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Fractional calculus in one-dimensional isotropic thermo-viscoelasticity

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A new fractional relaxation operator is derived using the methodology of fractional calculus. The governing coupled fractional differential equations in the frame of the thermo-viscoelasticity with fractional order heat transfer are applied to the one-dimensional problem with heat sources. Laplace transform and state space techniques are used to get the solution. According to the numerical results and its graphs, conclusion about the new theory of thermo-viscoelasticity has been constructed. The theories of coupled thermo-viscoelasticity and of generalized thermo-viscoelasticity with one relaxation time follow as limit cases.

1. Introduction

Linear viscoelasticity remains an important area of research not only due the advent and use of polymers, but also because most solids, when subjected to dynamic loading, exhibit viscous effects [1]. The stress–strain law for many materials such as polycrystalline metals and high polymers can be approximated by the linear viscoelasticity theory [2]. The mechanical-model representation of linear viscoelastic behavior results was investigated by Gross [3], Staverman and Schwarzl [4], Alfrey and Gurnee [5] and Ferry [6]. One can refer to Ilioushin and Pobedria [7] for the formulation of a mathematical theory of thermal viscoelasticity and for the solutions of some boundary value problems as well as to Pobedria [8] for the coupled problems in continuum mechanics. A description of the linear theory of viscoelastic behavior of materials and theoretical formulation derived from continuum mechanics viewpoint can be found in Christensen's work [9].

The modification of the heat-conduction equation from diffusive to a wave type may be affected either by a microscopic consideration of the phenomenon of heat transport or, in a phenomenological way, by modifying the classical Fourier law of heat conduction. The first is due to Cattaneo [10], who obtained a wave-type heat equation by postulating a new law of heat conduction to replace the classical Fourier law. Lord and Shulman [11] introduced the theory of generalized thermoelasticity with one relaxation time for the special case of an isotropic body. This theory was extended by Dhaliwal and Sherief [12] to include the anisotropic case. In this theory, a modified law of heat conduction including both the heat flux and its time derivative replaces the conventional Fourier's law. The heat equation associated with this theory is hyperbolic and hence eliminates the paradox of infinite speeds of propagation inherent in both the uncoupled and the coupled theories of thermoelasticity.

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Nomenclature

λ, μ	Lame' constants
C_E	specific heat at constant strains
$K = \lambda + (2/3)\mu$	bulk modulus
$C_0^2 = \frac{K}{\rho}$	longitudinal wave speed
ε_{ij}	components of strain tensor
e_{ij}	components of strain deviator tensor
σ_{ij}	components of stress tensor
S_{ij}	components of stress deviator tensor
$e = \varepsilon_{ii}$	dilatation
k, k_{ij}	thermal conductivity
$R(t, \beta)$	relaxation functions
T	absolute temperature
u_i	components of displacement vector
α_T	coefficient of linear thermal expansion
$\gamma = 3K\alpha_T$	
δ_{ij}	Kronecker's delta
T_0	reference temperature
$\varepsilon = \frac{\gamma^2 T_0}{k\eta_0 \rho C_0^2}$	thermal coupling parameter
$\Theta = T - T_0$	such that $\lvert\Theta/T_0\rvert \ll 1$
ρ	mass density
τ	the ratio of the shear viscosity to Young's modulus
τ_0	relaxation time
α, β	orders of the fractional derivatives
$_C D_t^\alpha$	Caputo fractional derivative
t	times
E	Young's moduli
$E_\alpha(x)$	Mittag-Leffler function
$\Gamma(.)$	Gamma function
S	entropy

Several generalizations to the coupled theory are introduced. One can refer to Ignaczak [13] and to Chandrasekharaiah [14] for a review, presentation of generalized theories. Hetnarski and Ignaczak in their survey article [15] examined five generalizations to the coupled theory and obtained a number of important analytical results. The mathematical aspects of Lord and Shulman [11] and convolution variational principles are explained and illustrated in detail in the work of Ignaczak and Ostoja-Starzewski [16].

The generalized thermo-viscoelasticity models ignoring the relaxation effects of the volume, with one relaxation time and with two relaxation times, are established by El-Karamany and Ezzat [17] and Ezzat et al. [18]. Among the theoretical contributions to the subject are the proofs of uniqueness theorems under different conditions by Ezzat and El Karamany [19], whereas the propagation of discontinuities of solutions in the generalized theory was investigated by El-Karamany and Ezzat [20]. The boundary-element formulation and reciprocal and uniqueness theorems in linear micropolar electro-magnetic thermoelasticity with two relaxation times were presented by El-Karamany and Ezzat [21,22]. Ezzat [23] investigated the relaxation effects on the volume properties of an electrically conducting viscoelastic material.

Fractional calculus has been used successfully to modify many existing models of physical processes. One can state that the whole theory of fractional derivatives and integrals was established in the second half of the 19th century. The first application of fractional derivatives was given by Abel, who applied fractional calculus in the solution of an integral equation that arises in the formulation of the tautochrone problem. The generalization of the concept of derivative and integral to a non-integer order has been subjected to several approaches and some various alternative definitions of fractional derivatives appeared in [24–27]. In the last few years, fractional calculus was applied successfully in various areas to modify many existing models of physical processes, e.g., chemistry, biology, modeling and identification, electronics, wave propagation and viscoelasticity [13,28–31]. Fractional order models often work well, particularly for dielectrics and viscoelastic materials over extended ranges of time and frequency [32,33]. In heat transfer and electrochemistry, for example, the half-order fractional integral is the natural integral operator connecting the applied gradients (thermal or material) with the diffusion of ions of heat [34,35]. One can refer to Padlubny [36] for a survey of applications of fractional calculus.

A quasi-static uncoupled theory of thermoelasticity based on the fractional heat-conduction equation was put forward by Povstenko [37]. The theory of thermal stresses based on the heat-conduction equation with the Caputo time-fractional derivative is used by Povstenko [38] to investigate thermal stresses in an infinite body with a circular cylindrical hole. Sherief et al. [39] introduced a new model of thermoelasticity using fractional calculus, proved a uniqueness theorem, and derived a reciprocity relation and a variational principle. Youssef [40] introduced another new model of fractional heat-conduction equation, proved a uniqueness theorem and presented a one-dimensional application. Ezzat [41,42] established a new model of fractional heat-conduction equation by using the new Taylor series expansion of time-fractional order which had been developed by Jumarie [43]. El-Karamany and Ezzat [44] introduced two models where the fractional derivatives and integrals are used to modify the Cattaneo heat-conduction law and, in the context of the two-temperature thermoelasticity theory, uniqueness and reciprocal theorems are proved, the convolutional variational principle is given and used to prove a uniqueness theorem with no restrictions imposed on the elasticity or thermal conductivity tensors, except symmetry conditions. The fractional order theory of a perfect conducting thermoelastic medium was investigated by Ezzat and El-Karamany [45,46].

