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Exact solutions of the problem of free-boundary unsteady flows

Solutions exactes du problème des écoulements instationnaires à surface libre

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ABSTRACT

Some approach to the solution of boundary value problems for finding functions that are analytical in a wedge is proposed. If the ratio of the angle at the wedge vertex to the number π is rational, then the boundary value problem is reduced to the finite system of ordinary differential equations. Such approach, applied to the problem of inertial motion of a liquid wedge, made it possible to sum the series with small denominators arising in the problem and find four exact examples of self-similar flows with a free boundary.

RÉSUMÉ

Une approche permettant de déterminer des fonctions analytiques dans un secteur angulaire et respectant certaines conditions aux limites est proposée. Si le rapport de l'angle du secteur au nombre π est rationnel, le problème aux limites est réduit à un système fini d'équations différentielles ordinaires. Une telle approche, appliquée au problème du mouvement inertiel d'un secteur liquide, a permis de sommer la série comportant des petits dénominateurs intervenant dans le problème et de trouver quatre exemples exacts d'écoulements auto-semblables comportant une surface libre.

1. Statement of the problem

Very few exact solutions of the problem of planar unsteady flows of an ideal incompressible fluid with free boundaries are known. One of the examples is a flow with a linear velocity field [1,2].

In the present work, new solutions are proposed. The following problem is considered. Initially, at time $t = 0$, the fluid occupies the wedge with an apical angle α (Fig. 1). The origin of the Cartesian coordinate system x, y is placed at the vertex of the wedge.

A boundary of the flow region consists of two arcs. The first one is a rigid fixed wall $y = 0$, the other one is a free surface. Capillarity and gravitation are absent. The wall is added for the sake of symmetry. If we remove it, then we have a liquid wedge with an angle 2α. It is necessary to find the free surface and the complex velocity $U(z,t) = u(x,y,t) - \mathrm{i}v(x,y,t)$ for $t > 0$. Here $z = x + \mathrm{i}y$ and $u(x,y,t)$, $v(x,y,t)$ are the velocity components. The fluid movement is caused by the initial velocity field, which is supposed to be quadratic: $U(z,0) = Az^2$.

The problem contains only one dimensional real-valued parameter A. Therefore, the problem is self-similar. The solution of the problem depends only on the sign of this parameter.

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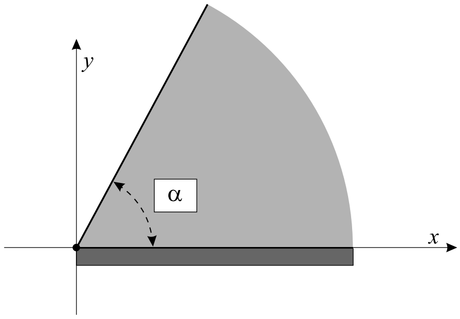

Fig. 1. Initial liquid configuration. Along x-axis disposes rigid wall.

2. Power series in time

We will seek a solution in the form of a power series in time. It follows, from the dimension considerations:

$$U(z,t) = Az^2\big(1 + a_1 Azt + a_2(Azt)^2 + a_3(Azt)^3 + \cdots\big) \tag{1}$$

Consequently, $U(0,t) = 0$. This means that a particle of the liquid initially situated at the vertex of the wedge always remains there. No dimensionless function of time exists. Therefore, the apical angle is always constant.

The impermeability condition at $y = 0$ implies that the coefficients a_j are real-valued. They can be obtained from the pressure constancy condition on the free surface as follows:

$$a_1 = -\frac{2}{\cos 4\alpha}, \qquad a_2 = \frac{5}{2}\frac{3\cos\alpha - \cos 7\alpha}{\cos 4\alpha \cos 5\alpha}$$
$$a_3 = \frac{-\frac{7}{2}\cos\alpha - 5\cos 3\alpha - \frac{23}{2}\cos 5\alpha - 7\cos 7\alpha + \frac{7}{2}\cos 9\alpha + 4\cos 11\alpha + \frac{23}{4}\cos 13\alpha - \frac{1}{4}\cos 19\alpha}{\cos^2 4\alpha \cos 5\alpha \cos 6\alpha}, \quad \ldots \tag{2}$$

Small denominators arise in the solution. For example, at $\alpha = \pi/8$, acceleration and pressure are initially infinite. It is not clear whether it has some physical sense or it was caused by the inadequacy of the series.

