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# Thin linearly viscoelastic Kelvin–Voigt plates

## *Plaques minces viscoélastiques linéaires de Kelvin–Voigt*

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ABSTRACT

A mathematical model for thin viscoelastic Kelvin–Voigt plates is derived through an asymptotic analysis when the thickness goes to zero. The model involves Kirchhoff–Love kinematics, but the mechanical behavior is no longer of Kelvin–Voigt type: an additional term of delayed memory appears like in homogenization.


RÉSUMÉ

On propose un modèle mathématique pour les plaques minces viscoélastiques linéaires de Kelvin–Voigt par une étude asymptotique lorsque l'épaisseur tend vers zéro. Le modèle met en jeu une cinématique de Kirchhoff–Love, mais le comportement n'est plus de type Kelvin–Voigt: comme en homogénéisation, un terme additionnel de mémoire longue apparaît.


## 1. Setting the problem

The reference configuration of the thin linearly viscoelastic Kelvin–Voigt plate is the closure in $\mathbb{R}^3$ of the set $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$ whose outward unit normal is $n^\varepsilon$. Here $\varepsilon$ is a small positive number and $\omega$ a bounded domain of $\mathbb{R}^2$ with a Lipschitz boundary $\partial\omega$. The lateral part of the plate $\partial\omega \times (-\varepsilon, \varepsilon)$ is denoted $\Gamma^\varepsilon_{\mathrm{lat}}$, while $\Gamma^\varepsilon_\pm = \omega \times \{\pm\varepsilon\}$ refers to the upper or lower face, respectively. The plate is clamped on a portion $\Gamma^\varepsilon_D := \gamma_D \times (-\varepsilon, \varepsilon)$ of its lateral face, with $\gamma_D$ of positive length, and subjected to body forces and surface forces on $\Gamma^\varepsilon_N := \partial\Omega^\varepsilon \setminus \Gamma^\varepsilon_D$ of density $f^\varepsilon$ and $g^\varepsilon$, respectively. The equations determining the quasi-static evolution, during the time interval $[0, T]$, of the plate in an initial state $u^\varepsilon_0$ read as:

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$$\begin{cases} \operatorname{div}\sigma^{\varepsilon} + f^{\varepsilon} = 0 \text{ in } \Omega^{\varepsilon} \times (0,T), \sigma^{\varepsilon} n^{\varepsilon} = g^{\varepsilon} \text{ on } \Gamma_N^{\varepsilon} \times (0,T), u^{\varepsilon} = 0 \text{ on } \Gamma_D^{\varepsilon} \times (0,T) \\ \sigma^{\varepsilon} = a^{\varepsilon} e(u^{\varepsilon}) + b^{\varepsilon} e(\dot{u}^{\varepsilon}) \text{ in } \Omega^{\varepsilon} \times (0,T) \\ u^{\varepsilon}(.,0) = u_0^{\varepsilon} \text{ in } \Omega^{\varepsilon} \end{cases} \tag{1}$$

where $u^{\varepsilon}, \sigma^{\varepsilon}, e(u^{\varepsilon}), a^{\varepsilon}$ and $b^{\varepsilon}$ denote the displacement, the stress tensor, the linearized strain tensor, the elasticity and viscosity tensor field, respectively, while the upper dot stands for the time derivative. Under suitable and realistic assumptions on the data, (1) can be formulated in terms of an ordinary differential equation governed by a bounded, selfadjoint and negative operator, which yields existence and uniqueness for $u^{\varepsilon}$.

To derive a simplified and accurate model, the true question is to study the behavior of $u^{\varepsilon}$ when $\varepsilon$, *regarded as a parameter*, tends to zero. As in the linearly elastic case [1], it is convenient to proceed to a change of coordinates and unknowns. One comes down to a fixed open set $\Omega = \omega \times (-1,1)$ through the mapping $\pi^{\varepsilon}$:

$$x = (\widehat{x}, x_3) \in \overline{\Omega} \mapsto \pi^{\varepsilon} x = (\widehat{x}, \varepsilon x_3) \in \overline{\Omega}^{\varepsilon} \tag{2}$$

where $\widehat{\xi} = (\xi_1, \xi_2)$ if $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. We also drop the index $\varepsilon$ for the images by $(\pi^{\varepsilon})^{-1}$ of the geometric sets defined previously. The following assumptions on the loading are the "time-dependent" version of the classical assumptions leading to a limit kinematics of Kirchhoff–Love:

$$(\mathbf{H}_1): \quad \begin{cases} \exists (f,g) \in C^0([0,T]; L^2(\Omega)^3 \times L^2(\Gamma_N)^3) \text{ such that} \\ \widehat{f}^{\varepsilon}(\pi^{\varepsilon} x, t) = \varepsilon \widehat{f}(x,t), \quad f_3^{\varepsilon}(\pi^{\varepsilon} x, t) = \varepsilon^2 f_3(x,t), \ \forall (x,t) \in \Omega \times [0,T] \\ \widehat{g}^{\varepsilon}(\pi^{\varepsilon} x, t) = \varepsilon^2 \widehat{g}(x,t), \quad g_3^{\varepsilon}(\pi^{\varepsilon} x, t) = \varepsilon^3 g_3(x,t), \ \forall (x,t) \in (\Gamma_N \cap \Gamma_{\pm}) \times [0,T] \\ \widehat{g}^{\varepsilon}(\pi^{\varepsilon} x, t) = \varepsilon \widehat{g}(x,t), \quad g_3^{\varepsilon}(\pi^{\varepsilon} x, t) = \varepsilon^2 g_3(x,t), \ \forall (x,t) \in (\Gamma_N \cap \Gamma_{lat}) \times [0,T] \end{cases}$$

where $t$ denotes the time. Similarly, the following assumptions on the elasticity and viscosity tensors are in order:

$$(\mathbf{H}_2): \quad \begin{cases} a^{\varepsilon}(\pi^{\varepsilon} x) = a(x), \quad b^{\varepsilon}(\pi^{\varepsilon} x) = b(x) \text{ with } a, b \in L^{\infty}(\Omega, \operatorname{Lin}(S^3)), \text{ and} \\ \exists \kappa > 0 : a(x) e \cdot e \geqslant \kappa |e|_{S^3}^2, b(x) e \cdot e \geqslant \kappa |e|_{S^3}^2, \ \forall e \in S^3, \text{ a.e. } x \in \Omega, \end{cases}$$

where $\operatorname{Lin}(S^3)$ denotes the space of linear mappings from $S^3$ into $S^3$, $S^3$ being the space of $3 \times 3$ symmetric matrices. It will be convenient to write $S^3 = \hat{S} \oplus S^{\perp}$ with $\hat{S} := \{e \in S^3; e_{i3} = 0, 1 \leqslant i \leqslant 3\}$, $S^{\perp} := \{e \in S^3; e_{\alpha\beta} = 0, 1 \leqslant \alpha, \beta \leqslant 2\}$, and to denote the projection of $e$ on $\hat{S}$ and $S^{\perp}$ by $\hat{e}$ and $e^{\perp}$, respectively.

Then one associates a *scaled* displacement $u(\varepsilon) = S(\varepsilon) u^{\varepsilon}$, defined on $\overline{\Omega} \times [0,T]$, with the true physical displacement $u^{\varepsilon}$, defined on $\overline{\Omega}^{\varepsilon} \times [0,T]$, by:

$$\widehat{u^{\varepsilon}}(x^{\varepsilon}, t) = \varepsilon \big(\widehat{u(\varepsilon)}\big)(x,t), \qquad u_3^{\varepsilon}(x^{\varepsilon}, t) = \big(u(\varepsilon)\big)_3(x,t), \quad \forall (x^{\varepsilon}, t) = (\pi^{\varepsilon} x, t) \in \overline{\Omega}^{\varepsilon} \times [0,T] \tag{3}$$

Thus (1) is formally equivalent to

$$\begin{cases} u(\varepsilon) \in C^1([0,T]; H); \quad u(\varepsilon)(.,0) = u_0(\varepsilon) := S(\varepsilon) u_0^{\varepsilon} \\ \int_{\Omega} a(x) e(\varepsilon, u(\varepsilon))(x,t) \cdot e(\varepsilon, v)(x) \, dx + \int_{\Omega} b(x) e(\varepsilon, \dot{u}(\varepsilon))(x,t) \cdot e(\varepsilon, v)(x) \, dx \\ \quad = \int_{\Omega} f(x,t) \cdot v(x) \, dx + \int_{\Gamma_N} g(x,t) \cdot v(x) \, ds, \quad \forall (v,t) \in H \times [0,T] \end{cases} \tag{4}$$

where $H$ is the subspace of $H^1(\Omega)^3$ whose elements vanish on $\Gamma_D$ equipped with the inner product:

$$(u,v)_{\varepsilon} = \int_{\Omega} b(x) e(\varepsilon, u)(x) \cdot e(\varepsilon, v)(x) \, dx \tag{5}$$

with

$$e(\varepsilon, v)_{\alpha\beta} = e_{\alpha\beta}(v), \qquad e(\varepsilon, v)_{\alpha 3} = \varepsilon^{-1} e_{\alpha 3}(v), \quad 1 \leqslant \alpha, \qquad \beta \leqslant 2, \qquad e_{33}(\varepsilon, v) = \varepsilon^{-2} e_{33}(v) \tag{6}$$