The current work is an attempt to derive a solution for the fractional relaxation differential equation. Also, the mathematically rigorous constitutive equation for a generalized viscoelastic model for an isotropic medium is obtained with the fractional relaxation operator $\hat{\boldsymbol{R}}_\beta$, which was obtained by using the methodology of fractional calculus. A new model of thermo-viscoelasticity by using the latter methodology has been applied to the one-dimensional problem with heat sources. The solution is obtained using the state–space approach. The first writers who introduce the state–space formulation in thermoelastic problems were Bahar and Hetnarski [47]. A review of this method is presented in [48]. The inversion of the Laplace transform will be computed numerically using a method based on a Fourier expansion technique [34]. The limit cases of the dynamic coupled and Lord–Shulman thermoelasticity theories are presented in the numerical results section. According to the numerical results and its graphs, we studied the effect for the coupled system of fractional derivative parameters α, β about the new theory on all the studied fields.

2. Derivation solution of the fractional relaxation equation

We consider the fractional ordinary differential equation which governs the relaxation function $R(t,\beta)$ as [35,49]:

$$\tau^{\beta}\frac{\mathrm{d}^{\beta}R(t,\beta)}{\mathrm{d}^{\beta}t}+R(t,\beta)=0, \quad 0<\beta<1 \tag{1}$$

where τ is a positive constant for the ratio of the shear viscosity to Young's modulus, β is the fractional power, and the field variable $R(t,\beta)$ is assumed to be a causal relaxation function of time.

In the former case, Eq. (1) must be equipped with a single *initial condition*, say $R(0,\beta)=R_0$, and in the latter with two initial conditions, say $R(0^+,\beta)=R_0$ land $\dot{R}(0^+,\beta)=\zeta_0$. We assume $\zeta_0=0$, in order to ensure the continuous dependence of the solution of Eq. (1) on the parameter β also in the transition from $\beta=-1$ to $\beta=1$.

Applying the Laplace transform technique, since $\zeta_0=0$, the image solution turns out to be:

$$\bar{R}(s,\beta)=R_0\frac{s^{\beta-1}}{s^{\beta}+\tau^{-\beta}} \tag{2}$$

To invert (2), we can use the series expansion theorem or the Bromwich formula [50]; we obtain:

$$R(t,\beta)=R_0E_{\beta}\big[(t/\tau)^{\beta}\big] \tag{3}$$

where E_{β} denotes the Mittag-Leffler function [36,51]. In the complex plane, this function is defined by the following series and integral representations:

$$E_{\beta}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(n\beta+1)}, \quad \beta>0,\ z\in\mathbb{C} \tag{4}$$

$$E_{\beta}(x)=\frac{1}{2\pi\mathrm{i}}\int_{Ha}\frac{\xi^{\beta-1}\mathrm{e}^{\xi}}{\xi^{\beta}-z}\,\mathrm{d}\xi, \quad \beta>0,\ z\in\mathbb{C} \tag{5}$$

In Eq. (5) *Ha* denotes the Hankel path, a loop which starts from $-\infty$ along the lower side of the negative real axis, encircles the circular disc$|\xi|\leqslant|z|^{1/\beta}$ in the positive sense and ends at $-\infty$ along the upper side of the negative real axis.

For the present purposes, our interest in this function is limited to the negative real axis, i.e. $E_{\beta}(-x)$ with $x=(t/\tau)^{\beta}\geqslant 0$ for $0<\beta\leqslant 1$; the main properties are as follows:

(1) for $0<\beta\leqslant 1$, $E_{\beta}(-x)$ turns out to be positive and completely monotonic, i.e.:

$$(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}E_{\beta}(-x)\geqslant 0, \quad 0<\beta\leqslant 1 \tag{6}$$

(2) the following asymptotic expansion is valid:

$$E_{\beta}(-x)\simeq-\sum_{n=0}^{\infty}\frac{(-x)^{-n}}{\Gamma(1-n\beta)}, \quad x\to\infty,\ 0<\beta\leqslant 1 \tag{7}$$

The property (6) of complete monotonic is better understood by the following equivalent representations for our solution

$$E_{\beta}\big[-(t/\tau)^{\beta}\big]=\int_0^{\infty}\mathrm{e}^{-t/v}\xi(v,\beta)\,\mathrm{d}v=\int_0^{\infty}\mathrm{e}^{-qt}\eta(q,\beta)\,\mathrm{d}q, \quad 0<\beta\leqslant 1 \tag{8}$$

where $\xi(v,\beta)$ and $\eta(q,\beta)$ are non-negative locally integrable functions in $\mathbb{R}^+$, referred to as spectrum of relaxation times and spectrum of relaxation frequencies, respectively. The explicit expressions of the spectra can be derived using the integral representation equation (5) and turn out to be related to:

$$S(y,\beta)=\frac{1}{\pi y}\left(\frac{\sin\pi\beta}{y^{\beta}+y^{\beta}+2\cos\pi\beta}\right)=\begin{cases}\xi(v,\beta)\tau, & y=v/\tau\\ \eta(q,\beta)/\tau, & y=q\tau\end{cases} \tag{9}$$

Plots of the function $S(y,\beta)$ that we may refer to as the normalized spectrum of relaxation are shown in Fig. 1 for some values of β.

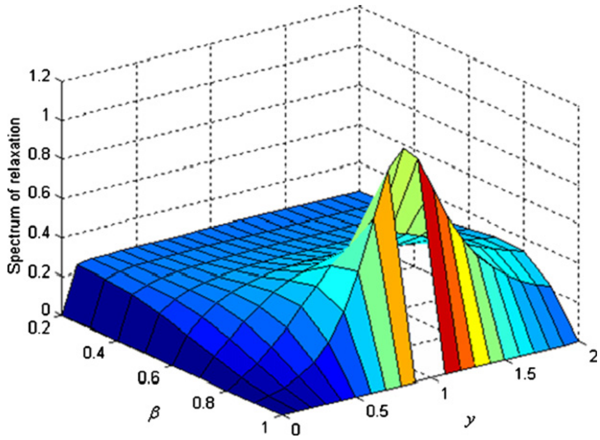

Fig. 1. The normalized spectrum of relaxation $S(y,\beta)$ for various values of β.

