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Links between effective tensors for fiber-reinforced elastic composites

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A R T I C L E   I N F O

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Predicting the effective elasticity of a composite material based on the elasticity of the constituent materials is extremely difficult, even when the microstructure is known. In this paper we consider a link between effective elastic tensors of composites with the same microgeometry but different constituent materials. Information about the effective tensor of one composite can then be used to determine the other. The general theory of exact relations allows us to identify all such links in principle. Here we describe a special set of links, for which one of the composites can be chosen arbitrarily. Several applications are considered and a number of microstructure-independent relations satisfied by the effective elastic tensors is derived.

1. Introduction

The great benefit of composite materials lies in the fact that the undesirable properties of the constituent materials may not be present in the resulting composite. Here we use the term composite to describe any material with heterogeneous structure on the micro-scale that behaves like a homogeneous material on a macro-scale. Whether by layering, laminating, injecting, or encasing different constituent materials, engineers have developed new composites with effective properties that cannot be found in nature. There now exist materials that are both lightweight and strong and therefore useful in applications ranging from orthopedic casting to aerospace technology. Modern ski construction uses composites to obtain a unique combination of flexibility and torsional rigidity. Yet exactly when and how a composite will retain the properties of its constituent materials and when and how it will not remains a largely unanswered question. Certainly the microstructure of the composite material plays a significant role. However, there are also relations that hold regardless of microstructure.

For example, an elastic composite material made of two isotropic materials may in many cases be anisotropic. However, if the two constituent materials have the same shear modulus, μ, then the composite will be isotropic with shear modulus μ, regardless of the microstructure of the material [1]. Furthermore, the bulk modulus κ^* of such a composite will be given by

$$\frac{1}{3\kappa^* + 4\mu} = \frac{\theta_1}{3\kappa_1 + 4\mu} + \frac{\theta_2}{3\kappa_2 + 4\mu} \tag{1}$$

where θ_1 and θ_2 represent the volume fractions and κ_1 and κ_2 represent the bulk moduli of the two constituents. Given the high cost of producing composite materials, the ability to obtain results such as these without conducting expensive tests is valuable.

An *exact relation* describes a material property that is maintained in the construction of composites, regardless of microstructure. That is, it represents a characteristic of tensors for constituent materials that is always present in the effective tensors of composites made with those materials. In Hill's example above, the set of isotropic materials with a given shear modulus forms an exact relation. It is therefore helpful to be aware of exact relations both so that we may take advantage

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of them when we hope to maintain a characteristic and also so that we may avoid them when we want to change a given property.

Much of the early work on exact relations was on *uniform field relations* (UFR). These arise whenever there exist constant stress and strain (in the case of elasticity) tensors σ and ϵ such that

$$C(x)\epsilon = \sigma \quad \forall x$$

In this case the uniform fields σ and ϵ also satisfy $C^*\epsilon = \sigma$, where C^* is the effective tensor. So the set of materials satisfying a certain uniform field relation form an exact relation. The exact relation in (1) stems from this idea. So does the exact relation identified by Hill in [2] and described in [3] regarding materials that exhibit cubic symmetry. If C is the elasticity tensor of such a material and I represents the 3×3 identity matrix, then there exists $\kappa > 0$ such that $CI = \kappa I$. So we can think of κ as representing the bulk modulus of the material even though the material is not fully isotropic. A simple uniform field argument tells us that the effective bulk modulus of a statistically isotropic polycrystal made with this material is the same as the bulk modulus of the pure crystal. This result was generalized by He [4] to all materials that respond isotropically to isotropic stress or strain.

Hill's work in elasticity was followed by results from Lurie, Cherkaev, and Fedorov [5–7] and Francfort and Tartar [8]. Cribb [9], Rosen [10], Hashin [11], Schulgasser [12], and Dvorak [13] found exact relations in the context of thermoelasticity. Dvorak also specifically applied uniform field arguments to fiber-reinforced elastic composites [14]. Exact results for piezoelectric composites were discovered by Benveniste and Dvorak [15–18] while Milgrom and Shtrikman studied thermoelectricity [19–21]. Benveniste also found exact relations specifically for polycrystalline composites in the context of thermopiezoelectricity [22]. An excellent summary of exact relations can be found in [23]. Finally in [24–26], the elegant mathematical theory of exact relations and links was developed, allowing us to find all exact relations in a wide range of physical contexts including all of the above. In [27], To used this theory to find all exact relations for 3D conductors exhibiting the Hall effect, all of which were of the UFR type.

In this paper we use the theory of exact relations to obtain information about fiber-reinforced elastic composites. Fiber-reinforced composites are those whose microstructure is independent of the longitudinal coordinate and can therefore be described by a single transverse cross section. Furthermore, we focus our attention on *polycrystalline* exact relations. In other words, we require that if a material with elastic tensor C is an admissible constituent, then so is the rotated material $\mathcal{R} \cdot C$ where $\mathcal{R} \in SO(2)$ represents a rotation in the transverse plane. The image we have in mind is a composite created by injecting anisotropic fibers into an isotropic matrix. Fiber rotations around the longitudinal axis can be arbitrary. The exact relations we seek are the ones that hold regardless of fiber position, orientation, and cross section.

The theory described here takes a geometric point of view, seeing exact relations as surfaces in the space of elasticity tensors. It utilizes an explicit diffeomorphism that maps all such surfaces to $SO(2)$-invariant subspaces with special algebraic properties. Tools from representation theory help to identify all subspaces that satisfy these properties, and hence all exact relations.

Here we apply the theory of exact relations to compute *links* between tensors. When two composites have the same microstructure but different constituent materials, their effective tensors may be related. In this case we say that a link exists between the two composites. For example, in [28], it was found that for any two-dimensional local conductivity tensor $\sigma(x)$, if a second tensor is defined by

$$\sigma'(x) = \frac{\sigma(x)}{\det \sigma(x)}$$

then the same relation holds between the effective tensors of the two composites:

$$\sigma'^* = \frac{\sigma^*}{\det \sigma^*} \tag{2}$$

We choose to focus on links because in general they give more information than exact relations. Links map exact relations to exact relations. They can also generate exact relations. For example, the set of tensors unchanged by a link forms an exact relation.

By re-characterizing links as exact relations in a higher dimensional space, we can use the theory of exact relations to find and describe links. For the sake of completeness, we will begin with a brief overview of the theory of exact relations and links as well as the tools needed to complete the calculations for the case of fiber-reinforced elastic composites. We then use this theory to compute a special set of links that establishes equivalence between exact relation surfaces passing through different points. Finally, we apply our link to the case of a composite material made from two transversely isotropic materials and the case of a polycrystalline composite made from one orthotropic or tetragonal material.

