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On the fractional generalization of Eringen's nonlocal elasticity for wave propagation

Noël Challamel [a,*], Dušan Zorica [b], Teodor M. Atanacković [c], Dragan T. Spasić [c]

[a] Université européenne de Bretagne, University of South Brittany UBS, UBS – LIMATB, centre de recherche, rue de Saint Maude, BP92116, 56321 Lorient cedex, France
[b] Mathematical Institute, Serbian Academy of Arts and Sciences, Kneza Mihaila 36, 11000 Beograd, Serbia
[c] Department of Mechanics, Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6, 21000 Novi Sad, Serbia

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A B S T R A C T

A fractional nonlocal elasticity model is presented in this Note. This model can be understood as a possible generalization of Eringen's nonlocal elastic model, with a free non-integer derivative in the stress–strain fractional order differential equation. This model only contains a single length scale and the fractional derivative order as parameters. The kernel of this integral-based nonlocal model is explicitly given for various fractional derivative orders. The dynamical properties of this new model are investigated for a one-dimensional problem. It is possible to obtain an analytical dispersive equation for the axial wave problem, which is parameterized by the fractional derivative order. The fractional derivative order of this generalized fractional Eringen's law is then calibrated with the dispersive wave properties of the Born–Kármán model of lattice dynamics and appears to be greater than the one of the usual Eringen's model. An excellent matching of the dispersive curve of the Born–Kármán model of lattice dynamics is obtained with such generalized integral-based nonlocal model.

1. Introduction

Nonlocal elastic models have been introduced in the 1960s for modelling elastic wave dispersion in crystals or in heterogeneous materials (see, for instance, Maugin [1]). Since these pioneering works, nonlocality has found a lot of applications especially for modelling scale effects from micro to macro structures. The mechanics community has recently found a new specific interest in the use of nonlocal theory with the active development of researches in the field of small scale structures such as micro and nanostructures (see, for instance, Elishakoff et al. [2]). One efficient model in the theory of nonlocal elasticity is the model of Eringen [3] defined from a differential equation between the stress and the strain variables. This model only depends on one length scale to be calibrated. The application of such a differential-based nonlocal model has been able to reproduce quiet accurately the dispersive wave properties of the Born–Kármán model of lattice dynamics (Born and Kármán [4]), a specific property that cannot be captured by a local stress–strain model (see for instance Eringen [3], or Eringen [5] for the calibration of his nonlocal model with respect to the lattice dynamics model – see, more generally, Eringen [6]). The aim of this Note is to improve such a differential-based model by relaxing the integer order of the differential equation in the stress–strain relationship, and introducing a symmetrized Caputo fractional derivative. It is expected to better match the Born–Kármán model with such an additional degree-of-freedom.

* Corresponding author.
E-mail addresses: noel.challamel@univ-ubs.fr (N. Challamel), dusan_zorica@mi.sanu.ac.rs (D. Zorica), atanackovic@uns.ac.rs (T.M. Atanacković), spasic@uns.ac.rs (D.T. Spasić).

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Fractional derivatives have been already applied for modelling nonlocal elastic phenomena, as developed for instance by Lazopoulos [7] in the static case where a fractional derivative strain measure is introduced in the energy formulation. Fractional derivatives have been also recently used to model nonlocal elastic media for wave propagation applications, leading to some new spatial nonlocality kernels (see, for instance, Cottone et al. [8]; Atanacković and Stanković [9]; Carpinteri et al. [10]; Michelitsch [11]; Michelitsch et al. [12]). The nonlocality is introduced in the elastic stress–strain constitutive law by Cottone et al. [8] or Carpinteri et al. [10], whereas Atanacković and Stanković [9] modified the kinematics strain–displacement relationship with an alternative nonlocal formulation. Interestingly, Atanacković and Stanković [9] obtained the same kind of equation for the wave propagation, even if introduced with different physical arguments (see also the discussion in Carpinteri et al. [10]). It is worth mentioning that Michelitsch [11] or Michelitsch et al. [12] also recently treated a similar wave equation using a nonlocal Laplacian operator in the wave equation.