In order to show plots of the solution of Eq. (1), we list some explicit formulas useful for numerical computation. The series representation obtained from Eq. (4) is:

$$R(t,\beta)=R_0E_\beta\big[(t/\tau)^\beta\big]=\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n\beta+1)}\big[(t/\tau)^{-n\beta}\big] \tag{10}$$

but is suitable only for short times, since it exhibits a very slow numerical convergence. For sufficiently large times, one can seek the matching with the asymptotic expansion obtained from Eq. (7), which reads:

$$R(t,\beta)=R_0E_\beta\big[(t/\tau)^\beta\big]=\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n\beta+1)}\big[(t/\tau)^{-n\beta}\big],\quad t\to\infty^+ \tag{11}$$

We note that the above expansion can be formed and obtained by expanding the Laplace transforms (see Appendix B) in positive powers of s and then inverting term-by-term. In this respect, we have to consider the analytical continuation of the Laplace transform to the half-plane $\mathrm{Re}(\lambda)\leqslant 0$.

We attribute particular attention to the cases where the index of derivation is effectively a rational number, i.e. $\beta=p/q$ with $p,q\in\mathbb{N}$. Here, on the basis of Ref. [30], our solutions turn out to be expressed in terms of exponentials and incomplete Gamma functions, according to the formulas listed hereafter in our notation:

$$R(t,\beta)=R_0E_\beta\big[(t/\tau)^\beta\big]=\frac{R_0}{p}\sum_{h=0}^{p-1}E_{1/q}\big[\in_{h,p}(t/\tau)^{p/q}\big],\quad \in_{h,p}=\mathrm{e}^{\left(\frac{\mathrm{i}(2h+1)\pi}{p}\right)} \tag{12}$$

and

$$R(t,\beta)=R_0E_{1/q}\big[-c(t/\tau)^{1/q}\big]=R_0\left[1+\sum_{h=0}^{q-1}\frac{c^k}{(c^q)^{k/q}}\frac{\gamma(k/qc^qt)}{\Gamma(k/q)}\right],\quad c\in\mathbb{C} \tag{13}$$

where γ denotes the incomplete Gamma function. It should be noted that in Eq. (13), $c^k=(c^q)^{k/q}$ unless c^q is a positive real number.

In order to gain some insight into the effect of the order of derivation, plots of the function $R(t)$ are shown in Fig. 2, for some (rational) values of β $(0<\beta\leqslant 1)$. The fractional solutions exhibit very different behaviors. In particular, we note the leading asymptotic behaviors for $t\to 0^+$ and $t\to\infty^+$ derived from Eqs. (10) and (11) – see Eq. (1.35), p. 301 in [52]:

$$R(t,\beta)=R_0\begin{cases}1-\frac{(t/\tau)^\beta}{\Gamma(1+\beta)} & \text{as } t\to 0^+\\ \frac{(t/\tau)^{-\beta}}{\Gamma(1-\beta)} & \text{as } t\to\infty^+\end{cases} \tag{14}$$

The solution $R(t,\beta)$ of the fractional relaxation equation $0<\beta<1$ exhibits for small times a much faster decay (the derivative tends to $-\infty$ in comparison with -1), and for large times a much slower decay (algebraic decay in comparison with exponential decay). In view of its slow decay, the phenomenon of fractional relaxation is usually referred to as a super-slow process [49,52].

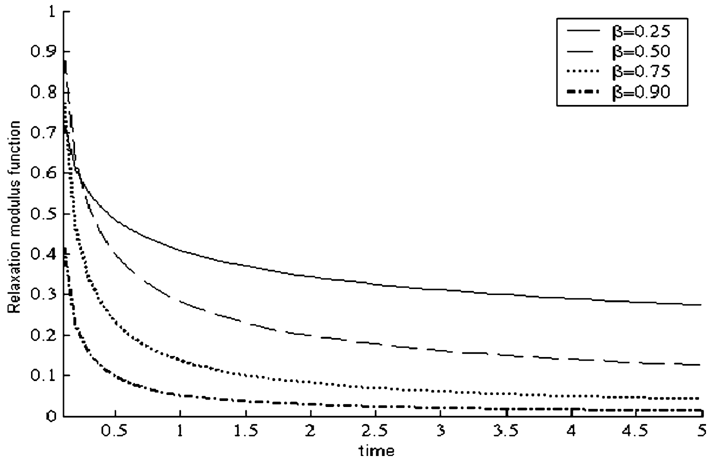

Fig. 2. The relaxation moduli function vs. time t.

3. The governing equations

The constitutive relations for an anisotropic viscoelastic solid are given by [9]:

$$\sigma_{ij} = \int_0^t G_{ijkl}(t-\tau)\frac{\partial \varepsilon_{kl}}{\partial \tau}\,\mathrm{d}\tau - \int_0^t \gamma_{ij}(t-\tau)\frac{\partial \theta}{\partial \tau}\,\mathrm{d}\tau = (G_{ijkl} * \dot{\varepsilon}_{kl}) - (\gamma_{ij} * \dot{\theta})$$

$$S = \int_0^t \beta(t-\tau)\frac{\partial \theta}{\partial \tau}\,\mathrm{d}\tau + \int_0^t \gamma_{ij}(t-\tau)\frac{\partial \varepsilon_{ij}}{\partial \tau}\,\mathrm{d}\tau = (\beta * \dot{\theta}) + (\gamma_{ij} * \dot{\varepsilon}_{ij})$$

where $G_{ijkl}(\boldsymbol{x}, t), \gamma_{ij}(\boldsymbol{x}, t)$, and $\beta(\boldsymbol{x}, t)$ are fourth-order, second-order and zero-order relaxation tensors. In addition, it is assumed that the following symmetry relations hold:

$$G_{ijkl} = G_{klij} = G_{jikl} = G_{ijlk}, \qquad \gamma_{ij} = \gamma_{ji}, \quad \text{on } V \times [0, \infty)$$

Substituting $G_{ijkl}(t) = \frac{1}{3}(G_2(t) - G_1(t))\delta_{ij}\delta_{kl} + \frac{1}{2}G_1(t)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, $\gamma_{ij}(t) = \gamma(t)\delta_{ij}$ and considering $G_1(t) = R(t)$, $G_2 = 3K$ and $\gamma = 3K\alpha_T$, we get the following system of governing equations of generalized linear thermo-viscoelastic interactions in a homogeneous isotropic medium with one relaxation time.