2. Theory of exact relations

Without loss of generality, let us assume that the fibers in our composite are oriented vertically. Let

$$\mathcal{D} := \left\{ \mathbf{n} = (n_1, n_2, 0) \in \mathbb{R}^3 :\ \|\mathbf{n}\| = 1 \right\}$$

represent the set of directions in the transverse plane. Let $\mathrm{Sym}(\mathbb{R}^3)$ denote the Euclidean space of 3×3 symmetric matrices with the inner product

$$\langle E_1, E_2\rangle = \frac{1}{2}\mathrm{Tr}(E_1E_2)$$

For each $\mathbf{n} = (n, 0) = (n_1, n_2, 0) \in \mathcal{D}$, we define

$$\mathcal{E}_{\mathbf{n}} = \{\mathbf{a}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{a}\colon\ \mathbf{a}\in\mathbb{R}^3\}$$
$$\mathcal{J}_{\mathbf{n}} = \{\sigma\in\mathrm{Sym}(\mathbb{R}^3)\colon\ \sigma\mathbf{n} = 0\}$$

where for each $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$, we define the 3×3 matrix $\mathbf{a}\otimes\mathbf{b} = a_ib_j$. Note that $\mathrm{Sym}(\mathbb{R}^3) = \mathcal{E}_{\mathbf{n}}\oplus\mathcal{J}_{\mathbf{n}}$ for each $\mathbf{n}$. Let $\mathrm{Sym}(\mathrm{Sym}(\mathbb{R}^3))$ denote the set of symmetric linear maps from $\mathrm{Sym}(\mathbb{R}^3)$ to itself. Now fix an arbitrary reference tensor C_0 in the set of positive definite elasticity tensors,

$$\mathcal{T} = \{C\in\mathrm{Sym}(\mathrm{Sym}(\mathbb{R}^3))\colon\ \langle C\xi,\xi\rangle > 0 \text{ for all } \xi\in\mathrm{Sym}(\mathbb{R}^3),\ \xi\neq 0\}$$

Let $\Gamma(\mathbf{n})$ be the orthogonal projection onto $C_0^{1/2}\mathcal{E}_{\mathbf{n}}$. Following [29] we define:

$$W_{\mathbf{n}}(C) = \left[I - \left(I - C_0^{-1/2}CC_0^{-1/2}\right)\Gamma(\mathbf{n})\right]^{-1}\left(I - C_0^{-1/2}CC_0^{-1/2}\right) \tag{3}$$

This map has a special property in the case of laminates. If C^* represents the effective tensor of a laminate made with materials C_1 and C_2 taken in volume fractions f_1 and f_2, then

$$W_{\mathbf{n}}(C^*) = f_1W_{\mathbf{n}}(C_1) + f_2W_{\mathbf{n}}(C_2) \tag{4}$$

where $\mathbf{n}$ is the direction of lamination. If the surface $\mathcal{M}$ is an exact relation then certainly $\mathcal{M}$ is closed under lamination. Thus (4) implies that $W_{\mathbf{n}}(\mathcal{M})$ is a convex subset of some affine subspace, $\Pi_{\mathbf{n}}\subset\mathrm{Sym}(\mathrm{Sym}(\mathbb{R}^3))$, whose dimension is the same as $\mathcal{M}$. In fact, if we pick $C_0\in\mathcal{M}$, then $\Pi_{\mathbf{n}}$ is a vector space. Furthermore, from [26] we know that $W_{\mathbf{n}}$ is a diffeomorphism on $\mathcal{T}$, the subspaces $\Pi_{\mathbf{n}}$ do not depend on $\mathbf{n}$, and $\Pi := \Pi_{\mathbf{n}}$ (for any $\mathbf{n}$) has a special algebraic structure, which we will now describe.

Fix $\mathbf{n}_0\in\mathcal{D}$ and define

$$\mathcal{A} = \mathrm{Span}\{\Gamma(\mathbf{n}) - \Gamma(\mathbf{n}_0)\colon\ \mathbf{n}\in\mathcal{D}\}$$

Then $\mathcal{A}$ encodes the fiber-reinforced structure of the composite and does not depend on the choice of $\mathbf{n}_0$. In fact, if

$$\bar{\Gamma} := \frac{1}{2\pi}\int_0^{2\pi}\Gamma(\cos(t), \sin(t), 0)\,\mathrm{d}t$$

then

$$\mathcal{A} = \mathrm{Span}\{\Gamma(\mathbf{n}) - \bar{\Gamma}\colon\ \mathbf{n}\in\mathcal{D}\} \tag{5}$$

For each $A\in\mathcal{A}$, define a product $*^A$ on $\mathrm{Sym}(\mathrm{Sym}(\mathbb{R}^3))$ by:

$$K_1 *^A K_2 = \frac{1}{2}(K_1AK_2 + K_2AK_1)$$

for all $K_1, K_2\in\mathrm{Sym}(\mathrm{Sym}(\mathbb{R}^3))$. If $\mathcal{M}$ is an exact relation and Π is the vector space containing $W_{\mathbf{n}}(\mathcal{M})$ as described above, then

$$K_1 *^A K_2 \in \Pi \tag{6}$$

for all $A\in\mathcal{A}$ and for all $K_1, K_2\in\Pi$. The product $*^A$ is commutative and non-associative. It is called a *Jordan product* [30]. We say Π is a *Jordan multi-algebra* since Π is closed with respect to a whole family of Jordan multiplications.

That Π satisfies (6) follows from the stability of $\mathcal{M}$ with respect to making laminates and is certainly necessary for $\mathcal{M}$ to be an exact relation. In addition to this necessary condition, we have a related sufficient condition. We say that Π satisfies the *j-chain property* if for all $K_1, K_2, \ldots, K_j\in\Pi$ and $A_1, A_2, \ldots, A_{j-1}\in\mathcal{A}$,

$$K_1A_1K_2A_2\cdots A_{j-1}K_j + K_jA_{j-1}\cdots A_2K_2A_1K_1\in\Pi \tag{7}$$

In [26] it was shown that if Π satisfies the j-chain property for $j = 2, 3$, and 4, then it represents the image of an exact relation. However, so far every subspace found to satisfy the 2-chain property has satisfied 3- and 4-chain properties. Note also that $(K_1AK_2 + K_2AK_1) \in \Pi$ for all $K_1, K_2 \in \Pi$ and for all $A \in \mathcal{A}$ if and only if $KAK \in \Pi$ for all $K \in \Pi$ and for all $A \in \mathcal{A}$.

Because we are interested in polycrystalline exact relations and links, *transversely isotropic* tensors will be important. For each

$$\mathcal{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

in $SO(2)$ let

$$\tilde{\mathcal{R}} = \begin{bmatrix} \mathcal{R} & 0 \\ 0 & 1 \end{bmatrix}$$

so that for an arbitrary

$$E = \begin{bmatrix} \mathbf{E} & e \\ e^T & \varepsilon \end{bmatrix} \tag{8}$$

in $\mathrm{Sym}(\mathbb{R}^3)$, we can define

$$\mathcal{R} \cdot E = \tilde{\mathcal{R}} E \tilde{\mathcal{R}}^T = \begin{bmatrix} \mathcal{R}\mathbf{E}\mathcal{R}^T & \mathcal{R}e \\ (\mathcal{R}e)^T & \varepsilon \end{bmatrix} \tag{9}$$

Now if we define the action of $\mathcal{R}$ on $C \in \mathcal{T}$ by

$$\mathcal{R} \cdot (CE) = (\mathcal{R} \cdot C)(\mathcal{R} \cdot E)$$

for all $E \in \mathrm{Sym}(\mathbb{R}^3)$, then $\mathcal{R} \cdot C$ describes the elasticity tensor of the rotated material. We say C is transversely isotropic if $\mathcal{R} \cdot C = C$.