More precisely, in Atanacković and Stanković [9], Hooke's law was generalized by replacing the displacement gradient by the symmetrized Caputo spatial fractional derivative (see Eq. (2.17) of Atanacković and Stanković [9]). In Michelitsch [11,12] the generalization of the classical theory is performed on the level of the potential energy, which then led to a fractional derivative of the Marchaud type. It is known (see, for instance, Samko et al. [13]) that, under certain conditions on function involved, the Marchaud and Riemann–Liouville and Caputo fractional derivatives are the same. However, in general the approaches of Atanacković and Stanković [9] and Michelitsch [11] (see also Michelitsch et al. [12]) may be different.

In all these approaches, the nonlocal kernel for defining the nonlocal variables is chosen in a fractional power law decaying functions which can be physically supported for some specific applications. It can be shown that these attenuation functions can be introduced from fractional derivative theory. The new generalization that we suggest in this paper based on the fractional generalization of Eringen's model cannot be cast in the framework associated with a fractional power law.

2. A fractional differential nonlocal model

The following fractional differential equation can be postulated as a generalization of Eringen's differential model:

$$\sigma - l_c^{\alpha} D_x^{\alpha} \sigma = E\varepsilon \tag{1}$$

where l_c is a characteristic length responsible for nonlocal effects. σ and ε are respectively the uniaxial stress and uniaxial strain, and E is the Young modulus. The local stress–strain law is found as a particular case when the characteristic length vanishes $l_c = 0$. D_x^{α} is the symmetrized Caputo fractional derivative of order $\alpha \in [n-1; n[$, $n \in N$, defined by (see Kilbas et al. [14]):

$$\begin{aligned}
&D_x^{\alpha}\sigma = \frac{1}{2}\left({}_{-\infty}^{\;\;C}D_x^{\alpha} + {}_x^C D_{\infty}^{\alpha}\right)\sigma \quad \text{where} \\
&{}_{-\infty}^{\;\;C}D_x^{\alpha}\sigma = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{x} \frac{\sigma^{(n)}(\xi)}{(x-\xi)^{\alpha-n+1}}\, d\xi \\
&{}_x^C D_{\infty}^{\alpha}\sigma = (-1)^n \frac{1}{\Gamma(n-\alpha)} \int_{x}^{\infty} \frac{\sigma^{(n)}(\xi)}{(\xi-x)^{\alpha-n+1}}\, d\xi
\end{aligned} \tag{2}$$

where Γ is Euler's gamma function and where $\sigma^{(n)}(\xi) = \frac{d^n}{d\xi^n}\sigma(\xi)$. The case $\alpha = 2$ corresponds to the well known Eringen's nonlocal model (Eringen [3,6]). The fractional nonlocal model is generalized around Eringen's model based on $\alpha = 2$, i.e. only $\alpha \in [1; 2[$ and $\alpha \in [2; 3[$ are studied in this Note.

The kinematics is simply given by:

$$\varepsilon = \frac{\partial u}{\partial x} \tag{3}$$

The axial dynamic equation of motion of a bar of section S is obtained through Newton's law:

$$\frac{\partial N}{\partial x} = \rho S \frac{\partial^2 u}{\partial t^2} \quad \text{with } N = \sigma S \tag{4}$$

For a bar with uniform cross section $S(x) = S_0$, this partial differential equation is equivalently written as:

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \tag{5}$$

3. Mathematical properties of the model

The symmetrized Caputo derivative of order $\alpha \in [1; 2[$, by Eq. (2) reads

$$D_x^\alpha \sigma = \frac{1}{2\Gamma(2-\alpha)} \int_{-\infty}^{+\infty} \frac{\sigma^{(2)}(\xi)}{|x-\xi|^{\alpha-1}}\,\mathrm{d}\xi = \frac{1}{2\Gamma(2-\alpha)} \frac{1}{|x|^{\alpha-1}} * \sigma^{(2)}(x) \tag{6}$$

while in the case $\alpha \in [2; 3[$ it is written as:

$$D_x^\alpha \sigma = \frac{1}{2\Gamma(3-\alpha)} \int_{-\infty}^{+\infty} \frac{\sigma^{(3)}(\xi)\operatorname{sgn}(x-\xi)}{|x-\xi|^{\alpha-2}}\,\mathrm{d}\xi = \frac{1}{2\Gamma(3-\alpha)} \frac{\operatorname{sgn}(x)}{|x|^{\alpha-2}} * \sigma^{(3)}(x) \tag{7}$$