(1) The constitutive equation is given by [7,18,21]:

$$S_{ij}(\bar{x}, t) = \int_0^t R(t-\xi)\frac{\partial e_{ij}(\bar{x}, \xi)}{\partial \xi}\,\mathrm{d}\xi = \hat{\boldsymbol{R}}_\beta(e_{ij}) \tag{15}$$

where $R(t)$ is the relaxation function such that $R(\infty) > 0$.

$$S_{ij} = \sigma_{ij} - \sigma\delta_{ij}, \quad e = \varepsilon_{kk}, \qquad \bar{x} = (x_1, x_2, x_3) \tag{16}$$

(2) The kinematic relations write:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad e_{ij} = \varepsilon_{ij} - \frac{e}{3}\delta_{ij}, \quad i, j = 1, 2, 3 \tag{17}$$

with the assumption

$$\frac{\partial \sigma_{ij}(\bar{x}, t)}{\partial t} = 0, \qquad \frac{\partial \varepsilon_{ij}(\bar{x}, t)}{\partial t} = 0, \quad -\infty \leqslant t < 0 \tag{18}$$

For large times refers a much slower decay, $R(t)$ is the positive decreasing fractional relaxation function of time defined as follows:

$$R(t; \beta) = R_0\frac{(t/\tau)^{-\beta}}{\Gamma(1-\beta)}, \quad 0 < \beta \leqslant 1,\ t \to \infty^+ \tag{19}$$

and by combining Eqs. (15) and (19), one arrives at:

$$S_{ij}(\bar{x},t) = \frac{R_0 \tau^\beta}{\Gamma(1-\beta)} \int_0^t (t-\xi)^{-\beta} \frac{\partial e_{ij}(\bar{x},\xi)}{\partial \xi} \, \mathrm{d}\xi = \hat{\boldsymbol{R}}_\beta(e_{ij}), \quad 0 < \beta \leqslant 1, \; t \to \infty^+ \tag{20}$$

The right-hand side of Eq. (20) represents the fractional integral (FI). To see this, we start from the expression of an (FI) [25,53] given by:

$$_cD_t^{-\alpha} f(\bar{x},t) = J^\alpha f(\bar{x},t) = \frac{1}{\Gamma(\alpha)} \int_c^t \frac{f(\bar{x},\xi)}{(t-\xi)^{1-\alpha}} \, \mathrm{d}\xi \tag{21}$$

where $\alpha > 0$. Eq. (21) includes two special (FI) forms. For $c = 0$, one recovers the Riemann–Liouville fractional integral formulated as the Laplace convolution. For α is a positive integer, Eq. (21) presents a multiple (Cauchy) integral of order $-\alpha$.

Now, the Caputo fractional derivative [54] of order $\alpha > 0$, is obtained by:

$$_cD_t^\alpha f(\bar{x},t) = J^{n-\alpha} D_t^n f(\bar{x},t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-1-\alpha} \frac{\partial^n}{\partial \xi^n} f(\bar{x},\xi) \, \mathrm{d}\xi$$

where $n - 1 < \gamma \leqslant n$ and $n \in \mathbb{N}$. We take $n = 1$ and $c = 0$, to obtain:

$$D_t^\alpha f(\bar{x},t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\xi)^{-\alpha} \frac{\partial f(\bar{x},\xi)}{\partial \xi} \, \mathrm{d}\xi, \quad \gamma \in (0,1] \tag{22a}$$

Using the shorthand notation:

$$\frac{\partial^\beta f(\bar{x},t)}{\partial t^\beta} \equiv D_t^\beta f(\bar{x},t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\xi)^{-\beta} \frac{\partial f(\bar{x},\xi)}{\partial \xi} \, \mathrm{d}\xi \tag{22b}$$

where [44]:

$$\lim_{\beta \to 1} \frac{\partial^\beta f(\bar{x},t)}{\partial t^\beta} = \frac{\partial f(\bar{x},t)}{\partial t} \tag{22c}$$

we can rewrite Eq. (20) as:

$$S_{ij}(\bar{x},t) = R_0 \tau^\beta \frac{\partial^\beta}{\partial t^\beta}(e_{ij}) = \hat{\boldsymbol{R}}_\beta(e_{ij}), \quad 0 < \beta < 1 \tag{23}$$

According to the concept of the eigenvalue, the convolutional operator $\hat{\boldsymbol{R}}_\beta(g(\bar{x},t))$ is defined, for any function of class C_1, as:

$$\hat{\boldsymbol{R}}_\beta(g) = R_0 \tau^\beta \frac{\partial^\beta g(\bar{x},t)}{\partial t^\beta} = \frac{R_0 \tau^\beta}{\Gamma(1-\beta)} \int_0^t (t-\xi)^{-\beta} \frac{\partial g(\bar{x},\xi)}{\partial \xi} \, \mathrm{d}\xi, \quad 0 < \beta \leqslant 1 \tag{24}$$

(3) The stress–strain temperature relation:

$$\sigma = K\big[e - 3\alpha_T(T - T_0)\big] \tag{25}$$

where:

$$\sigma = \frac{\sigma_{kk}}{3}, \qquad \sigma_{ij} = \sigma_{ji}, \qquad S_{ij} = \sigma_{ij} - \sigma\delta_{ij}$$

substituting Eq. (25) into Eq. (15), we obtain:

$$\sigma_{ij} = \hat{\boldsymbol{R}}_\beta\left(\varepsilon_{ij} - \frac{e}{3}\delta_{ij}\right) + Ke\delta_{ij} - \gamma\Theta\delta_{ij} \tag{26}$$

(4) The equation of motion:

$$\rho\ddot{u}_i = \sigma_{ij,j}, \quad i,j = 1,2,3 \tag{27}$$

From Eq. (25) and Eq. (27), one obtains:

$$\rho \ddot{u}_i = \hat{\boldsymbol{R}}_\beta \left(\frac{1}{2}\nabla^2 u_i + \frac{1}{6}e_{,i}\right) + Ke_{,i} - \gamma \Theta_{,i} \tag{28}$$

(6) The fractional heat-conduction law.

We have the linearized equation of energy:

$$q_{j,j} = -\rho T_0 \dot{S} + Q \tag{29}$$

where the entropy S may be written in terms of temperature and strain tensor in an isotropic medium as follows:

$$\rho T_0 S = \rho C_E \Theta + \gamma \varepsilon_{ij} \tag{30}$$

We apply the new Taylor series expansion of time-fractional order α which was developed by [43] starting from the classical Fourier law of heat conduction $q_j(x_i, t + \tau_0) = -k\Theta_{,j}$; one obtains [42]:

$$\left(1 + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right) q_i = -k_{ij}\Theta_{,j} \tag{31}$$

where $\tau_0 \ll 1$ is the thermal relaxation time.

By taking the divergence of both sides of Eq. (31) and using Eq. (29) and its time derivative, we arrive at the equation of modified fractional heat conduction for an isotropic medium, in our case, namely,

$$\left(1 + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)(\rho C_E \dot{\Theta} + \gamma T_0 \dot{e}_{ij} - Q) = k\Theta_{,ii}, \quad 0 < \alpha \leqslant 1 \tag{32}$$

Throughout this paper, a rectangular coordinates system (x_1, x_2, x_3) is employed. Here $\bar{x} = (x_1, x_2, x_3)$ is the position vector and t is the time. All the function is considered to be a function of $(\bar{x}, t)$. A superposed dot denotes differentiation with respect to time, a comma followed by index i denotes the derivative with respect to x_i. The summation notation is used and microrotations are ignored.