Suppose $\mathcal{M}$ is an exact relation. If our reference tensor $C_0 \in \mathcal{M}$ is transversely isotropic, then

$$\mathcal{R} \cdot W_{\mathbf{n}}(C) = W_{\mathcal{R}\mathbf{n}}(\mathcal{R} \cdot C)$$

for all $C \in \mathcal{M}$. Therefore Π is rotation-invariant if and only if $\mathcal{M}$ is polycrystalline. Thanks to the representation theory of two-dimensional rotations, we can easily find all rotation-invariant subspaces of $\mathrm{Sym}(\mathrm{Sym}(\mathbb{R}^3))$. We then investigate which of these subspaces satisfy the algebraic condition (6). Finally, we need to invert $W_{\mathbf{n}}$ to describe the polycrystalline exact relations in physical variables. Hereafter when we say exact relation we will mean polycrystalline exact relation. In [31] we have found a complete list of solutions of (6), containing over 200 exact relations. The work of converting them to physical variables and exploring their consequences remains to be done.

3. Theory of links

We can now apply these ideas to the concept of *links*. Given local elasticity tensors of two composites $C(x)$ and $C'(x)$, a function $\mathcal{G}$ on $\mathcal{T} \times \mathcal{T}$ is called a link if $\mathcal{G}(C^*, C'^*) = 0$ whenever $\mathcal{G}(C(x), C'(x)) = 0$. In the example from Mendelson above, we have a link given by $\mathcal{G}(\sigma, \sigma') = \frac{\sigma}{\det \sigma} - \sigma'$. It is easy to see that links are simply exact relations in $\mathcal{T} \times \mathcal{T}$ when we define

$$\hat{\mathcal{M}} = \{(C, C') \colon \mathcal{G}(C, C') = 0\}$$

and note that $\hat{\mathcal{M}}$ represents a link if $(C, C') \in \hat{\mathcal{M}}$ implies $(C^*, C'^*) \in \hat{\mathcal{M}}$. We can therefore use the theory described in Section 2 to find links. If we have a pair of fixed transversely isotropic tensors $(C_1, C_2) \in \hat{\mathcal{M}}$ we can map this link to the set

$$\hat{W}_{\mathbf{n}}(\hat{\mathcal{M}}) = \{(W^1_{\mathbf{n}}(C), W^2_{\mathbf{n}}(C')) \colon (C, C') \in \hat{\mathcal{M}}\} \tag{10}$$

where $W^i_{\mathbf{n}}$ is defined as in (3) using $C_0 = C_i$ for $i = 1, 2$. Once again, $\hat{W}_{\mathbf{n}}(\hat{\mathcal{M}})$ is a convex subset of some subspace $\hat{\Pi}$, which has the same dimension as $\mathcal{M}$.

Let $\Gamma_i(\mathbf{n})$ be the orthogonal projection onto $C_i^{1/2}\mathcal{E}_{\mathbf{n}}$. Define

$$\hat{\mathcal{A}} = \mathrm{Span}\{(\Gamma_1(\mathbf{n}) - \bar{\Gamma}_1, \Gamma_2(\mathbf{n}) - \bar{\Gamma}_2) \colon \mathbf{n} \in \mathcal{D}\} \tag{11}$$

Rewriting $\hat{A} \in \hat{\mathcal{A}}$ and $\hat{K} \in \hat{\Pi}$ as matrices, the general theory of exact relations tells us that if $\hat{\mathcal{M}}$ represents a link, then $\hat{\Pi}$ is a subspace satisfying

$$\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \in \hat{\Pi}$$

for all $(A_1, A_2) = \hat{A} \in \hat{\mathcal{A}}$ and for all $(K_1, K_2) = \hat{K} \in \hat{\Pi}$.

We will focus in this paper on a special set of links corresponding to

$$\hat{\Pi} = \{(K, \Phi(K)) :\ K \in \mathrm{Sym}(\mathrm{Sym}(\mathbb{R}^3))\} \tag{12}$$

where Φ is a linear bijection. Furthermore, since $\hat{\mathcal{M}}$ is an exact relation, we have from (6) that

$$\begin{bmatrix} K & 0 \\ 0 & \Phi(K) \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} \tilde{K} & 0 \\ 0 & \Phi(\tilde{K}) \end{bmatrix} + \begin{bmatrix} \tilde{K} & 0 \\ 0 & \Phi(\tilde{K}) \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} K & 0 \\ 0 & \Phi(K) \end{bmatrix}$$

is in $\hat{\Pi}$ for all $K, \tilde{K} \in \mathrm{Sym}(\mathrm{Sym}(\mathbb{R}^3))$ and for all $(A_1, A_2) \in \hat{\mathcal{A}}$. But this implies

$$\Phi(K *^{A_1} \tilde{K}) = \Phi(K) *^{A_2} \Phi(\tilde{K}) \tag{13}$$

for all $(A_1, A_2) \in \hat{\mathcal{A}}$, and for all $K, \tilde{K} \in \mathrm{Sym}(\mathrm{Sym}(\mathbb{R}^3))$.

Because we are focusing on polycrystalline exact relations, the subspace $\hat{\Pi}$ is $SO(2)$-invariant and therefore

$$\mathcal{R} \cdot \Phi(K) = \Phi(\mathcal{R} \cdot K) \tag{14}$$

for all $\mathcal{R} \in SO(2)$ and for all $K \in \mathrm{Sym}(\mathrm{Sym}(\mathbb{R}^3))$. We can find all Φ satisfying (13) and (14) using representation theory. These maps are described in Section 4. Then we will use the maps $W^1_{\mathbf{n}}$ and $W^2_{\mathbf{n}}$ to convert $\hat{\Pi}$ back to physical variables. This inversion is far from routine. The procedure is described in Section 5. The set of links $\hat{\mathcal{M}}$ corresponding to all such Φ is then described in Section 6.

4. Algebraic structure

We find it convenient to write elasticity tensors as 3×3 block matrices. If we think of $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$, then we may decompose

$$\mathrm{Sym}(\mathbb{R}^3) = \mathrm{Sym}(\mathbb{R}^2) \oplus \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}) \oplus \mathrm{Sym}(\mathbb{R})$$

where $\mathrm{Hom}(\mathbb{R}^2, \mathbb{R})$ represents the set of linear maps from $\mathbb{R}^2$ to $\mathbb{R}$. First let us write the three-dimensional stress and strain matrices

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \quad \text{and} \quad \epsilon = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix}$$

in the block form

$$\sigma = \begin{bmatrix} \tilde{\sigma} & s \\ s^T & \varsigma \end{bmatrix} \quad \text{and} \quad \epsilon = \begin{bmatrix} \tilde{\epsilon} & e \\ e^T & \delta \end{bmatrix}$$

where

$$\tilde{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}, \qquad \tilde{\epsilon} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{xy} & \epsilon_{yy} \end{bmatrix}$$

$$s = \begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}, \qquad e = \begin{bmatrix} \epsilon_{xz} \\ \epsilon_{yz} \end{bmatrix}, \qquad \varsigma = \sigma_{zz}, \qquad \delta = \epsilon_{zz}$$

