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# Effects of rough boundary on the heat transfer in a thin-film flow

## *Effets de la frontière rugueuse sur le transfert de chaleur dans un écoulement en film mince*

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ABSTRACT

In this Note, a heat flow through a rough thin domain filled with fluid (lubricant) is studied. The domain's thickness is considered as the small parameter $\varepsilon$, while the roughness is defined by a periodical function with a period of order $\varepsilon^2$. We assume that the lubricant is cooled by the exterior medium and we describe the heat exchange on the rough part of the boundary by Newton's cooling law. Depending on the magnitude of the heat transfer coefficient with respect to $\varepsilon$, we obtain three different macroscopic models via formal asymptotic analysis. We identify the critical case explicitly acknowledging both roughness-induced effects and the effects of the surrounding medium on heat transfer at main order. We illustrate the obtained results by some numerical simulations.



RÉSUMÉ

Dans cette Note, on étudie un flux de chaleur dans un domaine rugueux de faible épaisseur rempli de liquide (lubrifiant). On considère l'épaisseur du domaine comme le petit paramètre $\varepsilon$, tandis que la rugosité est définie par une fonction périodique de période d'ordre $\varepsilon^2$. On suppose que le lubrifiant est refroidi par le milieu extérieur et que l'échange de chaleur est décrit sur la partie rugueuse de la frontière par la loi de refroidissement de Newton. En fonction de la valeur du coefficient de transfert de chaleur par rapport à $\varepsilon$, on obtient trois différents modèles macroscopiques via une analyse asymptotique formelle. On identifie le cas critique, reconnaissant explicitement les effets induits par la rugosité et les effets du milieu environnant sur le transfert de chaleur à l'ordre principal. Les résultats obtenus sont illustrés à l'aide de simulations numériques.



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## 1. Introduction

Lubrication is mostly concerned with the behavior of a lubricant flowing through a narrow gap. It appears naturally in many engineering applications consisting of moving machine parts, e.g. in journal bearings or computer disk drives. Assuming that the height of the gap is given by:

$$h_\varepsilon(x) = \varepsilon h(x) \tag{1}$$

and using $\varepsilon > 0$ as a small parameter, a simple asymptotic approximation can be easily derived providing a well-known Reynolds equation for the pressure of the fluid. Its derivation goes back to 1886 and the pioneering work of O. Reynolds [1]. The formal relationship between the Navier–Stokes equations and the Reynolds equation in a thin domain was given in the 1950s by Wannier [2] and Elrod [3]. A rigorous mathematical justification of the Reynolds equation for a Newtonian flow between two plain surfaces can be found in [4]. Some more recent results on that subject can be found in [5–7].

Numerous works combine the lubrication phenomena with the analysis of the roughness effects. Formulating the boundary roughness using a periodic function, the *isothermal* thin-film flow has been extensively studied for different rugosity profiles. Throughout the literature (see, e.g., [8–10]), the most usual assumption is that the size of the roughness is at least of the same order as the film thickness, namely:

$$h_\varepsilon(x) = \varepsilon h\left(x, \frac{x}{\varepsilon^\beta}\right), \quad \beta \leqslant 1 \tag{2}$$

For such $\beta$, the effective model (at main order) turns out to be the classical Reynolds equation. That motivated Bresch and co-authors [11] to recently consider a rather specific (but physically relevant, see e.g. [12]) case:

$$h_\varepsilon(x) = \varepsilon h\left(x, \frac{x}{\varepsilon^2}\right) \tag{3}$$

leading to the explicit correction of the Reynolds equation by the roughness-induced term. Interesting generalizations of such framework can be found in [13,14] and they merit careful reading (see also [15]).

In the present paper, we consider a *non-isothermal* fluid flow in a rough thin domain:

$$\Omega_\varepsilon = \left\{(x, z) \in \mathbb{R}^2 \times \mathbb{R}\colon x \in \omega,\ 0 < z < h_\varepsilon(x)\right\} \tag{4}$$

where the height $h_\varepsilon$ has the form (3). The lubricant is assumed to be a heat-conducting incompressible viscous fluid and the flow is governed by the stationary Navier–Stokes equations coupled with the temperature equation. In order to consider the situation appearing naturally in industrial applications, we suppose that the lubricant is cooled by some exterior medium with temperature $T_\mathrm{h}$. In view of that, the heat exchange (occurring on the rough part of the boundary $z = h_\varepsilon(x)$) is described by the Robin boundary condition resulting from Newton's cooling law. The bottom part of the boundary ($z = 0$) is maintained on the constant temperature $T_\mathrm{b}$ described by the standard Dirichlet condition. We study the thermodynamic part of the system (assuming that the hydrodynamic part is known) and our goal is to derive an effective model describing the heat transfer. So far, the authors have studied the similar problem only in the classical case $h_\varepsilon(x) = \varepsilon h(x, \frac{x}{\varepsilon})$, when the ratio between the size of the rugosities and the mean height of the domain is of order one (see, e.g., [16–18]). We believe that the case (3) is the more demanding one from the point of view of asymptotic analysis due to technical difficulties caused by the specific height profile.