Limiting cases

(i) In the theory of thermoelasticity

(1) Eqs. (24), (26) and (32) in the limiting case when $\beta = 0$, $\tau_0 = 0$, $R_0 = 2\mu$ transforms to the works of Biot [55] and Povstenko [37].

(ii) In the theory of generalized thermoelasticity

(2) Eqs. (24), (26) and (32) in the limiting case when $\beta = 0, \alpha = 1, \tau_0 > 0, R_0 = 2\mu$ transforms to the work of Lord and Shulman [11].

(iii) In the theory of fractional generalized thermoelasticity

(3) Eqs. (24), (26) and (32) in the limiting case when $\beta = 0, 0 < \alpha \leqslant 1, \tau_0 > 0, R_0 = 2\mu$ transform to the works of Sherief et al. [39], Ezzat [42] and Ezzat and El-Karamany [45].

(iv) In the theory of thermo-viscoelasticity

(4) Eqs. (24), (26) and (32) in the limiting case when $\beta = 1, \tau_0 = 0$ transform to the work of Pobedria [8].

(v) In the theory of generalized thermo-viscoelasticity

(2) Eqs. (24), (26) and (32) in the limiting case $\beta = 0, \alpha = 1, \tau_0 > 0$ transform to the work of Ezzat et al. [56].

4. Application

4.1. Analysis and state–space approach

We shall consider a thermo-viscoelastic solid occupying the region $x \geqslant 0$; for the one-dimensional problems, all the considered functions will depend only on the space variable x and the time t. The governing equations for generalized thermo-viscoelasticity are given below.

(1) The components of the displacement vector are, in the one-dimensional medium:

$$u_x = u(x, t), \qquad u_y = u_z = 0 \tag{33}$$

(2) The strain component takes the form:

$$e = \varepsilon_{xx} = \frac{\partial u}{\partial x} \tag{34}$$

(3) The fractional heat-conduction law writes:

$$\left(1 + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)(\rho C_E \dot{\Theta} + \gamma T_0 \dot{e} - Q) = k\frac{\partial^2 \Theta}{\partial x^2}, \qquad 0 < \alpha \leqslant 1 \tag{35a}$$

(4) The equation of motion is:

$$\rho\frac{\partial^2 u}{\partial t^2}=\left(\frac{2}{3}\hat{\boldsymbol{R}}_\beta+K\right)\frac{\partial^2 u}{\partial x^2}-\gamma\frac{\partial\Theta}{\partial x},\quad 0<\beta\leqslant 1 \tag{36a}$$

(5) The constitutive equation in linear form yields:

$$\sigma_{xx}=\left(\frac{2}{3}\hat{\boldsymbol{R}}_\beta+K\right)\frac{\partial u}{\partial x}-\gamma\Theta \tag{37a}$$

(6) The fractional heat flux equation writes:

$$\left(1+\frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)q_x=-k\frac{\partial\Theta}{\partial x} \tag{38a}$$

Let us introduce the following non-dimensional variables:

$$x^*=C_0\eta_0 x,\qquad u^*=C_0\eta_0 u,\qquad t^*=C_0^2\eta_0 t,\qquad \tau_0^*=C_0^2\eta_0\tau_0,\qquad \eta_0=\frac{\rho C_E}{k},\qquad \sigma_{ij}^*=\frac{1}{K}\sigma_{ij}$$
$$C_0^2=\frac{K}{\rho},\qquad \Theta^*=\frac{\gamma\Theta}{\rho C_0^2},\qquad Q^*=\frac{Q\gamma}{k\rho C_0^4\eta_0^2},\qquad q_i^*=\frac{q_i\gamma}{k\rho C_0^3\eta_0},\qquad R_0^*=\frac{2}{3K}R_0$$

Using the homogeneity and scale change properties of fractional derivatives [27], Eqs. (35a)–(38a) take the following form (after dropping the asterisks for the sake of convenience):

$$\frac{\partial^2\Theta}{\partial x^2}=\left(\frac{\partial}{\partial t}+\frac{\tau_0^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)(\Theta+\varepsilon e)-\left(1+\frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)Q \tag{35b}$$

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}-\frac{\partial\Theta}{\partial x}+\frac{R_0\tau^\beta}{\Gamma(1-\beta)}\int_0^t(t-\xi)^{-\beta}\frac{\partial}{\partial\xi}\left(\frac{\partial^2 u(x,\xi)}{\partial x^2}\right)\mathrm{d}\xi \tag{36b}$$

$$\sigma_{xx}=\frac{\partial u}{\partial x}-\Theta+\frac{R_0\tau^\beta}{\Gamma(1-\beta)}\int_0^t(t-\xi)^{-\beta}\frac{\partial}{\partial\xi}\left(\frac{\partial u(x,\xi)}{\partial x}\right)\mathrm{d}\xi \tag{37b}$$

$$\left(1+\frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)q_x=-\frac{\partial\Theta}{\partial x} \tag{38b}$$

The calculations will be carried out for the case:

$$R(t,\beta)=\frac{R_0}{\Gamma(1-\beta)}(t/\tau)^{-\beta},\quad 0<\beta\leqslant 1 \tag{39}$$

Applying the Laplace transform with parameter s (denoted by a bar) of both sides for the non-dimensional equations (35b)–(37b), we arrive at the following set of equations:

$$\left(\frac{\partial^2}{\partial x^2}-ws^2\right)\bar{u}(x,s)=w\frac{\partial\bar{\Theta}}{\partial x} \tag{40}$$

$$\left(\frac{\partial^2}{\partial x^2}-s-\frac{\tau_0^\alpha}{\alpha!}s^{\alpha+1}\right)\bar{\Theta}=\varepsilon s\left(1+\frac{\tau_0^\alpha}{\alpha!}s^\alpha\right)\frac{\partial\bar{u}}{\partial x}-\left(1+\frac{\tau_0^\alpha}{\alpha!}s^\alpha\right)\bar{Q} \tag{41}$$

$$\bar{\sigma}_{xx}=\frac{1}{w}\frac{\partial\bar{u}}{\partial x}-\bar{\Theta} \tag{42}$$

Since $L\{t^{-\beta}\}=\Gamma(1-\beta)/s^{1-\beta}$, the Laplace transform of the relaxation modulus can be written under the form:

$$L\big\{R(t,\beta)\big\}=R_0(s\tau)^\beta\left(\frac{1}{s}\right),\quad 0<\beta\leqslant 1 \tag{43}$$

where all the initial state functions are equal to zero.