Both Eq. (6) and Eq. (7) generalize the second derivative, since ${}_{-\infty}^{\;C}D_x^\alpha \to \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ as $\alpha \to 2$, as well as ${}_{x}^{C}D_\infty^\alpha \to \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ as $\alpha \to 2$. Note that as $\alpha \to 1$ and $\alpha \to 3$, we have $D_x^\alpha \to 0$, since ${}_{-\infty}^{\;C}D_x^\alpha \to \frac{\mathrm{d}}{\mathrm{d}x}$, and ${}_{x}^{C}D_\infty^\alpha \to -\frac{\mathrm{d}}{\mathrm{d}x}$ as $\alpha \to 1$ and ${}_{-\infty}^{\;C}D_x^\alpha \to \frac{\mathrm{d}^3}{\mathrm{d}x^3}$, and ${}_{x}^{C}D_\infty^\alpha \to -\frac{\mathrm{d}^3}{\mathrm{d}x^3}$ as $\alpha \to 3$.

For harmonic wave propagation, the corresponding solutions can be written in a complex form as $u(x,t) = u_0 \exp(i\omega t - ikx)$. Injecting this expression in Eq. (5) and Eq. (3) gives:

$$\frac{\partial \sigma}{\partial x} = -\rho\omega^2 u \quad \text{and} \quad \varepsilon = -iku \tag{8}$$

Further, we use the Fourier transform:

$$\hat{f}(k) = \Im\big[f(x)\big](k) = \int_{-\infty}^{+\infty} f(x) e^{-ikx}\,\mathrm{d}x \tag{9}$$

We have the following formulae:

$$\begin{aligned}
&\Im\left[\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x)\right](k) = (ik)^n \hat{f}(k), \qquad \Im\big[f(x) * g(x)\big](k) = \hat{f}(k)\hat{g}(k) \\
&\Im\left[\frac{1}{|x|^\beta}\right](k) = 2\Gamma(1-\beta)\sin\frac{\beta\pi}{2}\frac{1}{|k|^{1-\beta}}, \quad \beta \in [0; 1] \\
&\Im\left[\frac{\operatorname{sgn}(x)}{|x|^\beta}\right](k) = -2i\Gamma(1-\beta)\cos\frac{\beta\pi}{2}\frac{\operatorname{sgn}(k)}{|k|^{1-\beta}}, \quad \beta \in [0; 1]
\end{aligned} \tag{10}$$

Applying the Fourier transform to Eq. (8), we obtain:

$$\frac{\hat{\sigma}}{\hat{\varepsilon}} = \frac{\rho\omega^2}{k} \tag{11}$$

Next, we apply the Fourier transform to the symmetrized Caputo fractional derivative (6), (7) and using Eqs. (10) in the both cases, i.e., for $\alpha \in [1; 3[$, obtain:

$$\Im\big[D_x^\alpha \sigma\big] = \cos\frac{\alpha\pi}{2}|k|^\alpha \hat{\sigma} \tag{12}$$

so that Eq. (1), in the Fourier domain becomes:

$$\frac{\hat{\sigma}}{\hat{\varepsilon}} = \frac{E}{1 - \cos\frac{\alpha\pi}{2}(l_c|k|)^\alpha} \tag{13}$$

with $\alpha \in [1; 3[$. When combining Eq. (13) with Eq. (11), we finally obtain the following dispersive equation:

$$\omega = \pm k c_0 \sqrt{\frac{1}{1 - \cos(\frac{\alpha\pi}{2})(l_c|k|)^\alpha}} \tag{14}$$

where $c_0 = \sqrt{E/\rho}$ is the celerity of axial wave. Dispersive equation (14) is true for the all range of parameters $\alpha \in [1; 3[$.

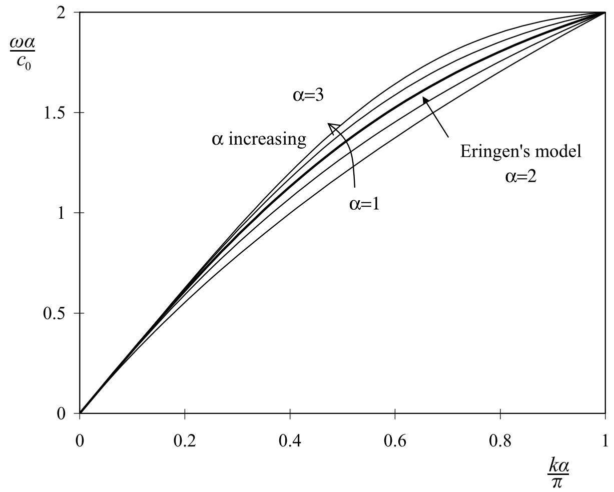

Fig. 1. Dispersion curve for the fractional nonlocal elasticity model – parameterized study for $\alpha \in \{1; 1.5; 2; 2.5; 3\}$.