Obviously, the linear operator $A$, defined by:

$$(Au, v)_{\varepsilon} = -\int_{\Omega} a(x) e(\varepsilon, u)(x) \cdot e(\varepsilon, v)(x) \, dx, \quad \forall v \in H \tag{7}$$

is bounded, selfadjoint and negative, while $F(\varepsilon)(t)$ in $H$ defined for all $t$ in $[0,T]$ by:

$$\big(F(\varepsilon)(t), v\big)_{\varepsilon} = \int_{\Omega} f(x,t) \cdot v(x) \, dx + \int_{\Gamma_N} g(x,t) \cdot v(x) \, ds, \quad \forall v \in H \tag{8}$$

belongs to $C^0([0,T]; H)$. Hence (4) is equivalent to:

$$\begin{cases} \dfrac{\mathrm{d}u(\varepsilon)}{\mathrm{d}t} = Au(\varepsilon) + F(\varepsilon) \quad \text{in } H \\ u(\varepsilon)(0) = u_0(\varepsilon) \end{cases} \tag{9}$$

which, classically, has a unique solution in $C^1([0,T];H)$.

## 2. A convergence result

Let $(\tilde{F}(\varepsilon), \tilde{f}, \tilde{g})$ be a $C^0([0,\infty); H \times L^2(\Omega)^3 \times L^2(\Gamma_N)^3)$ extension of $(F(\varepsilon), f, g)$ with compact support in $[0, T+1)$, then $u(\varepsilon)$, solution to (9), can be viewed as the restriction to $[0,T]$ of the solution $\tilde{u}(\varepsilon)$ to:

$$\begin{cases} \dfrac{\mathrm{d}\tilde{u}(\varepsilon)}{\mathrm{d}t} = A\tilde{u}(\varepsilon) + \tilde{F}(\varepsilon) \quad \text{in } H \\ \tilde{u}(\varepsilon)(0) = u_0(\varepsilon) \end{cases} \tag{10}$$

which does exist by the same arguments. As in [2], one will obtain a convergence result by studying the asymptotic behavior of the Laplace transform of $\tilde{u}(\varepsilon)$. If for any Banach space $X$, $\mathcal{L}z$ denotes the Laplace transform:

$$\mathcal{L}z(p) := \int_0^\infty \exp(-pt) f(t)\,\mathrm{d}t \tag{11}$$

of any function of $L^\infty(0,+\infty;X)$, one has:

$$\begin{cases} \mathcal{L}\tilde{u}(\varepsilon) \in H; \\ \int_\Omega (a+pb)e(\varepsilon, \mathcal{L}\tilde{u}(\varepsilon)) \cdot e(\varepsilon, v)\,\mathrm{d}x = \int_\Omega be(\varepsilon, u_0(\varepsilon)) \cdot e(\varepsilon, v)\,\mathrm{d}x + \int_\Omega \mathcal{L}\tilde{f} \cdot v\,\mathrm{d}x + \int_{\Gamma_N} \mathcal{L}\tilde{g} \cdot v\,\mathrm{d}s, \quad \forall v \in H \end{cases} \tag{12}$$

Similarly to [2], one makes the fundamental assumption of admissibility for the initial state:

$$(\mathbf{H}_3)\colon \quad \begin{cases} \exists (f_0, g_0) \in L^2(\Omega)^3 \times L^2(\Gamma_N)^3) \text{ such that} \\ \int_\Omega ae(\varepsilon, u_0(\varepsilon)) \cdot e(\varepsilon, v)\,\mathrm{d}x = \int_\Omega f_0 \cdot v\,\mathrm{d}x + \int_{\Gamma_N} g_0 \cdot v\,\mathrm{d}s, \quad \forall v \in H \end{cases}$$