In order to construct the asymptotic approximation for the temperature, we introduce a suitable change of variables (taking into account the rough oscillations) and employ a multiscale expansion technique. As a result, we obtain three possible macroscopic models (in a form of the explicit formulae), depending on the magnitude of the heat transfer coefficient $\alpha$ from the Newton cooling condition. More precisely:

- if $\alpha \ll \mathcal{O}(\frac{1}{\varepsilon})$, the effects of cooling are negligible. Consequently, the temperature $T_\mathrm{b}$ of the bottom part completely dominates the process and no effects of the roughness are observed;
- if $\alpha \gg \mathcal{O}(\frac{1}{\varepsilon})$, we obtain some roughness-induced effects, but the process is essentially dominated by the exterior temperature $T_\mathrm{h}$;
- critical case $\alpha = \mathcal{O}(\frac{1}{\varepsilon})$ when the influences of cooling and specific rugosity profile are of the same order and they both remain in the asymptotic solution.

To our knowledge, such result cannot be found in the context of tribology and we believe that it could be instrumental for improving the known engineering practice. In order to illustrate the obtained results, we also provide some numerical simulations comparing, in the critical case, the effective temperature for different rugosity profiles.

## 2. Description of the problem

We consider the following rough domain (see Fig. 1):

$$\Omega_\varepsilon = \left\{(x, z) \in \mathbb{R}^2 \times \mathbb{R}\colon x \in \omega,\ 0 < z < h_\varepsilon(x)\right\} \tag{5}$$

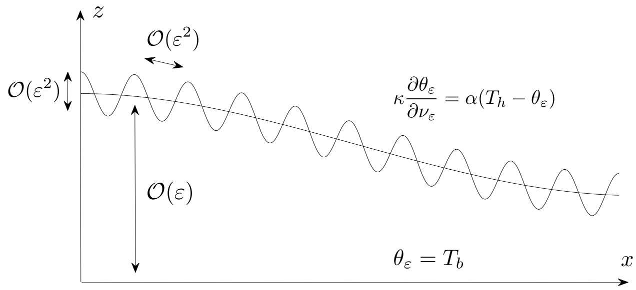


**Fig. 1.** The domain.

where $\omega \subset \mathbb{R}^2$ and

$$h_\varepsilon(x) = \varepsilon h_1(x) + \varepsilon^2 h_2\left(\frac{x}{\varepsilon^2}\right) \tag{6}$$

The positive function $h_1$ is the main-order part of the roughness, while periodic function $h_2$ describes the oscillating part.

We denote by $\Gamma_\varepsilon$ the rough top of the domain, that is:

$$\Gamma_\varepsilon = \left\{(x, z) \in \mathbb{R}^2 \times \mathbb{R}\colon\ x \in \omega,\ z = h_\varepsilon(x)\right\}$$

We also define $\Omega = \omega \times (0, 1) \subset \mathbb{R}^2 \times \mathbb{R}$, and denote by $\mathbb{T}^2$ the torus of dimension 2.

In this paper, we focus on the heat conduction equation describing the thermal effects on the fluid motion in $\Omega_\varepsilon$. Neglecting the buoyancy forces resulting from the thermic expansion of the fluid (see [19]), we can assume that the hydrodynamic part is known and given by the asymptotic approximation $\mathbf{u}_\varepsilon^0$ rigorously derived in [11]. Thus, we consider the following problem for the unknown fluid temperature $\theta_\varepsilon$:

$$\begin{cases} -\kappa \Delta \theta_\varepsilon + \rho_0 C_p (\mathbf{u}_\varepsilon^0 \nabla)\theta_\varepsilon = 0 & \text{in } \Omega_\varepsilon \\ \kappa \frac{\partial \theta_\varepsilon}{\partial \nu_\varepsilon} = \alpha(T_\mathrm{h} - \theta_\varepsilon) & \text{on } \Gamma_\varepsilon \\ \theta_\varepsilon = T_\mathrm{b} & \text{on } \omega \times \{0\} \\ \theta_\varepsilon = 0 & \text{on } \partial\omega \end{cases} \tag{7}$$