3. Conformal mapping

Formulae (2) for the coefficients a_j become complicated very rapidly as j increases. However, series (1) can be summed exactly for some values of α. In this paper, summation is performed for three values of α: $\pi/4$, $\pi/2$, $3\pi/4$. It turned out to be possible due to the usage of conformal mapping.

Let us consider in the auxiliary plane ζ a wedge with an angle α and a vertex at point $\zeta = 0$. Let $Z(\zeta,t)$ be a conformal mapping of this wedge onto the flow region and $U(\zeta,t)$ be a complex velocity. These functions have the representation $Z = g(\zeta)/(At)$, $U = f(\zeta)/(At^2)$. The boundary conditions to find unknown functions $f(\zeta)$, $g(\zeta)$ are of the form [3]:

$$\begin{array}{lll} \operatorname{Im} g_r(\overline{g} + f) = 0, & \operatorname{Re}(-2g_r f + f_r g + f_r \overline{f}) = 0 & \big(\zeta = r\mathrm{e}^{\mathrm{i}\alpha}\big) \\ \operatorname{Im} f = 0, & \operatorname{Im} g = 0 & (\operatorname{Im}\zeta = 0) \end{array} \tag{3}$$

4. Generating functions

Let us consider the functions:

$$P_j(\zeta) = g\big(\zeta \mathrm{e}^{2\mathrm{i}\alpha(j-1)}\big), \qquad Q_j(\zeta) = f\big(\zeta \mathrm{e}^{2\mathrm{i}\alpha(j-1)}\big) \tag{4}$$

From the boundary conditions (3), we obtain:

Lemma 4.1. *Functions $P_j(\zeta)$, $Q_j(\zeta)$ satisfy the infinite system of ordinary differential equations*

$$\frac{\mathrm{d}P_{j+1}}{\mathrm{d}\zeta}(P_j + Q_{j+1}) = \frac{\mathrm{d}P_j}{\mathrm{d}\zeta}(P_{j+1} + Q_j)$$
$$2\frac{\mathrm{d}P_{j+1}}{\mathrm{d}\zeta}Q_{j+1} + 2\frac{\mathrm{d}P_j}{\mathrm{d}\zeta}Q_j = \frac{\mathrm{d}Q_{j+1}}{\mathrm{d}\zeta}(P_{j+1} + Q_j) + \frac{\mathrm{d}Q_j}{\mathrm{d}\zeta}(P_j + Q_{j+1})$$

If α/π is a rational number, then after some finite number of rotations of the coordinate system we get the initial location of the system. Therefore, from (4) we have:

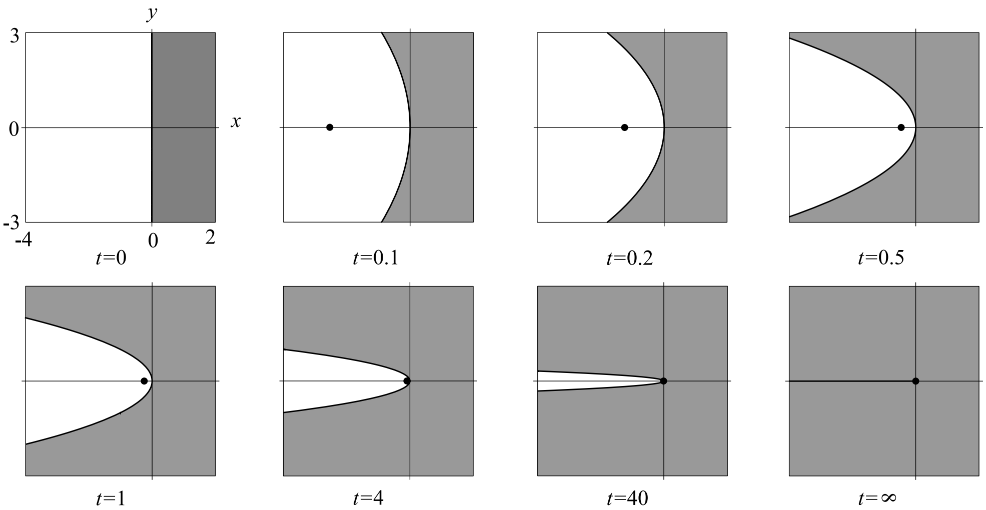

Fig. 2. Free boundary time evolution for $\alpha = \pi/2$.