Hence (12) can be rewritten:

$$\begin{cases} \theta(\varepsilon) := p\mathcal{L}\tilde{u}(\varepsilon) - u_0(\varepsilon) \in H; \quad \int_\Omega (a/p+b)e(\varepsilon, \theta(\varepsilon)) \cdot e(\varepsilon, v)\,\mathrm{d}x \\ \qquad = \int_\Omega \mathcal{L}\tilde{f} \cdot v\,\mathrm{d}x + \int_{\Gamma_N} \mathcal{L}\tilde{g} \cdot v\,\mathrm{d}s - 1/p(\int_\Omega f_0 \cdot v\,\mathrm{d}x + \int_{\Gamma_N} g_0 \cdot v\,\mathrm{d}s), \quad \forall v \in H \end{cases} \tag{13}$$

so that the study of the asymptotic behavior of $\tilde{u}(\varepsilon)$ reduces to two problems of asymptotic behavior of linearly elastic thin plates with elasticity tensors $a$ and $c(p) := a/p + b$. It is well known (for instance see [1] in the homogeneous isotropic case or [3] in the heterogeneous anisotropic case) that $u_0(\varepsilon), \theta(\varepsilon)$ converge strongly in $H^1(\Omega)^3$ toward $\bar{u}_0, \bar{\theta}$, which solve:

$$\begin{cases} \bar{u}_0 \in V_{KL}(\Omega); \int_\Omega a^{KL} e(\bar{u}_0) \cdot e(v)\,\mathrm{d}x = \int_\Omega f_0 \cdot v \mathrm{d}x + \int_{\Gamma_N} g_0 \cdot v\,\mathrm{d}s, \quad \forall v \in V_{KL}(\Omega) \\ \bar{\theta} \in V_{KL}(\Omega); \quad \int_\Omega c(p)^{KL} e(\bar{\theta}) \cdot e(v)\,\mathrm{d}x \\ \quad = \int_\Omega \mathcal{L}\tilde{f} \cdot v\,\mathrm{d}x + \int_{\Gamma_N} \mathcal{L}\tilde{g} \cdot v\,\mathrm{d}s - 1/p(\int_\Omega f_0 \cdot v\,\mathrm{d}x + \int_{\Gamma_N} g_0 \cdot v\,\mathrm{d}s), \quad \forall v \in V_{KL}(\Omega) \end{cases} \tag{14}$$

where

$$\begin{cases} V_{KL}(\Omega) := \{u \in H^1_{\Gamma_D}(\Omega)^3; e(u)^\perp = 0\} \\ a^{KL} := a_{\wedge\wedge} - a_{\wedge\perp}(a_{\perp\perp})^{-1} a_{\perp\wedge}, c(p)^{KL} := c(p)_{\wedge\wedge} - c(p)_{\wedge\perp}(c(p)_{\perp\perp})^{-1} c(p)_{\perp\wedge} \\ \widehat{(ae)} = a_{\wedge\wedge}\hat{e} + a_{\wedge\perp}e^\perp, (ae)^\perp = a_{\perp\wedge}\hat{e} + a_{\perp\perp}e^\perp, \quad \forall e \in S^3 \\ \widehat{(c(p)e)} = c(p)_{\wedge\wedge}\hat{e} + c(p)_{\wedge\perp}e^\perp, (c(p)e)^\perp = c(p)_{\perp\wedge}\hat{e} + c(p)_{\perp\perp}e^\perp, \quad \forall e \in S^3 \end{cases} \tag{15}$$

The key point concerning the asymptotic behavior of $u(\varepsilon)$ is that $(a/p+b)^{KL}$ does differ from $a^{KL}/p + b^{KL}$, indeed:

$$c(p)^{KL} = a^{KL}/p + b^{KL} + \mathcal{L}K(p) \tag{16}$$

where for all $e$ in $S^3$ one has:

$$\begin{cases} Ke := a_{\wedge\perp} w^e + b_{\wedge\perp}\dot{w}^e \\ w^e \in S^\perp; a_{\perp\perp} w^e + b_{\perp\perp}\dot{w}^e = 0, w^e(0) = (u^b)^\perp - (u^a)^\perp \\ (u^a)^\perp := (a_{\perp\perp})^{-1} a_{\perp\wedge}\hat{e}, (u^b)^\perp := (b_{\perp\perp})^{-1} b_{\perp\wedge}\hat{e} \end{cases} \tag{17}$$

The capital relation (16) is a trivial consequence of the identity:

$$p\mathcal{L}w^e + (u^a)^\perp = (u^{c(p)})^\perp (:= (c(p)_{\perp\perp})^{-1} c(p)_{\perp\wedge}\hat{e}) \tag{18}$$

which stems from the very definitions of $w^e$, $(u^a)^\perp$ and $(u^b)^\perp$. Hence, for all $p$ in $(0, +\infty)$, $\mathcal{L}\tilde{u}(\varepsilon)(p)$ converges strongly in $H^1(\Omega)^3$ toward the unique solution to:

$$\begin{cases} \check{u}(p) \in V_{KL}(\Omega); \int_\Omega (a^{KL} e(\check{u}(p)) + (b^{KL} + \mathcal{L}K(p)) e(p\check{u}(p) - u_0)) \cdot e(v) \, dx \\ \quad = \int_\Omega \mathcal{L}\tilde{f} \cdot v \, dx + \int_{\Gamma_N} \mathcal{L}\tilde{g} \cdot v \, ds, \quad \forall v \in V_{KL}(\Omega) \end{cases} \tag{19}$$