Then we can write the elasticity tensor C effecting the constitutive relation $\sigma = C\epsilon$ in the form

$$C = \begin{bmatrix} \mathsf{C} & \mathcal{C} & \mathbf{c} \\ \mathcal{C}^* & \mathbf{C} & c \\ \mathbf{c}^* & c^* & \gamma \end{bmatrix}$$

where

$$\mathsf{C} : \mathrm{Sym}(\mathbb{R}^2) \to \mathrm{Sym}(\mathbb{R}^2), \qquad \mathcal{C} : \mathbb{R}^2 \to \mathrm{Sym}(\mathbb{R}^2), \qquad \mathbf{c} : \mathbb{R} \to \mathrm{Sym}(\mathbb{R}^2), \qquad \mathbf{C} : \mathbb{R}^2 \to \mathbb{R}^2$$
$$c : \mathbb{R} \to \mathbb{R}^2, \qquad \mathcal{C}^* : \mathrm{Sym}(\mathbb{R}^2) \to \mathbb{R}^2, \qquad \mathbf{c}^* : \mathrm{Sym}(\mathbb{R}^2) \to \mathbb{R}, \qquad c^* : \mathbb{R}^2 \to \mathbb{R}, \qquad \gamma : \mathbb{R} \to \mathbb{R}$$

and where the dual spaces to $\mathrm{Sym}(\mathbb{R}^2)$, $\mathbb{R}^2$, and $\mathbb{R}$ are identified with themselves via the inner products

$$\langle \mathbf{E}_1, \mathbf{E}_2 \rangle = \frac{1}{2} \mathrm{Tr}(\mathbf{E}_1 \mathbf{E}_2), \qquad \langle e_1, e_2 \rangle = e_1 \cdot e_2, \quad \text{and} \quad \langle \delta_1, \delta_2 \rangle = \delta_1 \delta_2 \tag{15}$$

respectively. In addition, the maps C and $\mathbf{C}$ are symmetric and the maps $\mathbf{c}$, c, and γ can be identified with the corresponding images of $1 \in \mathbb{R}$. That is, we identify $\mathbf{c}$ with the matrix $\mathbf{c} \in \mathrm{Sym}(\mathbb{R}^2)$ into which the operator $\mathbf{c}$ maps $1 \in \mathbb{R}$, the map c with the vector $c \in \mathbb{R}^2$, and the map γ with the number $\gamma \in \mathbb{R}$.

With these conventions and notations, the constitutive relation $\sigma = C\epsilon$ can be written as

$$\tilde{\sigma} = \mathsf{C}\tilde{\epsilon} + \mathcal{C}e + \frac{\delta}{\sqrt{2}}\mathbf{c}, \qquad s = \mathcal{C}^*\tilde{\epsilon} + \mathbf{C}e + \frac{\delta}{\sqrt{2}}c, \qquad \varsigma = \frac{1}{\sqrt{2}}\mathrm{Tr}(\mathbf{c}\tilde{\epsilon}) + \sqrt{2}(c \cdot e) + \gamma\delta$$

The coefficients of $1/\sqrt{2}$ appear merely as consequences of aligning different forms of the inner product.

We are now equipped to describe the fixed transversely isotropic tensor, C_0, the projector $\Gamma(\mathbf{n})$, and the subspace $\mathcal{A} \subset \mathrm{Sym}(\mathrm{Sym}(\mathbb{R}^3))$. We will skip most of the details of the calculations which can be found in [31]. If G_0 is transversely isotropic, then we can write it as

$$G_0 = \begin{bmatrix} \mathsf{C}(\kappa_0, \mu_0) & 0 & \alpha_0\mathbf{I} \\ 0 & \rho_0\mathbf{I} & 0 \\ \alpha_0\mathbf{I} & 0 & \gamma_0 \end{bmatrix}$$

for some $\kappa_0, \mu_0, \alpha_0, \rho_0, \gamma_0 \in \mathbb{R}$, where $\mathsf{C}(\kappa, \mu)$ is a two-dimensional elastic isotropic tensor with bulk modulus $\kappa/2$ and shear modulus $\mu/2$. Let $C_0 = G_0^2$. If $c \in \mathbb{C}$ is written as $c = \alpha + i\beta$ where $\alpha, \beta \in \mathbb{R}$, let us define

$$\psi(c) = \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}$$

Then any matrix in $\mathrm{Sym}(\mathbb{R}^2)$ can be written as $r\mathbf{I} + \psi(c)$ for some $r \in \mathbb{R}$ and $c \in \mathbb{C}$. From (5) we see that, having computed the orthogonal projector $\Gamma(\mathbf{n})$ onto $G_0\mathcal{E}_{\mathbf{n}}$, what we are truly interested in is the difference:

$$\Gamma(\mathbf{n}) - \bar{\Gamma} = \frac{1}{\vartheta_0}\begin{bmatrix} \mathsf{A}_{2D}(\kappa_0\mu_0 v, -\frac{1}{2}(\kappa_0^2 + \alpha_0^2)v^2) & 0 & \psi(\alpha_0\mu_0 v) \\ 0 & \psi(\frac{1}{2}\vartheta_0 v) & 0 \\ \psi(\alpha_0\mu_0 v) & 0 & 0 \end{bmatrix} \tag{16}$$

where $\vartheta_0 = \kappa_0^2 + \alpha_0^2 + \mu_0^2$, $v = n^2$ ($n \in \mathbb{C} \cong \mathbb{R}^2$ comes from $\mathbf{n} = (n, 0)$, $\|n\| = 1$), and we define $\mathsf{A}_{2D}(z, w) : \mathrm{Sym}(\mathbb{R}^2) \to \mathrm{Sym}(\mathbb{R}^2)$ by the action

$$\mathsf{A}_{2D}(z, w)\big(r\mathbf{I} + \psi(c)\big) = \langle z, c\rangle\mathbf{I} + \psi(rz + \bar{c}w)$$

Now, as seen in (5), $\mathcal{A}$ consists of the real span of all tensors of the form in (16). It is a simple exercise to show that the real span of $\{(v, v^2):\ v \in \mathbb{C},\ \|v\| = 1\}$ is simply $\mathbb{C}^2$. From this it follows that we may write the subspace

$$\mathcal{A} = \left\{\begin{bmatrix} \mathsf{A}_{2D}(2\kappa_0\mu_0 z, w) & 0 & \psi(2\alpha_0\mu_0 z) \\ 0 & \psi(\vartheta_0 z) & 0 \\ \psi(2\alpha_0\mu_0 z) & 0 & 0 \end{bmatrix} :\ w, z \in \mathbb{C}\right\}$$

To make our calculations easier, it is desirable to put $\mathcal{A}$ in a simpler form. Suppose B_1 is an arbitrary transversely isotropic tensor, not necessarily symmetric. Let $\Pi' = B_1^{-T}\Pi B_1^{-1}$ and let $\mathcal{A}' = B_1\mathcal{A}B_1^T$. Then

$$K_1 *^A K_2 \in \Pi \quad \Leftrightarrow \quad K_1' *^{A'} K_2' \in \Pi'$$

for all $A' = B_1 A B_1^T \in \mathcal{A}'$ and for all $K_1' = B_1^{-T}K_1B_1^{-1}$, $K_2' = B_1^{-T}K_2B_1^{-1} \in \Pi'$. In the case of fiber-reinforced periodic composites, we find that by choosing an appropriate B_1, we can simplify $\mathcal{A}'$ so that

$$\mathcal{A}' = \left\{\begin{bmatrix} \mathsf{A}_{2D}(z, w) & 0 & 0 \\ 0 & \psi(z) & 0 \\ 0 & 0 & 0 \end{bmatrix} :\ w, z \in \mathbb{C}\right\} \tag{17}$$

This simplification was key in calculating the complete list of solutions to (6) in [31].