We shall choose as state variables the temperature increment θ, the displacement component in the x-direction and their gradients; then Eqs. (40)–(42) can be written in matrix form as:

$$\frac{\mathrm{d}\bar{v}(x,s)}{\mathrm{d}x}=\boldsymbol{A}(s)\bar{v}(x,s)+\boldsymbol{B}(x,s) \tag{44}$$

where

$$\boldsymbol{A}(s)=\begin{pmatrix}0&0&1&0\\0&0&0&1\\gs&0&0&s\varepsilon g\\0&ws^2&w&0\end{pmatrix},\qquad \bar{v}(x,s)=\begin{pmatrix}\bar{\theta}(x,s)\\\bar{u}(x,s)\\\bar{\theta}'(x,s)\\\bar{u}'(x,s)\end{pmatrix} \tag{45}$$

and

$$\boldsymbol{B}(x,s)=-gQ\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \tag{46}$$

where $g=(1+\frac{\tau_0^\alpha}{\alpha!}s^\alpha)$ and $w=\frac{1}{(M_\beta s^\beta+1)}$, $M_\beta=R_0\tau^\beta$.

The formal solution of system (44) can be written in the form:

$$\bar{v}(x,s)=\exp\bigl(\boldsymbol{A}(s)x\bigr)\left[\bar{v}(0,s)+\int_0^x\exp\bigl(-\boldsymbol{A}(s)x\bigr)\boldsymbol{B}(z,s)\,\mathrm{d}z\right] \tag{47}$$

In the special case, when there is no source acting inside the medium, Eq. (47) simplifies to

$$\bar{v}(x,s)=\exp\bigl(\boldsymbol{A}(s)x\bigr)\bar{v}(0,s) \tag{48}$$

We shall use the well-known Cayley–Hamilton theorem to find the form of the matrix $\exp(\boldsymbol{A}(s)x)$; then the characteristic equation of the matrix $\boldsymbol{A}(s)$ can be written as:

$$k^4-\bigl[ws^2+sg(1+w\varepsilon)\bigr]k^2+wgs^3=0 \tag{49}$$

The roots of this equation, namely, k_1^2 and k_2^2 satisfy the relations:

$$k_1^2+k_2^2=ws^2+sg(1+w\varepsilon) \tag{50}$$

$$k_1^2k_2^2=wgs^3 \tag{51}$$

The Taylor series expansion of the matrix exponential has the form:

$$\exp\bigl(A(s)x\bigr)=\sum_{n=0}^{\infty}\frac{(A(s)x)^n}{n!} \tag{52}$$

Using Cayley–Hamilton theorem again, we can express higher orders of the matrix A in terms of $I,\ A,\ A^2$ and A^3, where I is the unit matrix of fourth order, thus, the infinite series in Eq. (52) can be reduced to:

$$\exp\bigl(A(s)x\bigr)=L(x,s)=a_0I+a_1A_1+a_2A_1^2+a_3A_1^3 \tag{53}$$

where (a_0-a_3) are some coefficients depending on x and s.

By the Cayley–Hamilton theorem, the characteristic roots $\pm k_1$ and $\pm k_2$ of the matrix A must satisfy Eq. (53); thus,

$$\exp(\pm k_1x)=a_0\pm a_1k_1+a_2k_1^2\pm a_3k_1^3 \tag{54a}$$

$$\exp(\pm k_2x)=a_0\pm a_1k_2+a_2k_2^2\pm a_3k_2^3 \tag{54b}$$

Substituting expressions Eq. (72) [Appendix A] into Eq. (54) and computing A^2 and A^3, we obtain:

$$\exp\bigl(A(s)x\bigr)=l(x,s)=\bigl[l_{ij}(x,s)\bigr],\quad i,j=1,2,3,4 \tag{55}$$

It is worth mentioning here that Eqs. (50) and (51) have been used repeatedly in order to write the above entries in the simplest possible form. We shall stress here that the above expression for the matrix exponential is a formal one. In the actual physical problem, the space is divided into two regions accordingly as $x\geqslant0$ or $x\leqslant0$. Inside the region $0\leqslant x<\infty$, the positive exponential terms, not bounded at infinity, must be suppressed. Thus, for $x\geqslant0$, we should replace each $\sinh(k_ix)$ by $-\frac{\mathrm{e}^{-k_ix}}{2}$ and each $\cosh(k_ix)$ by $\frac{\mathrm{e}^{-k_ix}}{2}$. In the region $x\leqslant0$, the negative exponentials are suppressed instead.

It is possible to solve a broad class of one-dimensional problem of generalized thermo-viscoelasticity with one relaxation time.

5. Plane distribution of heat sources

We also assume that there is of continuous heat sources located at the plane surface $x = 0$. The intensity of the heat sources is thus given by:

$$Q(x,t) = Q_0 H(t)\delta(x) \tag{56}$$

where Q_0 is a constant and $\delta(x)$ is Dirac's delta function.

Taking Laplace transform, we obtain:

$$\bar{Q}(x,s) = Q_0 \frac{\delta(x)}{s} \tag{57}$$

We now proceed to find the solution of the problem in the right half-space $x \geqslant 0$, using Eq. (44). The solution in the other half-space is obtained using the symmetric of the problem. Substituting $\boldsymbol{B}$ for $\bar{Q}$ in the expression, interesting the result in the right-hand side of Eq. (44), and applying the integral properties of the Dirac delta function, we get:

$$\bar{v}(x,s) = L(x,s)\big[\bar{v}(0,s) + H(s)\big] \tag{58}$$

$$H(s) = -\frac{Q_0 g}{4s}\begin{pmatrix} \frac{k_1k_2+ws^2}{k_1k_2(k_1+k_2)} \\ 0 \\ 1 \\ \frac{w}{(k_1+k_2)} \end{pmatrix} \tag{59}$$

Eq. (58) expresses the solution of the problem in the Laplace transform domain in terms of the vector $H(s)$, the applied heat source and the vector $\bar{v}(0,s)$, representing the conditions at the plane source of heat. In order to evaluate the components of this vector, we note first that, due to the symmetry of the problem, the displacement component vanishes at the plane source of heat, thus:

$$u(0,t) = 0, \qquad \bar{u}(0,s) = 0 \tag{60}$$

Gauss' divergence theorem will now be used to obtain the thermal condition at the plane source. We consider a short cylinder of unit base whose axis is perpendicular to the plane source of heat and whose bases lie on opposite sides of it. Taking limits as the height of the cylinder tends to zero and noting that there is no heat flux through the lateral surface, upon using the symmetry of the temperature field, we get:

$$q(0,s) = \frac{1}{2}H(t)Q_0 \quad \text{or} \quad \bar{q}(0,s) = \frac{Q_0}{2s} \tag{61}$$

We shall use the generalized fractional modification Fourier's law of heat conduction in the non-dimensional form, namely,

$$\left(1 + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)q_j = -\Theta_{,j} \tag{62}$$