All physical properties of the fluid are assumed to be constant, namely: viscosity $\mu$, heat conductivity $\kappa$, and specific heat capacity $C_p$. We denote by $\rho_0$ the referent density of the fluid and put $\rho_0 = 1$, $C_p = 1$, for the sake of notational simplicity. The lubricant inside $\Omega_\varepsilon$ is cooled by the exterior medium (with prescribed temperature $T_\mathrm{h}$) and that process is described by Newton's cooling condition $(7)_2$ at the top. We denote by $\nu_\varepsilon$ the exterior unit normal on $\Gamma_\varepsilon$, while $\alpha > 0$ corresponds to the heat transfer coefficient. The bottom part of the boundary is maintained on the constant temperature $T_\mathrm{b}$, $T_\mathrm{b} > T_\mathrm{h}$. The existence and uniqueness issues concerning the equations of heat-conducting incompressible viscous fluids have been successfully resolved (see, e.g., [20]), and thus are not addressed in this paper. Our goal is to find the macroscopic law describing the heat transfer governed by (7), using asymptotic analysis with respect to the small parameter $\varepsilon$.

## 3. Asymptotic analysis

We first introduce a fast variable $X = \frac{x}{\varepsilon^2}$ capturing the oscillating phenomena of the thin domain. In view of that, $h_\varepsilon$ can be written as:

$$h(x, X) = \varepsilon h_1(x) + \varepsilon^2 h_2(X)$$

Next we introduce a new vertical variable $Z = \frac{z}{h(x,X)}$ and, correspondingly, the new unknown function $\theta_\varepsilon(x, z) = \theta(x, X, Z)$. For the known velocity profile, according to [11], we must take $\mathbf{u}_\varepsilon^0(x, z) = (u_0(x, Z), 0)$, where $u_0(x, Z)$ is an explicit function of the *modified* Reynolds pressure (see (16)–(17) in [11]). Taking into account the above change of variables, we can rewrite the temperature equation $(7)_1$ in the $\varepsilon$-independent domain $\Omega \times \mathbf{R}^2$ as:

$$\begin{aligned} &-\kappa h^2 \Delta_x \theta - \frac{2\kappa}{\varepsilon^2} h^2 \nabla_x \cdot \nabla_X \theta - \frac{\kappa}{\varepsilon^4} h^2 \Delta_X \theta + \frac{2\kappa}{\varepsilon^2} h \nabla h \cdot Z \nabla_X \partial_Z u + \kappa h \Delta h Z \partial_Z \theta - \kappa |\nabla h|^2 Z \partial_Z \theta \\ &\quad + 2\kappa h \nabla h \cdot Z \nabla_x \partial_Z \theta - \kappa |\nabla h|^2 Z^2 \partial_Z^2 \theta - \kappa \partial_Z^2 \theta + h^2 u_0 \cdot \nabla_x \theta + \frac{1}{\varepsilon^2} h^2 u_0 \cdot \nabla_X \theta - h \nabla h \cdot u_0 Z \partial_Z \theta = 0 \end{aligned} \tag{8}$$

where

$$\nabla h(x,X)=\varepsilon\nabla_x h_1(x)+\nabla_X h_2(X),\qquad \Delta h(x,X)=\varepsilon\Delta_x h_1(x)+\frac{1}{\varepsilon^2}\Delta_X h_2(X)$$

Finally, the boundary condition $(7)_2$ at the top reads:

$$\partial_Z\theta=\alpha h(T_{\mathrm h}-\theta)\quad \text{for } Z=1 \tag{9}$$

Now we formally expand the unknown temperature:

$$\theta(x,X,Z)=\theta_0(x,X,Z)+\varepsilon\theta_1(x,X,Z)+\varepsilon^2\theta_2(x,X,Z)+\cdots \tag{10}$$

and plug it into (8)–(9). The leading-order term from Eq. (8) gives:

$$\frac{1}{\varepsilon^2}:\quad -\kappa h_1^2\Delta_X\theta_0=0 \tag{11}$$

Taking into account the boundary conditions with respect to $X$, we deduce $\nabla_X\theta_0=0$, i.e. $\theta_0=\theta_0(x,Z)$. In view of that, the next two terms yield:

$$\frac{1}{\varepsilon}:\quad \kappa h_1^2\Delta_X\theta_1=\kappa h_1\Delta_X h_2 Z\partial_Z\theta_0 \tag{12}$$