However, in the case of links we have two copies of $\mathcal{A}$, each of which depends on a different fixed transversely isotropic tensor. Therefore, as we simplified $\mathcal{A}$ using B_1, we will simplify $\hat{\mathcal{A}}$ using $\hat{B}$ so that

$$\hat{B}\hat{A}\hat{B}^T = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} B_1^T & 0 \\ 0 & B_2^T \end{bmatrix} = \begin{bmatrix} A' & 0 \\ 0 & A' \end{bmatrix}$$

and now $\hat{\mathcal{A}}' = \{[A', A']:\ A' \in \mathcal{A}'\}$. This will simplify our task of finding the maps Φ as in (12). Such Φ will satisfy (13) and (14), so they can also be regarded as symmetries (or automorphisms) of Eq. (6). As such, they form a group. In [31] we prove that all such Φ have the form $\Phi(K') = XK'X^T$ where

$$X \in \left\{ \begin{bmatrix} \mathsf{I} & 0 & 0 \\ 0 & \phi(\pm 1) & 0 \\ \sigma \mathbf{I} & 0 & \tau \end{bmatrix} : \sigma, \tau \in \mathbb{R},\ \tau \neq 0 \right\} \tag{18}$$

and I is the identity operator on $\mathrm{Sym}(\mathbb{R}^2)$.

5. Inversion formula

Inverting the maps $W_{\mathbf{n}}$ and $\hat{W}_{\mathbf{n}}$ and calculating exact relations and links in physical variables is not straightforward. While we could use the matrix notation above to write elasticity tensors as 6×6 matrices and ask a computer algebra program to calculate the inverse, we would then struggle to interpret the formula in terms of the block structure we have developed. Instead we solve a simpler problem. We take advantage of the following statement proved in [26]. If $M_0 \in \mathrm{Sym}(\mathrm{Sym}(\mathbb{R}^3))$ is such that

$$K'\left(B_1 \bar{\Gamma} B_1^T - M_0\right) K' \in \Pi'$$

for all $K' \in \Pi'$, then

$$\mathcal{M} = \left\{ C = C_0 - C_0^{1/2} B_1^T \left(I + K' M_0\right)^{-1} K' B_1 C_0^{1/2} \colon\ K' \in \Pi' \right\} \tag{19}$$

Certainly $M_0 = B_1 \bar{\Gamma} B_1^T$ will work in every case, however simpler choices of M_0 are often possible.

For the case of links, this formula means we need M_1, M_2 such that

$$\begin{bmatrix} K & 0 \\ 0 & \Phi(K) \end{bmatrix} \begin{bmatrix} B_1 \bar{\Gamma}_1 B_1^T - M_1 & 0 \\ 0 & B_2 \bar{\Gamma}_2 B_2^T - M_2 \end{bmatrix} \begin{bmatrix} K & 0 \\ 0 & \Phi(K) \end{bmatrix} \in \hat{\Pi}$$

for all $K \in \mathrm{Sym}(\mathrm{Sym}(\mathbb{R}^3))$. That is, by definition of $\hat{\Pi}$, we need

$$\Phi\left(K\left(B_1 \bar{\Gamma}_1 B_1^T - M_1\right) K\right) = \Phi(K)\left(B_2 \bar{\Gamma}_2 B_2^T - M_2\right) \Phi(K) \tag{20}$$

for all $K \in \mathrm{Sym}(\mathrm{Sym}(\mathbb{R}^3))$. Using the definition $\Phi(K) = X K X^T$ with (20) and a quick simplification, this implies

$$B_1 \bar{\Gamma}_1 B_1^T - M_1 = X^T \left(B_2 \bar{\Gamma}_2 B_2^T - M_2\right) X \tag{21}$$

Using (19), write our two linked tensors C and C' as

$$C = C_1 - C_1^{1/2} B_1^T \left(K_1^{-1} + M_1\right)^{-1} B_1 C_1^{1/2} \tag{22}$$

$$C' = C_2 - C_2^{1/2} B_2^T \left(K_2^{-1} + M_2\right)^{-1} B_2 C_2^{1/2} \tag{23}$$

By definition of the link, $K_2 = \Phi(K_1)$. So solving (22) for K_1^{-1}, using the fact that $K_2^{-1} = X^{-T} K_1^{-1} X^{-1}$, and substituting this into (23), we have

$$C' = C_2 - C_2^{1/2} B_2^T \left(X^{-T}\left[B_1 C_1^{1/2} (C_1 - C)^{-1} C_1^{1/2} B_1^T - M_1\right] X^{-1} + M_2\right)^{-1} B_2 C_2^{1/2} \tag{24}$$

We show in [31] that $B_2 \bar{\Gamma}_2 B_2^T$ has a special block diagonal structure with zero in the bottom right block. Combining this fact with the description of X given in (18), we can see that $X^T B_2 \bar{\Gamma}_2 B_2^T X = B_2 \bar{\Gamma}_2 B_2^T$. Therefore, expanding (24) and selecting any pair (M_1, M_2) satisfying (21), terms cancel such that we can write

$$C' = C_2 - \left[I + \Delta(C) H\right]^{-1} \Delta(C) \tag{25}$$

where

$$H = C_2^{-1/2} B_2^{-1} \left(-B_1 \bar{\Gamma}_1 B_1^T + B_2 \bar{\Gamma}_2 B_2^T\right) B_2^{-T} C_2^{-1/2}$$

and

$$\Delta(C) = S - Q C Q^T$$

and where S and Q are transversely isotropic tensors defined by

$$S = C_2^{1/2} B_2^T X B_1^{-T} B_1^{-1} X^T B_2 C_2^{1/2} \tag{26}$$

and

$$Q = C_2^{1/2} B_2^T X B_1^{-T} C_1^{-1/2} \tag{27}$$

6. Link

Then, if $H \neq 0$, the link (25) takes the form

$$C' = \begin{bmatrix} a_1^2(\mathsf{G}_0\Theta(\mathsf{C})\mathsf{G}_0 - a_0\mathsf{G}_0) & a_1a_2\mathsf{G}_0\Theta(\mathsf{C})\mathcal{C} & a_4\mathbf{I} + a_1a_3\mathsf{G}_0\Theta(\mathsf{C})(\mathbf{c} + a_5\mathbf{I}) \\ (C'_{1,2})^T & a_2^2(\mathcal{C}^T\Theta(\mathsf{C})\mathcal{C} + \mathbf{C}) & a_2a_3(c + \mathcal{C}^T\Theta(\mathsf{C})(\mathbf{c} + a_5\mathbf{I})) \\ (C'_{1,3})^T & (C'_{2,3})^T & a_6 + a_3^2(\gamma + \langle\Theta(\mathsf{C})(\mathbf{c} + a_5\mathbf{I}), (\mathbf{c} + a_5\mathbf{I})\rangle) \end{bmatrix} \tag{28}$$

where $a_i \in \mathbb{R}$ for $i = 0, \dots, 6$ and

$$\Theta(\mathsf{C}) := \left(a_0^{-1}\mathsf{G}_0 - \mathsf{C}\right)^{-1} \tag{29}$$