4. Dispersive wave equation

This dispersive wave equation can be rewritten for positive k as:

$$\frac{\omega a}{c_0} = ka\sqrt{\frac{1}{1-\cos(\frac{\alpha\pi}{2})(ak)^{\alpha}(\frac{l_c}{a})^{\alpha}}} \tag{15}$$

where a is the distance between atoms, with reference to lattice dynamics behaviour. The present nonlocal model can be compared to the Born–Kármán model of lattice dynamics where the nearest-neighbour interactions are accounted (Eringen [3,5]). This model gives the following dispersion relation:

$$\frac{\omega_{bk} a}{c_0} = 2\sin\left(\frac{ka}{2}\right) \tag{16}$$

The foregoing nonlocal model is able to predict the Born–Kármán model for specified values of the length scale parameter l_c. This parameter could be identified from the characteristic prediction of the dispersive curve of lattice dynamics and Eqs. (15), (16):

$$\frac{\omega a}{c_0}(ka = \pi) = 2 \quad \Rightarrow \quad \cos\left(\frac{\alpha\pi}{2}\right)\left(\frac{l_c\pi}{a}\right)^{\alpha} = 1 - \frac{\pi^2}{4} \leqslant 0 \tag{17}$$

valid for $\alpha \in [1; 3[$. Inserting the solution of Eq. (17) into Eq. (15) leads to the dispersive wave relation:

$$\frac{\omega a}{c_0} = ka\sqrt{\frac{1}{1+(\frac{\pi^2}{4}-1)(\frac{ka}{\pi})^{\alpha}}} \tag{18}$$

Fig. 1 shows the dispersive curve parameterized by the fractional order α. It is shown that the nonlinearity of the dispersive curve in Fig. 1 is more pronounced for higher values of the fractional order α.

It is possible to match adequately the Born–Kármán model, by adding an optimality condition based on the mean square error. Introducing the dimensionless parameters:

$$\overline{\omega} = \frac{\omega a}{c_0} \quad \text{and} \quad \bar{k} = \frac{ka}{\pi} \tag{19}$$

in the previous expressions (16) and (18), we obtain the dimensionless dispersion functions:

$$\overline{\omega} = \pi\bar{k}\sqrt{\frac{1}{1+(\frac{\pi^2}{4}-1)\bar{k}^{\alpha}}} \quad \text{and} \quad \overline{\omega}_{bk} = 2\sin\left(\frac{\pi\bar{k}}{2}\right) \tag{20}$$

The optimal fitting is obtained from the integral equation:

$$\frac{\partial}{\partial\alpha}\left[\int_0^1 \left(\overline{\omega}_{bk}(\bar{k}) - \overline{\omega}(\bar{k})\right)^2 \mathrm{d}\bar{k}\right] = 0 \quad \Rightarrow \quad \frac{\partial}{\partial\alpha}\left[\int_0^1 \left(2\sin\left(\frac{\pi\bar{k}}{2}\right) - \frac{\pi\bar{k}}{\sqrt{1+(\frac{\pi^2}{4}-1)\bar{k}^{\alpha}}}\right)^2 \mathrm{d}\bar{k}\right] = 0 \tag{21}$$

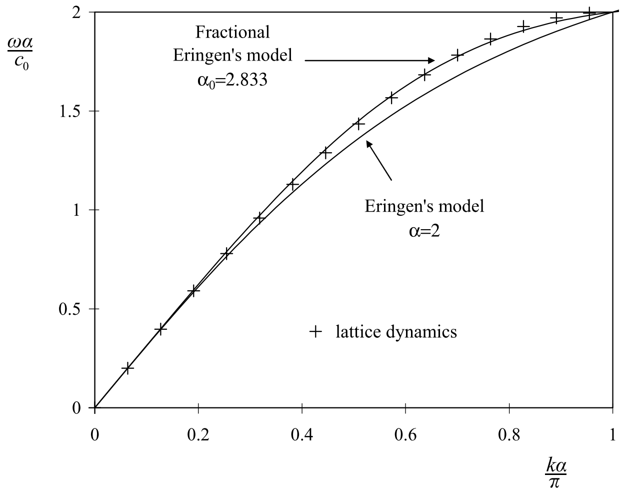

Fig. 2. Dispersion curve for the fractional nonlocal elasticity model – comparison with the Born–Kármán model of lattice dynamics.

