\sigma} &= \mathbb{D}^{\mathrm{eff}}\boldsymbol{e} - \boldsymbol{\alpha} p, & & \boldsymbol{\sigma} = \mathbb{A}\boldsymbol{e} - \hat{\boldsymbol{B}} p \\
\zeta &= \boldsymbol{\alpha} : \boldsymbol{e} + \frac{1}{\mu}(p - p_0), & \text{or} \quad & \zeta = \hat{\boldsymbol{B}} : \boldsymbol{e} + \hat{M}(p - p_0)
\end{aligned} \tag{15}$$

where $\boldsymbol{\sigma} = (\sigma_{ij})$ is the total stress, p is the fluid pressure whereby $p_0 := 0$ is the reference pressure, $\boldsymbol{e} = (e_{ij})$ is the strain tensor, i.e. $\boldsymbol{e} = \nabla^S \boldsymbol{u}$, and ζ is the fluid content increase (per unit volume with respect to the reference state). On comparing (15) with (13), the following relationships are evident: $\mathbb{D}^{\mathrm{eff}} = \mathbb{A}$, $\boldsymbol{\alpha} = \hat{\boldsymbol{B}}$ and $1/\mu = \hat{M}$, where the superscript β was dropped. Often the inverse relationships to (15) are used

$$\begin{aligned}
&\text{skeleton deformation} \quad \boldsymbol{e} = \mathbb{C}\boldsymbol{\sigma} + \boldsymbol{S}\zeta \\
&\text{fluid pressure} \quad p = -\boldsymbol{S} : \boldsymbol{\sigma} + K\zeta
\end{aligned} \tag{16}$$

where $\mathbb{C}$ is the compliance (fourth-order tensor) of the bulk material observed in an "undrained" loading, $\boldsymbol{S}$ is the Skempton tensor, and $K^{-1}\boldsymbol{S} : \boldsymbol{\sigma}$ expresses the fluid content increase for drained loading (i.e. "zero fluid pressure" condition applies). Coefficients in (16) are expressed in terms of the homogenized coefficients in (14), as follows

$$\begin{aligned}
&\mathbb{C} = \hat{M}\mathbb{D}\mathbb{A}^{-1} \\
&\boldsymbol{S} = \mathbb{D}\mathbb{A}^{-1} : \hat{\boldsymbol{B}} \qquad \text{where } \mathbb{D} = \left[\hat{M}\mathbb{I} + \mathbb{A}^{-1}(\hat{\boldsymbol{B}} \otimes \hat{\boldsymbol{B}})\right]^{-1} \\
&K = \frac{1}{\hat{M}}\left(1 - \hat{\boldsymbol{B}} : \mathbb{D}\mathbb{A}^{-1} : \hat{\boldsymbol{B}}\right)
\end{aligned} \tag{17}$$

It can be shown that $\mathbb{D}$ is symmetric, in turn $\mathbb{C}$ is symmetric as well. The undrained skeleton elasticity (i.e. when the fluid content is constant, $\zeta = 0$) can be computed, $\mathbb{A}_U := \mathbb{C}^{-1} = \mathbb{A} - K^{-1}\hat{\boldsymbol{B}} \otimes \hat{\boldsymbol{B}}$. The formulas (17) are consistent with the results of [7] obtained using the micromechanical approach.

4. Concluding remarks

Three concluding remarks are given below to comment on the "nested" homogenization approach and its possible extension for a class of double-porosity media and quasistatic problems.

1. Using the hierarchical homogenization, we developed the upscaled model of a nested poroelastic material, often called the "double-porosity" model. In our case we considered two levels (the microscopic and the mesoscopic) with connected pores; as the result, for a static problem there is just one scalar fluid pressure associated with all levels. We point out that formally the same expressions for the poroelastic coefficients and the same microscopic equations can be obtained for the case of a closed porosity, i.e. when $Y_c \subset Y$ and $\partial Y_c \cap \partial Y = \emptyset$ [4]. However, then the pressure is a (macroscopic) field, i.e. $p = p(x)$; the formulas for $\mathbb{A}^\alpha$, $\boldsymbol{B}^\alpha$ and M^α are in accordance with [8]. For the double (nested) porosity description one must distinguish particular cases of the pore connectivity which leads to different structure of the upscaled poroelastic equations. In contrast with the often used Eshelby approach, the homogenization theory allows to treat an arbitrary geometry and

topology of the pores, whereby the localization tensors and coefficients can be calculated as the response of the autonomous microscopic problems. The model can be adapted for mutually disconnected porosities, or for a combination of connected pores at one level and disconnected pores at the other level, even including weakly permeable interfaces separating the two porosities.

2. In the Introduction we pointed out that two different pressures describe the upscaled medium in case of mutually disconnected porosities associated with the micro- and mesoscopic levels. We did not treat this case which, however, is more straightforward – simply no flux appears between the channels and the matrix of the β level heterogeneity. Therefore, instead of (9)–(10), formulation (4) is adapted for the β level homogenization: the bilinear form of the elasticity equations on the left-hand side of $(4)_1$ is replaced by the left-hand side integrals of the homogenized α level equation $(8)_1$, whereby integration in $\Omega_m^{\beta,\varepsilon}$ applies and the matrix pressure $\bar{p}^\alpha$ is different from the β level pore pressure $\bar{p}^\beta$. Then Eq. $(4)_2$ is adopted wherein the pressure $\bar{p}^\beta$ is employed. Finally, the mass conservation $(8)_1$ with $\bar{p}^\alpha$ applies. By an upscaling technique, one obtains a two-pressure formulation where $\bar{p}^\beta$ can be a field when the β level porosity is formed by disconnected fluid inclusions. The same statement holds concerning the α level pressure and the corresponding α level microstructure. In general, the topology of structures at both levels and their mutual connectivity determines what are the pressure variables of the macroscopic description. In this context we refer the reader to [9] where a different treatment of the double porosity was considered. Our further research will focus on quasistatic and dynamic cases in this two-level structure setting.

3. Although we considered the static case only, so that no pressure gradients were considered, a slow flow leading to only modest pressure gradients can be covered by the model presented in Eq. (15); for this the fluid increase can be expressed by the Darcy flow. Namely, we can differentiate $(15)_2$ with respect to time and substitute there $\dot{\zeta} = -\nabla \cdot \boldsymbol{w} = \nabla \cdot (\boldsymbol{K} \nabla p)$ with the permeability $\boldsymbol{K} = (K_{ij})$ resulting from the homogenization of the Stokes problem in $\Omega_c^{\beta,\varepsilon}$, whereas the flow in the dual porosity related to the α level would be negligible.

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