As Laplace transform is one to one and $\tilde{u}(\varepsilon)$ is bounded in $C^1([0, +\infty); H^1(\Omega)^3)$, one has:

**Theorem 2.1.** *Under assumptions* $(\mathbf{H}_1)$–$(\mathbf{H}_3)$*, when* $\varepsilon$ *tends to* 0*, the family* $u(\varepsilon)_{\varepsilon>0}$ *of the unique solution to* (4) *converges weak star in* $W^{1,\infty}(0, T; H^1(\Omega)^3)$ *to the unique solution* $\bar{u}$ *to*:

$$\begin{cases} \bar{u} \in C^1([0, T]; V_{KL}(\Omega)); \quad \bar{u}(., 0) = \bar{u}_0 \\ \int_\Omega \{a^{KL} e(\bar{u}(t)) + b^{KL} e(\dot{\bar{u}}(t)) + \int_0^t K(t - \tau) e(\dot{\bar{u}}(\tau)) \, d\tau\} \cdot e(v) \, dx \\ \quad = \int_\Omega f(x, t) \cdot v(x) \, dx + \int_{\Gamma_N} g(x, t) \cdot v(x) \, ds, \quad \forall (v, t) \in V_{KL}(\Omega) \times [0, T] \end{cases} \tag{20}$$

Thus $\bar{u}$ is solution to an integro-differential equation where, from its very definition, the kernel $K$ has the same symmetries as $a$ and $b$, decreases exponentially fast with time, and vanishes if:

$$(a_{\perp\perp})^{-1} a_{\perp\wedge} = (b_{\perp\perp})^{-1} b_{\perp\wedge} \quad \text{or} \quad a_{\wedge\perp}(a_{\perp\perp})^{-1} = b_{\wedge\perp}(b_{\perp\perp})^{-1} \tag{21}$$

which is the case if $a$ is proportional to $b$. When assumption $(\mathbf{H}_3)$ is missing, an additional term taking into account the asymptotic behavior of the initial state appears in the expression of the scaled stress.

## 3. Concluding remarks

By proceeding to a descaling, $\bar{u}^\varepsilon = S(\varepsilon)^{-1}(\bar{u})$, of the limit field $\bar{u}$, one has that the true physical field of displacement $u^\varepsilon$ is asymptotically equivalent to a Kirchhoff–Love field $\bar{u}^\varepsilon$, element of:

$$V_{KL}(\Omega^\varepsilon) := \{u \in H^1_{\Gamma_D^\varepsilon}(\Omega^\varepsilon)^3; e(u)^\perp = 0\} \tag{22}$$

which satisfies:

$$\begin{cases} \bar{u}^\varepsilon \in C^1([0, T]; V_{KL}(\Omega^\varepsilon)); \bar{u}^\varepsilon(., 0) = \bar{u}_0^\varepsilon := S(\varepsilon)^{-1}(\bar{u}_0) \\ \int_{\Omega^\varepsilon} \{a^{KL}(\hat{x}, x_3/\varepsilon) e(\bar{u}^\varepsilon)(x, t) + b^{KL}(\hat{x}, x_3/\varepsilon) e(\dot{\bar{u}}^\varepsilon)(x, t) \\ \quad + \int_0^t K(\hat{x}, x_3/\varepsilon, t - \tau) e(\dot{\bar{u}}^\varepsilon)(x, \tau) \, d\tau\} \cdot e(v)(x) \, dx \\ \quad = \int_{\Omega^\varepsilon} f^\varepsilon(x, t) \cdot v(x) \, dx + \int_{\Gamma_N^\varepsilon} g^\varepsilon(x, t) \cdot v(x) \, ds, \quad \forall (v, t) \in V_{KL}(\Omega^\varepsilon) \times [0, T] \end{cases} \tag{23}$$

This asymptotic model, which involves a simpler kinematics than the genuine one, is no longer of Kelvin–Voigt type, but with fading memory. This phenomenon is comparable to that observed in homogenization by [4] and [2]; it is due to the fact that the limit processes required to derive the models are not additive with respect to the operators. As in the linearly elastic case, if the elasticity and viscosity tensors are even functions of the transverse coordinate $x_3$, a decoupling between membrane displacement and flexural displacement appears. Eventually, this study can be regarded as a justification of some intuitive model of thin viscoelastic plates (see [5] for instance); a formal derivation by the asymptotic expansion method can be found in [6,7].

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