Here the map $\mathsf{G}_0 : \mathrm{Sym}(\mathbb{R}^2) \to \mathrm{Sym}(\mathbb{R}^2)$ is defined by its action on $\mathbf{E} = \phi(\omega) + \psi(z) \in \mathrm{Sym}(\mathbb{R}^2)$:

$$\mathsf{G}_0\mathbf{E} = \phi(\omega) - \psi(z)$$

So we can think of G_0 as a two-dimensional isotropic elasticity tensor with bulk modulus $\frac{1}{2}$ and shear modulus $-\frac{1}{2}$. In other words, if we define the inner product of two symmetric 2×2 matrices as in (15) then

$$\langle \mathsf{G}_0\mathbf{E}, \mathbf{E}\rangle = \det \mathbf{E}$$

When $H = 0$, the link (25) becomes linear and has the form

$$C' = C_2 - S + QCQ^T \tag{30}$$

where S and Q are as defined in (26) and (27) and

$$F = C_2 - S = \begin{bmatrix} 0 & 0 & f_1\mathbf{I} \\ 0 & 0 & 0 \\ f_1\mathbf{I} & 0 & f_2 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} d_1\mathsf{I} & 0 & 0 \\ 0 & d_2\mathbf{I} & 0 \\ d_4\mathbf{I} & 0 & d_3 \end{bmatrix}$$

for some $f_1, f_2, d_1, d_2, d_3, d_4 \in \mathbb{R}$. The link is then

$$C' = \begin{bmatrix} d_1^2\mathsf{C} & d_1d_2\mathcal{C} & f_1\mathbf{I} + d_1d_3\mathbf{c} + d_1d_4\mathsf{C}\mathbf{I} \\ (C'_{1,2})^T & d_2^2\mathbf{C} & d_2d_3c + d_2d_4\mathcal{C}^T\mathbf{I} \\ (C'_{1,3})^T & (C'_{2,3})^T & f_2 + d_3^2\gamma + 2d_3d_4\langle\mathbf{c}, \mathbf{I}\rangle + d_4^2\langle\mathsf{C}\mathbf{I}, \mathbf{I}\rangle \end{bmatrix} \tag{31}$$

This special case corresponds to the limit of the general link when certain constants go to zero or infinity. Using the Neumann series expansion,

$$\Theta(\mathsf{C}) = a_0\mathsf{G}_0 + a_0^2\mathsf{G}_0\mathsf{C}\mathsf{G}_0 + O\left(a_0^3\right)$$

Fix $a_2 = d_2$ and $a_3 = d_3$ and let $a_0 \to 0$ and $a_1, a_4, a_5, a_6 \to \infty$ be such that

$$a_0a_1 \to d_1, \qquad a_0a_5 \to \frac{d_4}{a_3}, \qquad a_4 + a_0a_1a_3a_5 \to f_1, \qquad a_6 + a_0a_3^2a_5^2 \to f_2$$

Then the limit of the general case converges to the linear link $C' = F + QCQ^T$.

We make one further observation to clarify the relationship between the general case and the linear case: the essential nonlinearity in the general case (28) is manifested entirely in a_0. That is, we can rewrite any element of the general case as the composition of an element of the linear case and a special element of the general case involving only a_0. More explicitly, let $\mathcal{F}$ represent an arbitrary element of the general case (28) with parameters a_i. Then fix $\mathcal{F}_0$ to be a special case of (28) with parameters a_i^0 where $a_1^0 = a_2^0 = a_3^0 = 1$, $a_4^0 = a_5^0 = a_6^0 = 0$, and $a_0^0 = a_0$. That is,

$$\mathcal{F}_0(C) = \begin{bmatrix} a_0\mathsf{G}_0\Theta(\mathsf{C})\mathsf{C} & \mathsf{G}_0\Theta(\mathsf{C})\mathcal{C} & \mathsf{G}_0\Theta(\mathsf{C})\mathbf{c} \\ \mathcal{C}^T\Theta(\mathsf{C})\mathsf{G}_0 & \mathcal{C}^T\Theta(\mathsf{C})\mathcal{C} + \mathbf{C} & \mathcal{C}^T\Theta(\mathsf{C})\mathbf{c} + c \\ \mathbf{c}\Theta(\mathsf{C})\mathsf{G}_0 & \mathbf{c}\Theta(\mathsf{C})\mathcal{C} + c^T & \langle\Theta(\mathsf{C})\mathbf{c}, \mathbf{c}\rangle + \gamma \end{bmatrix}$$

If we let $\mathcal{F}_{lin}$ represent the linear link (31) with

$$d_1 = a_1, \qquad d_2 = a_2, \qquad d_3 = a_3, \qquad d_4 = a_0a_3a_5$$

$$f_1 = a_4 + a_0a_1a_3a_5, \quad \text{and} \quad f_2 = a_6 + a_0a_3^2a_5^2$$

then we have

$$\mathcal{F}_1\left(\mathcal{F}_{lin}(C)\right) = \mathcal{F}(C)$$

for all tensors $C \in \mathcal{T}$.

7. Applications

7.1. Two isotropic materials

Suppose we make a composite with two transversely isotropic materials, C_1 and C_2. We can write these as

$$C_i = \begin{bmatrix} \mathsf{C}_i(\kappa_i, \mu_i) & 0 & \alpha_i \mathbf{I}_2 \\ 0 & \rho_i \mathbf{I}_2 & 0 \\ \alpha_i \mathbf{I}_2 & 0 & \gamma_i \end{bmatrix}$$

The parameters above relate to the standard engineering constants in Voigt notation in the following ways:

$$\kappa_i = K_{11}^i + K_{12}^i, \qquad \mu_i = 2K_{66}^i, \qquad \rho_i = 2K_{44}^i, \qquad \gamma_i = K_{33}^i, \quad \text{and} \quad \alpha_i = \sqrt{2} K_{13}^i \tag{32}$$

Alternately, [14] uses the following six constants and one relation to describe transversely isotropic materials. The Young moduli in the longitudinal and transverse directions are given by

$$E_L^i = \frac{1}{\kappa_i}\left(\gamma_i \kappa_i - \alpha_i^2\right)$$

and

$$E_T^i = \frac{2\mu_i(\kappa_i \gamma_i - \alpha_i^2)}{(\kappa_i + \mu_i)\gamma_i - \alpha_i^2}$$