Taking the Laplace transform of both sides of Eq. (62),

$$\bar{q} = -\frac{\bar{\Theta}'}{(1 + (\tau_0^\alpha/\alpha!)s^\alpha)} \tag{63}$$

and using Eq. (63), we obtain the condition:

$$\left.\frac{\partial\bar{\Theta}}{\partial x}\right|_{x=0} = -\frac{Q_0 g}{2s} \tag{64}$$

Eqs. (60) and (61) give two components of the vector $\boldsymbol{B}(x,s)$. In order to obtain the remaining two components, we substitute 0 for x on both sides of Eq. (58), obtaining a system of linear equations whose solution gives:

$$\bar{\Theta}(0,s) = \frac{gQ_0(k_1k_2 + ws^2)}{2sk_1k_2(k_1+k_2)} \tag{65}$$

$$\bar{u}'(0,s) = \frac{gwQ_0}{2s(k_1+k_2)} \tag{66}$$

Inserting the values from Eqs. (65), and (66) into the right-hand side of Eq. (58) and performing the necessary matrix operations, we obtain:

$$\bar{\Theta}(x,s) = \frac{Q_0\big(1 + \frac{\tau_0^\alpha}{\alpha!}s^\alpha\big)}{2s(k_1^2 - k_2^2)}\left[\frac{(k_1^2 - ws^2)}{k_1}e^{\pm k_1 x} - \frac{(k_2^2 - ws^2)}{k_2}e^{\pm k_2 x}\right] \tag{67}$$

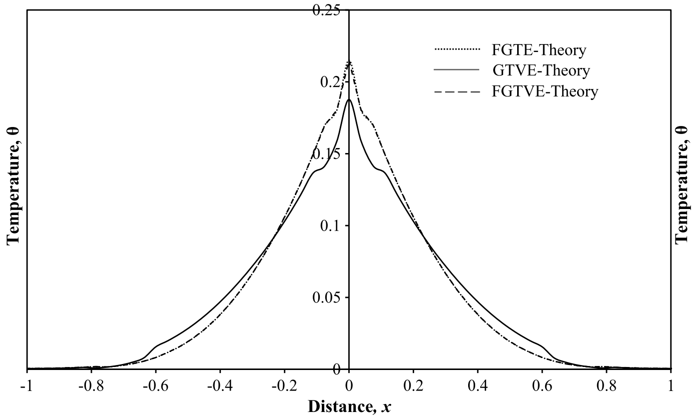

Fig. 3. Effect of fractional parameters α and β on the temperature distribution for the different theories.

$$\bar{u}(x,s) = \frac{\pm Q_0 w\left(1+\frac{\tau_0^\alpha}{\alpha!}s^\alpha\right)}{2s(k_1^2-k_2^2)}\left[e^{\pm k_1 x} - e^{\pm k_2 x}\right] \tag{68}$$

$$\bar{\sigma}(x,s) = \frac{Q_0 w s\left(1+\frac{\tau_0^\alpha}{\alpha!}s^\alpha\right)}{2(k_1^2-k_2^2)}\left[\frac{e^{\pm k_1 x}}{k_1} - \frac{e^{\pm k_2 x}}{k_2}\right] \tag{69}$$

In the above equations, the upper (plus) sign indicates the solution in the region where $x<0$, while the lower (minus) sign indicates the region where $x \geqslant 0$.

6. Inversion of Laplace transforms

In order to invert the Laplace transform in the above equations, we adopt a numerical inversion method based on a Fourier series expansion [34]. In this method, the inverse $g(t)$ of the Laplace transform $\bar{g}(s)$ is approximated by the relation:

$$g(t) = \frac{e^{ct}}{t_1}\left[\frac{1}{2}\bar{g}(c) + \mathrm{Re}\left(\sum_{k=1}^{N} e^{ik\pi t/t_1}\bar{g}(c+ik\pi/t_1)\right)\right], \quad 0 \leqslant t \leqslant 2t_1 \tag{70}$$

where N is a sufficiently large integer representing the number of terms in the truncated infinite Fourier series. N must be chosen such that:

$$e^{ct}\,\mathrm{Re}\left[e^{iN\pi t/t_1}\bar{g}(c+iN\pi/t_1)\right] \leqslant \varepsilon_1 \tag{71}$$

where ε_1 is a persecuted small positive number that corresponds to the degree of accuracy to be achieved. The parameter c is a positive free parameter that must be greater than the real parts of all singularities of $\bar{g}(s)$. The optimal choice of c was obtained according to the criteria described in [34].

7. Numerical results and discussion

A copper-like material was chosen for purposes of numerical evaluations, and the constants of the problem were taken from [57]

$$\varepsilon = 1.618, \quad \rho = 8954\ \mathrm{kg/m^3}, \quad T_0 = 293\ \mathrm{K}, \quad C_E = 383.1\ \mathrm{m^2/s^2\,K}, \quad \tau_0 = 0.02$$
$$\eta_0 = 8886.73\ \mathrm{s/m^2}, \quad \alpha_t = 1.78\times10^{-5}\ \mathrm{K^{-1}}, \quad C_0 = 3397.1\ \mathrm{m/s}$$
$$\lambda = 7.76\times10^{10}\ \mathrm{N/m^2}, \quad \mu = 3.86\times10^{10}\ \mathrm{N/m^2}$$

The investigation of the effect of the fractional derivative parameters α and β has been carried out in the preceding sections. The computations were performed for a value of time, namely, $t=1$. This enables us to represent the typical numerical results in Figs. 3–5, for the temperature Θ, displacement u and stress σ for various values of the parameters α and β. The graphs show curves predicted by the different theories of thermoelasticity. In these figures, dotted lines represent the solution corresponding to fractional generalized thermoelasticity (FGTE Theory) and continued lines represent the solution corresponding to generalized thermo-viscoelasticity (GTVE Theory), while broken lines represent the solution corresponding to new situation (FGTVE Theory).

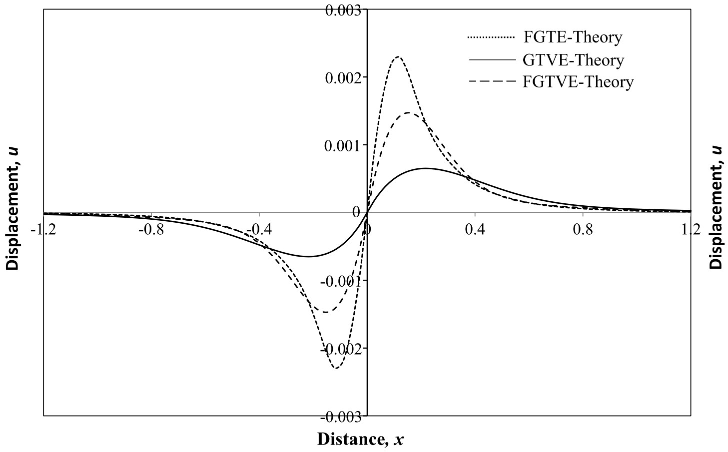

Fig. 4. Effect of fractional parameters α and β on the displacement distribution for the different theories.