$$\begin{aligned}1:\quad &-\kappa h_1^2\Delta_X\theta_2-2\kappa h_1h_2\Delta_X\theta_1+2\kappa h_1\nabla_X h_2\cdot Z\nabla_X\partial_Z\theta_1+\kappa h_1\Delta_X h_2 Z\partial_Z\theta_1\\&+\kappa h_2\Delta_X h_2 Z\partial_Z\theta_0-\kappa|\nabla_X h_2|^2 Z\partial_Z\theta_0-\kappa|\nabla_X h_2|^2Z^2\partial_Z^2\theta_0-\kappa\partial_Z^2\theta_0=0\end{aligned} \tag{13}$$

To obtain the effective equation satisfied by $\theta_0$, we take the mean value (with respect to $X$) of each term appearing in (13). Since $h_1$ does not depend on $X$, it can be easily verified that:

$$-\int_{\mathbb T^2}\kappa h_1^2\Delta_X\theta_2\,\mathrm dX=0 \tag{14}$$

Using (12), integration by parts and the fact that $h_2$ is $X$-periodic gives

$$-\int_{\mathbb T^2}2\kappa h_1h_2\Delta_X\theta_1\,\mathrm dX=-\int_{\mathbb T^2}2\kappa h_2\Delta_X h_2 Z\partial_Z\theta_0\,\mathrm dX=\int_{\mathbb T^2}2\kappa|\nabla_X h_2|^2Z\partial_Z\theta_0\,\mathrm dX=2\kappa MZ\partial_Z\theta_0 \tag{15}$$

Here and in the sequel we introduce:

$$M=\int_{\mathbb T^2}|\nabla_X h_2|^2\,\mathrm dX$$

as a new coefficient depending exclusively on the considered rugosity profile. We proceed in the similar manner as above to obtain:

$$\int_{\mathbb T^2}2\kappa h_1\nabla_X h_2\cdot Z\nabla_X\partial_Z\theta_1\,\mathrm dX=2\kappa MZ\partial_Z\theta_0+2\kappa MZ^2\partial_Z^2\theta_0 \tag{16}$$

$$\int_{\mathbb T^2}\kappa h_1\Delta_X h_2 Z\partial_Z\theta_1\,\mathrm dX=-\int_{\mathbb T^2}\kappa h_1\nabla_X h_2\cdot Z\nabla_X\partial_Z\theta_1\,\mathrm dX=-\kappa MZ\partial_Z\theta_0-\kappa MZ^2\partial_Z^2\theta_0 \tag{17}$$

$$\int_{\mathbb T^2}\kappa h_2\Delta_X h_2 Z\partial_Z\theta_0\,\mathrm dX=-\int_{\mathbb T^2}\kappa|\nabla_X h_2|^2Z\partial_Z\theta_0\,\mathrm dX=-\kappa MZ\partial_Z\theta_0 \tag{18}$$

$$-\int_{\mathbb T^2}\kappa|\nabla_X h_2|^2Z\partial_Z\theta_0\,\mathrm dX-\int_{\mathbb T^2}\kappa|\nabla_X h_2|^2Z^2\partial_Z^2\theta_0\,\mathrm dX=-\kappa MZ\partial_Z\theta_0-\kappa MZ^2\partial_Z^2\theta_0 \tag{19}$$

Applying (14)–(19) into (13) yields the equation for $\theta_0(x,Z)$:

$$-\partial_Z^2\theta_0+MZ\partial_Z\theta_0=0\quad \text{in } \Omega \tag{20}$$

For each (fixed) $x\in\omega$, the above equations can be explicitly solved as a simple second-order linear ODE with respect to $Z$. Since $\theta_0(x,0)=T_{\mathrm b}$ (see $(7)_3$), we obtain:

$$\theta_0(x,Z)=C(x)\int_0^Z \mathrm e^{\frac{M\xi^2}{2}}\,\mathrm d\xi+T_{\mathrm b},\qquad C(x)\in\mathbf R,\ (x,Z)\in\Omega \tag{21}$$

In order to uniquely determine $\theta_0$, we need to take into account Robin's boundary condition (9). Following the idea from [21,22], we compare the heat transfer coefficient $\alpha$ with the small parameter $\varepsilon$, leading to three characteristic situations:

**Case 1.** $\alpha \ll \mathcal{O}(\frac{1}{\varepsilon})$.

The main-order term from the boundary condition (9) gives $\partial_Z \theta_0(x,1) = 0$, meaning that the effects of cooling are negligible. Consequently, we get:

$$\theta_0(x,Z) = T_{\mathrm{b}}$$

and no effects of the roughness are observed in this case. The process is, in fact, dominated by the prescribed temperature of the bottom part.