The Poisson ratio for loading on the longitudinal axis is

$$\nu_L^i = \frac{\alpha_i}{\sqrt{2}\kappa_i}$$

while the Poisson ratio describing the orthogonal contraction within the transverse plane due to tension applied in the transverse plane is

$$\nu_T^i = \frac{(\kappa_i - \mu_i)\gamma_i - \alpha_i^2}{(\kappa_i + \mu_i)\gamma_i - \alpha_i^2}$$

Finally, the shear moduli in the longitudinal and transverse directions are simply

$$G_L^i = \frac{1}{2}\rho_i \quad \text{and} \quad G_T^i = \frac{1}{2}\mu_i$$

The relation indicating the dependence between these constants is

$$G_T^i = \frac{E_T^i}{2(1 + \nu_T^i)}$$

Our linear link maps these transversely isotropic materials to two new transversely isotropic materials given by

$$\begin{aligned} C_i' &= \begin{bmatrix} \mathsf{C}_i'(\kappa_i', \mu_i') & 0 & \alpha_i' \mathbf{I}_2 \\ 0 & \rho_i' \mathbf{I}_2 & 0 \\ \alpha_i' \mathbf{I}_2 & 0 & \gamma_i' \end{bmatrix} \\ &= \begin{bmatrix} d_1^2 \mathsf{C}_i(\kappa_i, \mu_i) & 0 & (f_1 + d_1 d_3 \alpha_i + d_1 d_4 \kappa_i)\mathbf{I}_2 \\ 0 & d_2^2 \rho_i \mathbf{I}_2 & 0 \\ (f_1 + d_1 d_3 \alpha_i + d_1 d_4 \kappa_i)\mathbf{I}_2 & 0 & f_2 + d_3^2 \gamma_i + 2 d_3 d_4 \alpha_i + d_4^2 \kappa_i \end{bmatrix} \end{aligned} \tag{33}$$

We can always set $\alpha_i' = 0$ so that the C_i' are block diagonal, which allows us to take advantage of the following lemma.

Lemma 7.1. *If C represents the elasticity tensor of a fiber reinforced composite and is block diagonal of the form*

$$C = \begin{bmatrix} \mathsf{C} & 0 & 0 \\ 0 & \mathbf{C} & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

then its effective tensor is of the form

$$C^* = \begin{bmatrix} \mathsf{C}^* & 0 & 0 \\ 0 & \mathbf{C}^* & 0 \\ 0 & 0 & \langle\gamma\rangle \end{bmatrix}$$

where C^* *and* $\mathbf{C}^*$ *represent the effective elasticity and conductivity tensors of a two-dimensional composite with local elasticity tensor* C *and local conductivity tensor* $\mathbf{C}$ *and with the same microstructure as the original fiber-reinforced composite's transverse cross section.*

Proof. Let $\mathbf{u} = (u_1, u_2, u_3) = (u', u_3)$ represent a deformation. Then we can write

$$e(\mathbf{u}) = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) \\ \frac{1}{2}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) & \frac{1}{2}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$
$$= \begin{bmatrix} e(u') & \frac{1}{2}(\frac{\partial u'}{\partial x_3} + \nabla' u_3) \\ \frac{1}{2}(\frac{\partial u'}{\partial x_3} + \nabla' u_3)^T & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \in \mathrm{Sym}(\mathbb{R}^3)$$

and any arbitrary $\xi \in \mathrm{Sym}(\mathbb{R}^3)$ as

$$\xi = \begin{bmatrix} \xi' & \bar{\xi} \\ \bar{\xi}^T & \sqrt{2}\xi_{33} \end{bmatrix}$$

We assume for fiber reinforced composites that C is independent of x_3. Let us now suppose that a solution $\mathbf{u}$ exists to

$$\nabla \cdot C\big(e(\mathbf{u}) + \xi\big) = \mathbf{0} \tag{34}$$

which is independent of x_3 as well. Then (34) becomes

$$\nabla' \cdot \left(\mathsf{C}\big(e(u') + \xi'\big) + \mathcal{C}\left(\frac{1}{2}\nabla' u_3 + \bar{\xi}\right) + \mathbf{c}\xi_{33}\right) = \mathbf{0} \tag{35}$$

$$\nabla' \cdot \left(\mathcal{C}^T\big(e(u') + \xi'\big) + \mathbf{C}\left(\frac{1}{2}\nabla' u_3 + \bar{\xi}\right) + c\xi_{33}\right) = 0 \tag{36}$$

where the top line is a vector equation while the bottom is a scalar equation.

Now, if C is block diagonal, then the top vector equation simplifies to

$$\nabla' \cdot \big(\mathsf{C}\big(e(u') + \xi'\big)\big) = \mathbf{0} \tag{37}$$

while the scalar equation on the bottom becomes

$$\nabla' \cdot \left(\mathbf{C}\left(\frac{1}{2}\nabla' u_3 + \bar{\xi}\right)\right) = 0 \tag{38}$$

But we know solutions u' and u_3 to (37) and (38) exist and are unique. Therefore the unique solution $\mathbf{u} = (u', u_3)$ to (34) is x_3-independent. Since the effective tensor is defined by

$$C^*\xi = \big\langle C\big(e(u) + \xi\big)\big\rangle$$

for all $\xi \in \mathrm{Sym}(\mathbb{R}^3)$ and C is block diagonal, we have

$$\begin{bmatrix} \mathsf{C}^*\xi' + \mathcal{C}^*\bar{\xi} + \mathbf{c}^*\xi_{33} \\ \mathcal{C}^{*T}\xi' + \mathbf{C}^*\bar{\xi} + c^*\xi_{33} \\ \mathbf{c}^{*T}\xi' + c^{*T}\bar{\xi} + \gamma^*\xi_{33} \end{bmatrix} = \left\langle \begin{matrix} \mathsf{C}(e(u') + \xi') \\ \mathbf{C}(\frac{1}{2}\nabla' u_3 + \bar{\xi}) \\ \gamma\xi_{33} \end{matrix} \right\rangle$$

which implies $\mathcal{C}^*, \mathbf{c}^*$, and c^* are all zero. Furthermore, the effective tensors C^* and $\mathbf{C}^*$ are defined by the formulas

$$\mathsf{C}^*\xi' = \big\langle \mathsf{C}\big(e(u') + \xi'\big)\big\rangle$$

and

$$\mathbf{C}^*\bar{\xi} = \left\langle \mathbf{C}\left(\frac{1}{2}\nabla' u_3 + \bar{\xi}\right)\right\rangle$$

for all $\xi' \in \mathrm{Sym}(\mathbb{R}^2)$ and all $\bar{\xi} \in \mathbb{R}^2$, while $\gamma^* = \langle\gamma\rangle$. $\square$