Lemma 4.2. *If $\alpha = \pi m/n$, where m, n are natural numbers, and $g(\zeta)$, $f(\zeta)$ are holomorphic functions at the point $\zeta = 0$, then $P_{n+1} = P_1$, $Q_{n+1} = Q_1$.*

Two above lemmas imply the following

Theorem 4.3. *Let $\alpha = \pi m/n$, where m, n are natural numbers, and $g(\zeta)$, $f(\zeta)$ are holomorphic functions at the point $\zeta = 0$. Then $g(\zeta)$, $f(\zeta)$ satisfy the boundary condition (3) if and only if $g = P_1$, $f = Q_1$, where (P_j, Q_j) are solutions of the system of ordinary differential equations*:

$$\begin{aligned}
&\frac{\mathrm{d}P_{j+1}}{\mathrm{d}\zeta}(P_j + Q_{j+1}) = \frac{\mathrm{d}P_j}{\mathrm{d}\zeta}(P_{j+1} + Q_j) \quad (j = \overline{1,n}) \\
&2\frac{\mathrm{d}P_{j+1}}{\mathrm{d}\zeta}Q_{j+1} + 2\frac{\mathrm{d}P_j}{\mathrm{d}\zeta}Q_j = \frac{\mathrm{d}Q_{j+1}}{\mathrm{d}\zeta}(P_{j+1} + Q_j) + \frac{\mathrm{d}Q_j}{\mathrm{d}\zeta}(P_j + Q_{j+1}) \quad (j = \overline{1,n}) \\
&P_{n+1} = P_1, \qquad Q_{n+1} = Q_1
\end{aligned} \tag{5}$$

5. Special cases

Solving system (5) for $\alpha = \pi/2$, we obtain:

$$g(\zeta) = \zeta + \zeta^2, \qquad f(\zeta) = \zeta^2, \qquad U(z,t) = \frac{(\sqrt{1 + 4zAt} - 1)^2}{4At^2} \tag{6}$$

The complex velocity contains a singular point of branching $x^* = -1/(4At)$. Since singular points cannot be inside the liquid, the solution (6) is valid only for $A > 0$. It is unknown how to construct a solution for $\alpha = \pi/2$ and $A < 0$.

On the free surface, $\zeta = \mathrm{i}r$. Taking the imaginary and real parts of the expression $Z = (\mathrm{i}r + (\mathrm{i}r)^2)/(At)$, we infer that the free surface is a parabola:

$$xAt = -(yAt)^2 \tag{7}$$

The evolution of (7) with increasing time at $A = 1$ is shown in Fig. 2. The dot denotes the location of the singularity x^*, which is initially outside the liquid and finally reaches the free surface. The initial half-plane is transformed into the plane with a rectilinear cut.

For both cases $\alpha = \pi/4$ and $\alpha = 3\pi/4$ system (5), consisting of 8 equations, has the same form and yields the same solutions:

$$g(\zeta) = \zeta - \zeta^2, \qquad f(\zeta) = \zeta^2, \qquad U(z,t) = \frac{(\sqrt{1 - 4zAt} - 1)^2}{4At^2}$$

However, the free surface is described by different formulae. Substituting $\zeta = r\mathrm{e}^{\mathrm{i}\pi/4}$ and $\zeta = r\mathrm{e}^{3\mathrm{i}\pi/4}$ into the obtained solution, we get the following forms of the free surface, respectively:

$$yAt = xAt - 2(xAt)^2, \qquad yAt = -xAt + 2(xAt)^2 \tag{8}$$

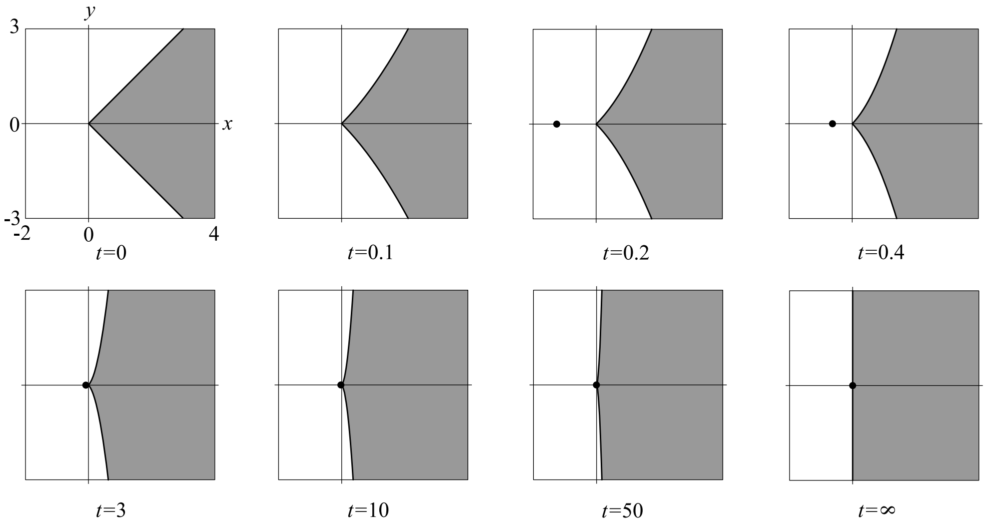

Fig. 3. Free boundary time evolution for $\alpha = \pi/4$.

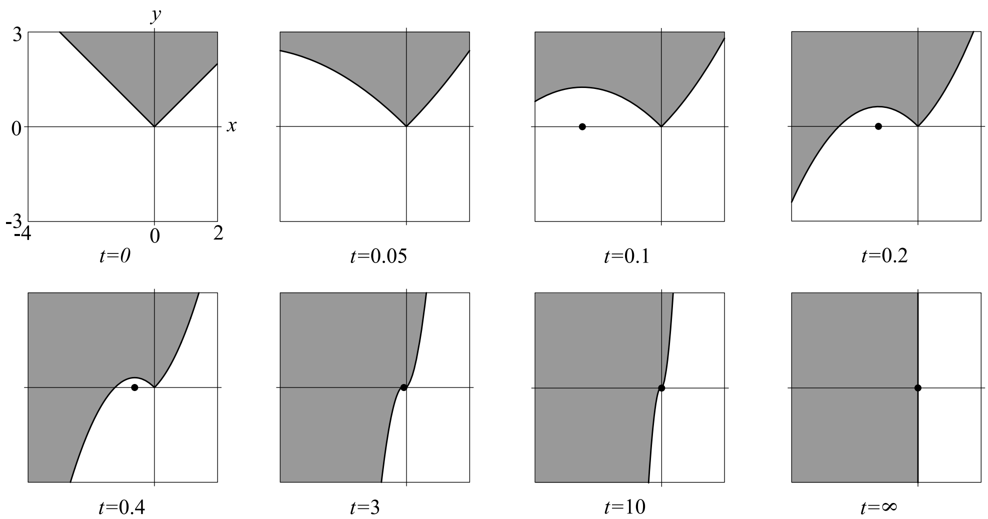

Fig. 4. The example of free boundary flow.

The point of branching $x^{**} = 1/(4At)$ should be outside the liquid; therefore we suppose that $A < 0$. It is unknown how to construct a solution for $\alpha = \pi/4$ and $\alpha = 3\pi/4$ at $A > 0$.

The form of the free surface described by the first formula in (8), and the singular point x^{**} for $A = -1$ are shown in Fig. 3. The flow, where the liquid wedge with the right angle is transformed into a half-plane, was found.

The flow of the liquid located between two free surfaces (8) for $A = -1$ is shown in Fig. 4. The liquid wedge with the right apical angle, whose bisector is along the y-axis, evolves to a half-plane with increasing time. Parameter A can be either positive or negative for such flow because the singularity x^{**} is anyway to be located outside the fluid.

A remarkable feature of the two flows, shown in Figs. 3 and 4, is the same form of the free surface. Therefore, we can obtain the flow for $\alpha = 3\pi/4$ if we join the upper half of the flow shown in Fig. 3 and the flow shown in Fig. 4. As a result, we have the evolution of the fluid wedge with the angle $3\pi/4$ into the wedge with the angle $3\pi/2$.

6. Conclusion

The proposed technique makes it also possible to obtain new exact solutions for other rational α/π ratios. Small denominators result from the resonance of the eigenfunctions of the linear boundary value problem, which can be stated in a small vicinity of the peak of the liquid wedge.

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