**Case 2.** $\alpha \gg \mathcal{O}(\frac{1}{\varepsilon})$.

The main-order term from (9) yields $\theta_0(x,1) = T_{\mathrm{h}}$, implying:

$$\theta_0(x,Z) = \frac{T_{\mathrm{h}} - T_{\mathrm{b}}}{\int_0^1 \mathrm{e}^{\frac{M\xi^2}{2}}\,\mathrm{d}\xi} \int_0^Z \mathrm{e}^{\frac{M\xi^2}{2}}\,\mathrm{d}\xi + T_{\mathrm{b}}$$

Though we obtained some roughness-induced effects (no effects of the main order $h_1$), the process is essentially dominated by the exterior temperature $T_{\mathrm{h}}$.

**Critical case.** $\alpha = \mathcal{O}(\frac{1}{\varepsilon})$.

Introducing $\gamma = \varepsilon\alpha = \mathcal{O}(1)$, the main-order term from (9) provides:

$$\partial_Z \theta_0(x,1) = \frac{\gamma}{\kappa} h_1(x)\big(T_{\mathrm{h}} - \theta_0(x,1)\big)$$

Using this condition in (21), we get:

$$\theta_0(x,Z) = \frac{\gamma h_1(x)(T_{\mathrm{h}} - T_{\mathrm{b}})}{\kappa\,\mathrm{e}^{\frac{M}{2}} + \gamma h_1(x)\int_0^1 \mathrm{e}^{\frac{M\xi^2}{2}}\,\mathrm{d}\xi} \int_0^Z \mathrm{e}^{\frac{M\xi^2}{2}}\,\mathrm{d}\xi + T_{\mathrm{b}}, \quad (x,Z) \in \Omega \tag{22}$$

This is, obviously, the most interesting case. The effects of cooling and the specific rugosity profile are of the same order and can be clearly seen in the asymptotic solution governing the macroscopic process.

**Remark 3.1.** If we consider a smooth domain without the rugosities, i.e. $h_\varepsilon(x) = \varepsilon h_1(x)$, we obtain a simple linear function as the asymptotic solution in the critical case. It can be easily recovered from (22) by putting $M = 0$. Moreover, using similar arguments as above, we can show that in the case where the narrow gap is smaller than the roughness, namely:

$$h_\varepsilon(x) = \varepsilon h_1(x) + \varepsilon^2 h_2\left(\frac{x}{\varepsilon^\beta}\right), \quad \beta < 1$$

no roughness-induced effects can be found at the main-order approximation. In fact, we obtain the same asymptotic behavior as in the case without roughness.

## 4. Numerical illustrations

In this section we present some numerical results. We focus on the critical case and compare the asymptotic solution $\theta_0$, given by (22), for different rugosity profiles obeying (6). Analogously to the case in [11], it is not hard to verify that the system:

$$\begin{cases} -\partial_Z^2 \theta_0 + MZ\partial_Z \theta_0 = 0, & \text{in } \Omega \\ \partial_Z \theta_0(x,1) = \dfrac{\gamma}{\kappa} h_1(x)\big(T_{\mathrm{h}} - \theta_0(x,1)\big), & \theta_0(x,0) = 0,\ x \in \omega \end{cases} \tag{23}$$

is energetically consistent for $M < 2$. Now, defining $h_1(x) = 1.3x^3 - 2x^2 + 1$, we present (see Fig. 2) three examples of rugosity profiles yielding different values of $M$. The left-hand figure provides $M = 0.1922$, the middle one gives $M = 1.2337$ while the right-hand figure, with a more oscillating boundary, yields $M = 1.7783$.