Since we do not need to change the two-dimensional elasticity block or the two-dimensional conductivity block, we may assume $d_1 = d_2 = 1$. Let us assume C_1 and C_2 are ordered such that $\kappa_1 > \kappa_2$. Then we can set

$$d_4 = -\frac{\alpha_1 - \alpha_2}{\kappa_1 - \kappa_2} \quad \text{and} \quad f_1 = -\alpha_1 + \frac{\alpha_1 - \alpha_2}{\kappa_1 - \kappa_2}\kappa_1$$

to ensure $\alpha_1' = \alpha_2' = 0$ and

$$f_2 = 2\frac{\alpha_1 - \alpha_2}{\kappa_1 - \kappa_2}\alpha_2$$

to ensure that $\kappa_i' > 0$ for $i = 1, 2$. Then, taking advantage of the lemma, we have the effective tensors

$$\left(C'\right)^* = \begin{bmatrix} \mathsf{C}^* & 0 & 0 \\ 0 & \mathbf{C}^* & 0 \\ 0 & 0 & \langle\gamma'\rangle \end{bmatrix}$$

and

$$C^* = \begin{bmatrix} \mathsf{C}^* & 0 & \mathbf{c}^* \\ 0 & \mathbf{C}^* & 0 \\ \mathbf{c}^* & 0 & \gamma^* \end{bmatrix} \tag{39}$$

where

$$\mathbf{c}^* = \frac{\Delta\alpha}{2\Delta\kappa}\left(\mathsf{C}^*\mathbf{I}\right) + \left(\langle\alpha\rangle - \frac{\Delta\alpha}{\Delta\kappa}\langle\kappa\rangle\right)\mathbf{I}, \qquad \gamma^* = \langle\gamma\rangle + \left(\frac{\Delta\alpha}{2\Delta\kappa}\right)^2\left(\left(\mathsf{C}^*\mathbf{I}, \mathbf{I}\right) - \langle 2\kappa\rangle\right)$$

and

$$\Delta\beta = \beta_1 - \beta_2$$

This result was found by Rosen and Hashin [10] in the context of two-dimensional thermoelasticity where C^* is the effective two-dimensional elasticity tensor, $\mathbf{c}^*$ is the effective thermal expansion tensor, and γ^* is the coefficient of specific heat.

If $\mu_1 = \mu_2 = \mu$, then $\mu_1' = \mu_2'$ and we can apply Hill's exact relation to the two-dimensional elasticity block. Note that we can rewrite (1) as

$$\frac{1}{\kappa^* + \mu} = \left\langle \frac{1}{\kappa + \mu} \right\rangle \tag{40}$$

Then (39) can be augmented by the explicit formulas

$$\mathsf{C}^* = \mathsf{C}\left(\kappa^*, \mu\right), \qquad \mathbf{c}^* = \alpha^*\mathbf{I}_2, \quad \text{and} \quad \gamma^* = \langle\gamma\rangle + \left(\frac{\Delta\alpha}{\Delta\kappa}\right)^2\left(\kappa^* - \langle\kappa\rangle\right)$$

where

$$\kappa^* = \left\langle \frac{1}{\kappa + \mu} \right\rangle^{-1} - \mu, \qquad \alpha^* = \langle\alpha\rangle + \frac{\Delta\alpha}{\Delta\kappa}\left(\kappa^* - \langle\kappa\rangle\right)$$

and $\mathsf{C}(\kappa^*, \mu)$ is the elasticity tensor of a two-dimensional isotropic composite.

7.2. Polycrystal made from an orthotropic monocrystal

We can also apply the link to the case of a polycrystal made from an *orthotropic* material. We define an orthotropic material to be one that can be rotated into an orientation in which the material is invariant to flips along each of the three coordinate axes. The tensor of such a material (rotated into the appropriate position) is therefore invariant with respect to rotations in the group

$$\mathcal{Q} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

A straightforward calculation shows us that $\mathcal{Q}$-invariant tensors can be written in the form

$$C_0 = \begin{bmatrix} \mathsf{C}_0 & 0 & \mathbf{c}_0 \\ 0 & \mathbf{C}_0 & 0 \\ \mathbf{c}_0 & 0 & \gamma_0 \end{bmatrix}$$

where C_0 sends diagonal matrices to diagonal matrices and $\mathbf{C}_0$ and $\mathbf{c}_0$ are diagonal. We would like to set the $\mathbf{c}$-block equal to zero. We may let $d_1 = d_2 = d_3 = 1$ and fix d_4 and f_1 so that

$$\mathbf{c}' = \mathbf{c}_0 + f_1 \mathbf{I} + d_4 \mathsf{C}_0 \mathbf{I} = 0 \tag{41}$$

This is possible if and only if $\mathbf{c}_0$ is a scalar multiple of the identity *or* $\mathbf{I}$ is *not* an eigenvector of C_0.

Since the linked tensor C_0' is now block diagonal, we may again apply the lemma to see that for a polycrystal made using C_0', the effective tensor $(C_0')^*$ is block diagonal. For the polycrystal made using C_0 we may then write C_0^* as

$$C_0^* = \begin{bmatrix} \mathsf{C}_0^* & 0 & \mathbf{c}_0^* \\ 0 & \mathbf{C}_0^* & 0 \\ \mathbf{c}_0^* & 0 & \gamma_0^* \end{bmatrix}$$

where if

$$\mathsf{C}_0 \mathbf{I} = \begin{bmatrix} \kappa_0 + \upsilon_0 & 0 \\ 0 & \kappa_0 - \upsilon_0 \end{bmatrix} \quad \text{and} \quad \mathbf{c}_0 = \begin{bmatrix} \alpha_0 + \zeta_0 & 0 \\ 0 & \alpha_0 - \zeta_0 \end{bmatrix}$$

then

$$\mathbf{c}_0^* = \alpha_0 \mathbf{I} + \frac{\zeta_0}{\upsilon_0}\big(\mathsf{C}_0^* \mathbf{I} - \kappa_0 \mathbf{I}\big) \quad \text{and} \quad \gamma_0^* = \gamma_0 + \frac{\zeta_0^2}{\upsilon_0^2}\big(\langle \mathsf{C}_0^* \mathbf{I}, \mathbf{I}\rangle - \kappa_0\big) \tag{42}$$

This result was shown for the isotropic case by Hashin [11] and generalized by Schulgasser [12], both in context of thermoelasticity.

In particular, if C_0 sends scalar matrices to scalar matrices, i.e. if $\upsilon_0 = 0$, then [4] tells us that C_0^* does as well and $\kappa^* = \kappa_0$. In order to establish (41) we will need $\mathbf{c}_0 = \alpha_0 \mathbf{I}$. In this case, (42) become

$$\mathbf{c}_0^* = \alpha_0 \mathbf{I} \quad \text{and} \quad \gamma_0^* = \gamma_0 \tag{43}$$

Furthermore, if the texture of the polycrystal is statistically isotropic, then, taking advantage of (2) on the $\mathbf{C}$-block, we have (43) and

$$\mathbf{C}_0^* = \mathbf{I}\sqrt{\det \mathbf{C}_0}$$

Of course this also holds for the special case when $\mathbf{C}_0$ itself is scalar. Tensors satisfying all of these conditions, i.e. tensors such that

$$\upsilon_0 = \zeta_0 = 0 \quad \text{and} \quad \mathbf{C}_0 = \rho_0 \mathbf{I}$$

represent materials that are *tetragonal*. Such materials have a fourfold rotational symmetry about the transverse axis. The above then tells us that the effective tensor of a fiber-reinforced polycrystalline composite made with one tetragonal material will itself be tetragonal and

$$\kappa^* = \kappa_0, \qquad \alpha^* = \alpha_0, \qquad \rho^* = \rho_0, \quad \text{and} \quad \gamma^* = \gamma_0$$

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