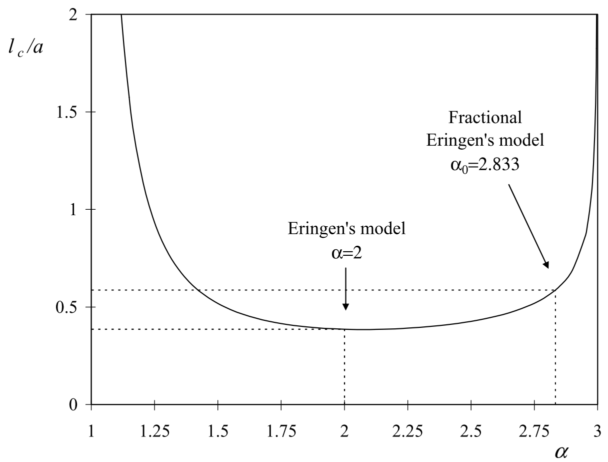

Fig. 3. Evolution of the length scale parameters with respect to the fractional derivative order.

The criterion expressing the optimality is then found from Eq. (21) as:

$$\int_0^1 \left[2\frac{\sin(\frac{\pi\bar{k}}{2})\bar{k}^{\alpha+1}\ln\bar{k}}{(\sqrt{1+(\frac{\pi^2}{4}-1)\bar{k}^\alpha})^3} - \frac{\pi\bar{k}^{\alpha+2}\ln\bar{k}}{(1+(\frac{\pi^2}{4}-1)\bar{k}^\alpha)^2} \right] \mathrm{d}\bar{k} = 0 \quad \Rightarrow \quad \alpha_0 \approx 2.833 \tag{22}$$

As shown in Fig. 2, an excellent matching of the dispersive curve of the Born–Kármán model of lattice dynamics is obtained with such generalized integral-based nonlocal model, using the numerical optimum $\alpha_0 = 2.833$ for the order of the fractional derivative. Fractional derivative theory appears to be an efficient engineering tool to calibrate the dispersive wave properties of the Born–Kármán model.

Furthermore, the length scale parameter can be easily identified thanks to Eq. (17):

$$\left(\frac{l_c}{a}\right)_0 = \frac{1}{\pi}\sqrt[\alpha_0]{\frac{4-\pi^2}{4\cos(\frac{\alpha_0\pi}{2})}} \approx 0.587 \quad \text{for } \alpha_0 = 2.833 \tag{23}$$

This value $l_c \cong 0.587a$ is not so far from the well-known value $l_c \cong 0.386a$ found in the case of Eringen's nonlocal model based on $\alpha = 2$ (see also Eringen [3,5]; Challamel et al. [15]). More generally, the length scale parameter is shown to be dependent on the order of the fractional derivative, as highlighted by Fig. 3. The length scale $l_c \approx 0.386a$ obtained for $\alpha = 2$ is almost the minimum value, even if the exact minimum is obtained for a slightly larger value $\alpha \approx 2.076$ associated with $l_c \approx 0.384a$.

We note that the present fractional generalization of Eringen's model is an alternative efficient engineering tool, and has the advantage and the simplicity to be based on only one length scale, whereas the recent model of Challamel et al. [15] which also fits well the Born–Kármán model, is based on two length scales.

5. Conclusions

A fractional nonlocal elasticity model is studied in this paper. This model can be understood as a possible generalization of Eringen's nonlocal elastic model, by including a new free parameter: the order of non-integer derivative in the stress–strain fractional order differential equation. This model only contains a single length scale and the fractional derivative order. It is possible to obtain an analytical dispersive equation for the axial wave problem, which is parameterized by the fractional derivative order. The optimized fractional derivative model shows a perfect matching with the dispersive wave properties of the Born–Kármán model of lattice dynamics. For realistic engineering applications, it has been assumed that the order of the fractional derivative α is comprised between 1 and 3, but other values of the fractional derivative order can be chosen as well. For higher values of the fractional derivative constant, the sign of the coefficient that affects the fractional derivative term is then dependent on the order of the fractional derivative. The static and dynamic properties of such a nonlocal model have now to be evaluated for general loading and geometry configurations.

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