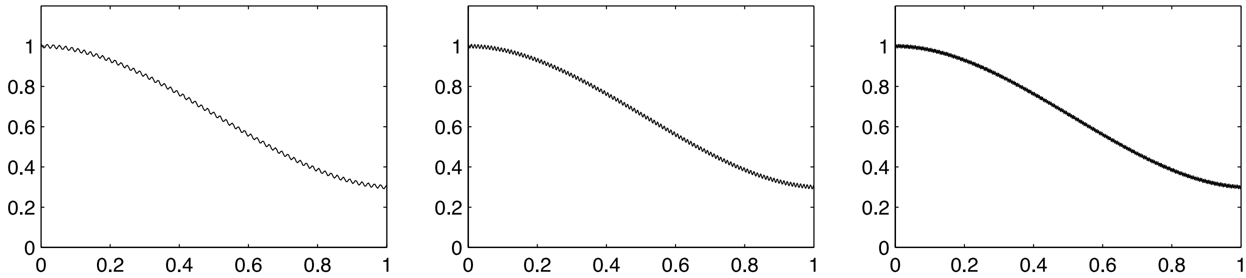


**Fig. 2.** (Left) $h(x) = h_1(x) + 0.01\cos(2000\pi x)$, (middle) $h(x) = h_1(x) + 0.01\cos(5000\pi x)$, (right) $h(x) = h_1(x) + 0.01\cos(6000\pi x)$.

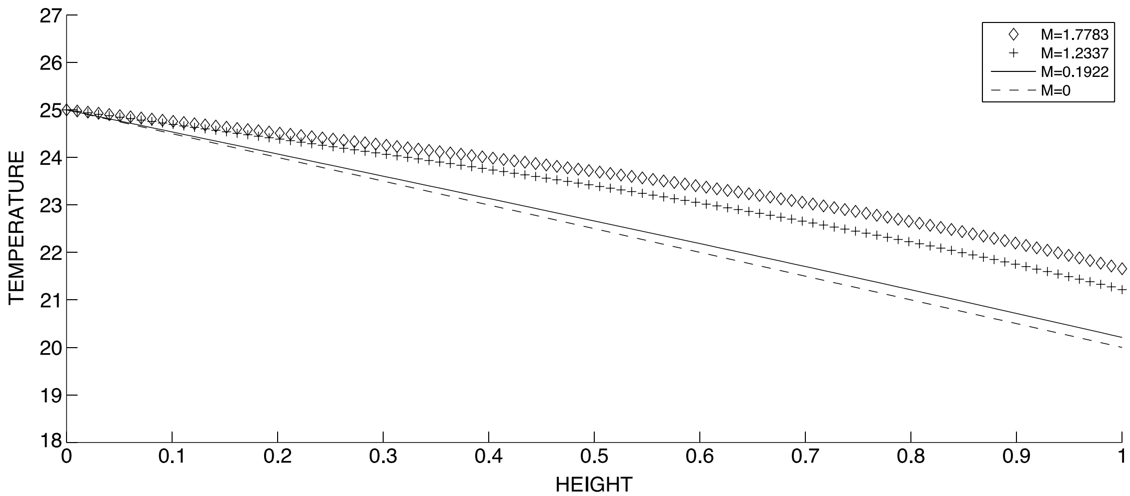


**Fig. 3.** The effective temperature for different rugosity profiles.

Finally, we illustrate in Fig. 3 the behavior of the asymptotic solution $\theta_0$ for a fixed $x \in \omega$, comparing it for three rugosity profiles given above. For this, we use the following data: $T_h = 10$, $T_b = 25$, $\gamma = 1$, $\kappa = 1$. Note that the simple linear function corresponding to $\theta_0$ for $M = 0$ (case without rugosity) has also been plotted. We clearly observe the effects of rugosities on the asymptotic behavior of the fluid temperature: as the coefficient $M$ increases (meaning that we have more oscillating boundary), the effective temperature becomes more different from a standard linear approximation corresponding to the case without rugosity.

## 5. Concluding remarks

In this Note, an effective model describing the heat transfer through a rough thin domain has been proposed. The domain's thickness is considered as the small parameter $\varepsilon$, while the roughness is defined by a periodical function with a period of order $\varepsilon^2$. Starting from the mixed boundary-value problem for the fluid temperature and assuming that the velocity distribution is known, we derive three possible asymptotic models in a form of the explicit formulae. We establish the strong connection between the magnitude of the heat transfer coefficient and the asymptotic behavior of the heat flow. Though the derivation was just formal, it provides a very good platform for understanding the direct influence of the rough boundary on the thermodynamic part of the lubrication process. Of course, from the strictly mathematical point of view, a formally derived model should be rigorously justified by proving some kind of convergence of the original solution towards the asymptotic one. A possible way to proceed is to use a variant of the usual two-scale convergence (see, e.g., [23]) and adapt it to our situation, similarly to the case in [11]. The other approach can be to compute the correctors $\theta_1$ and $\theta_2$ in the asymptotic expansion (10) and try to derive satisfactory $L^2$ or $H^1$ error estimates. The latter is the subject of our current investigation.

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