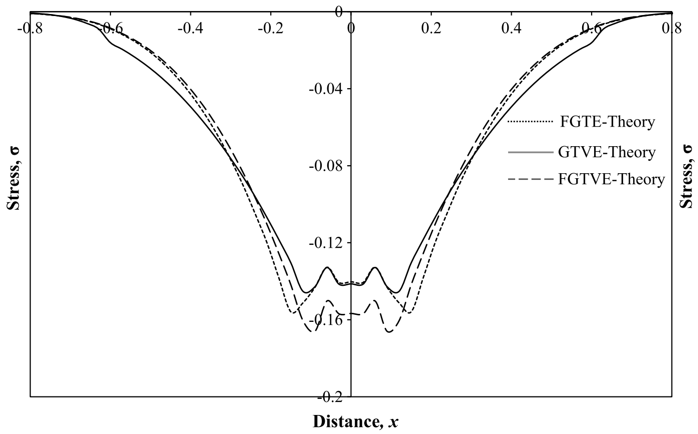

Fig. 5. Effect of fractional parameters α and β on stress distribution for the different theories.

Hence we conclude with the following points:

(i) in all figures, it is noticed that the fractional orders α and β have significant effects on all fields, but β has not effects on temperature distribution;
(ii) it is noticed that all the waves reach the steady state depending on the value of the fractional orders α and β;
(iii) in Fig. 1, we notice that the temperature increment in the fractional order theories is a continuous function, which means that the particles transport the heat to the other particles easily and this makes the decreasing rate of the temperature greater than in the other theory, which predicted through the generalized one. In the fractional order theories, we notice that the thermal wave cut the x-axis more rapidly than in the other ones. We learn from this figure that the temperature distribution decreases with increasing α;
(iv) in Fig. 2, exhibiting the space variation of the displacement for different theories, we notice that, in the fractional order theory, the displacement cut the x-axis more rapidly than in the other curves obtained through the generalized one when β increases.

8. Conclusion

The main goal of this work was to introduce a new fractional mathematical thermo-viscoelastic model for an isotropic medium, which could describe the behavior of viscoelastic materials using a few parameters. Some theorems of thermo-viscoelasticity follow as limit cases. Some comparisons have been shown in the figures to estimate the effects of the fractional-order parameters on all the studied fields. According to this new theory, we have to construct a new classification for materials according to their fractional parameter α, where this parameter becomes a new indicator of its ability to conduct heat in a viscoelastic medium, and fractional parameter β becomes a new indicator of its ability to restrict displacement and stress in the same medium.

Appendix A

The solution of the system (44) is given by:

$$a_0 = \frac{k_1^2 \cosh(K_2 x) - k_2^2 \cosh(K_1 x)}{k_1^2 - k_2^2} \tag{72a}$$

$$a_1 = \frac{k_1^3 \sinh(k_1 x) - k_2^3 \sinh(k_2 x)}{k_1 k_2 (k_1^2 - k_2^2)} \tag{72b}$$

$$a_2 = \frac{\cosh(k_2 x) - \cosh(k_1 x)}{k_1^2 - k_2^2} \tag{72c}$$

$$a_3 = \frac{k_2 \sinh(k_1 x) - k_1 \sinh(k_2 x)}{k_1 k_2 (k_1^2 - k_2^2)} \tag{72d}$$

where the components $[l_{ij}(x,s)]$ are defined as:

$$\begin{aligned}
l_{11} &= \frac{1}{k_1^2 - k_2^2}\big[\big(gs - k_2^2\big)\cosh(k_1 x) - \big(gs - k_1^2\big)\cosh(k_2 x)\big] \\
l_{12} &= \frac{g\varepsilon w s^3}{k_1^2 - k_2^2}\left[\frac{\sinh(k_1 x)}{k_1} - \frac{\sinh(k_2 x)}{k_2}\right] \\
l_{13} &= \frac{1}{k_1^2 - k_2^2}\left[\frac{(k_1^2 - ws^2)\sinh(k_1 x)}{k_1} - \frac{(k_2^2 - ws^2)\sinh(k_2 x)}{k_2}\right] \\
l_{14} &= \frac{s\varepsilon g}{k_1^2 - k_2^2}\big[\cosh(k_1 x) - \cosh(k_2 x)\big] \\
l_{21} &= \frac{gws}{k_1^2 - k_2^2}\left[\frac{\sinh(k_1 x)}{k_1} - \frac{\sinh(k_2 x)}{k_2}\right] \\
l_{23} &= \frac{w}{k_1^2 - k_2^2}\big[\cosh(k_1 x) - \cosh(k_2 x)\big] \\
l_{24} &= \frac{1}{k_1^2 - k_2^2}\left[\frac{(k_1^2 - gs)\sinh(k_1 x)}{k_1} - \frac{(k_2^2 - gs)\sinh(k_2 x)}{k_2}\right] \\
l_{31} &= sgl_{13}, \qquad l_{32} = \varepsilon g s^2 l_{23} \\
l_{33} &= \frac{1}{k_1^2 - k_2^2}\big[\big(k_1^2 - ws^2\big)\cosh(k_1 x) - \big(k_2^2 - ws^2\big)\cosh(k_2 x)\big] \\
l_{34} &= \frac{s\varepsilon g}{k_1^2 - k_2^2}\big[k_1 \sinh(k_1 x) - k_2 \sinh(k_2 x)\big] \\
l_{41} &= \frac{w}{\varepsilon} l_{14}, \qquad l_{42} = ws^2 l_{24}, \qquad l_{43} = \frac{w}{s\varepsilon g} l_{34} \\
l_{44} &= \frac{1}{k_1^2 - k_2^2}\big[\big(k_1^2 - gs\big)\cosh(k_1 x) - \big(k_2^2 - gs\big)\cosh(k_2 x)\big]
\end{aligned} \tag{73}$$

Appendix B

The Caputo-derivative Laplace transform is defined as:

$$L\big\{D_C^\alpha g(t)\big\} = \big\{s^\alpha \bar{g}(s)\big\} - \sum_{K=0}^{n-1} f^k\big(0^+\big) s^{\alpha - 1 - k}, \quad n - 1 < \alpha < n$$

Note that as we use the Caputo fractional derivative omitting the index C, it should be noted that this relation is valid if initial conditions are properly taken into account [49,52].

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