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Refined theory and decomposed theorem of transversely isotropic thermoporoelastic beam

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ABSTRACT

The refined theory of transversely isotropic beam is analyzed. Based on the transversely isotropic thermoporoelastic theory, a refined theory for bending beam is derived using the general solution and the Lur'e method without ad hoc assumptions. First, the expressions for all of the displacements and stress components of a transversely isotropic thermoporoelastic beam were obtained in terms of four functions with one independent variable. Second, using homogeneous boundary condition, the refined equation and the decomposed form of the thermoporoelastic beam were obtained. Finally, the approximate equations and solutions for the beam under general anti-symmetric loadings were derived from the refined theory.

1. Introduction

The consolidation theory of porosity media was extended from one to three dimensions by Biot [1], and the consolidation theory was improved more perfectly. Huang [2] and others gave the analytical solution of pore water pressure, stress and displacement of the two-dimensional consolidation problem [3–6].

Without ad hoc assumptions, Cheng [7] developed the refined theory for bending of isotropic plates directly from the three-dimensional theory of elasticity by using the solution of the plates and Lur'e method [8]. Under homogeneous boundary conditions, the refined theory of plate is exact and consists of three parts: the bi-harmonic equation, the shear equation and the transcendental equation. A parallel development on the plate theory was constructed by Gregory. In 1992, Gregory [9] provided a rigorous proof of the decomposed form of isotropic plates. The two-plate theory has been extended to the study of various material boards, such as transversely isotropic [10], thermoelastic [11], magnetoelastic [12], and piezoelectric plates [13]. In 2005, the connection between the refined theory and the decomposed theorem of an isotropic elastic plate was analyzed by Zhao and Wang [14]. The equivalence of the refined theory and the decomposed theorem of an isotropic plate were obtained. In 2007, the refined theory of thermoelastic rectangular plates [15] and thermoelastic plane problems [16] were obtained by Gao and Zhao.

In this paper, the research into the refined theory is extended to the study of the transversely isotropic thermoporoelastic beam. In the next section, the basic equations and notations are stated. In Section 3, the decomposed theorem under homogeneous boundary conditions is studied. The decomposed form is consistent with the interior state, the transcendental state, the pore pressure state and the thermal state. In Sections 4, 5, the approximate equations and the solutions for the beam under general anti-symmetric loadings are derived directly from the refined theory.

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2. Equations and notations

A transversely isotropic thermoporoelastic beam occupies the region:

$$\Omega = \{(x,z) \mid x \in D,\ |z| \leqslant t\} \tag{1}$$

where D is the cross-section of the beam, which has thickness $2t$, the z-axis being perpendicular to the isotropic plane of the medium in a Cartesian system (x,z). The constitutive equations for the transversely isotropic body in the two-dimensional linear elasticity are described to be:

$$\begin{aligned}
&\sigma_{xx} = C_{11}\frac{\partial u}{\partial x} + C_{13}\frac{\partial w}{\partial z} - \alpha_1 P - \beta_1 T, \qquad \sigma_{zz} = C_{13}\frac{\partial u}{\partial x} + C_{33}\frac{\partial w}{\partial z} - \alpha_3 P - \beta_3 T\\
&\sigma_{zx} = C_{44}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \qquad P = M\left(\xi - \alpha_1\frac{\partial u}{\partial x} - \alpha_3\frac{\partial w}{\partial z} + \beta_m T\right)
\end{aligned} \tag{2}$$

where σ_{xx}, σ_{zz} are the normal stresses, σ_{zx} is the shear stress, and u and w are displacements in the respective Cartesian directions, P and T are changes in the pore pressure and temperature. ξ is the variation of the fluid content. C_{ij}, α_1 (α_3, M) and β_1 (β_3, β_m) are the elastic moduli, Biot's effective stress coefficients and thermal constants. It is noted that C_{ij}, α_i, β_i can be expressed in terms of engineering contents such as Young's moduli, Poisson's ratio, etc.

The general solution of thermoporoelastic beam has the following expression [17]:

$$u = -\sum_{i=1}^{4}\frac{\partial\psi_i}{\partial x}, \qquad w = \sum_{i=1}^{4}\mu_{i1}\frac{\partial\psi_i}{s_i\partial z}, \qquad P = \mu_{32}\frac{\partial^2\psi_3}{s_3^2\partial z^2}, \qquad T = \mu_{43}\frac{\partial^2\psi_4}{s_4^2\partial z^2} \tag{3}$$

where:

$$\begin{aligned}
&\mu_{i1} = s_i\frac{a_2 s_i^4 - b_2 s_i^2 + c_2}{a_1 s_i^4 - b_1 s_i^2 + c_1} \quad (i = 1,2,3,4), \qquad \mu_{12} = \mu_{22} = \mu_{42} = \mu_{13} = \mu_{23} = \mu_{33} = 0\\
&\mu_{32} = \frac{(a_0 s_3^4 - b_0 s_3^2 + c_0)(\lambda_{33} s_3^2 - \lambda_{11})}{a_1 s_3^4 - b_1 s_3^2 + c_1}, \qquad \mu_{43} = \frac{(a_0 s_4^4 - b_0 s_4^2 + c_0)(\kappa_{33} s_4^2 - \kappa_{11})}{a_1 s_4^4 - b_1 s_4^2 + c_1}
\end{aligned} \tag{4}$$

$$\begin{aligned}
&a_0 = C_{33}C_{44}, \qquad b_0 = C_{11}C_{33} - C_{13}^2 - 2C_{13}C_{44}, \qquad c_0 = C_{11}C_{44}\\
&a_1 = (C_{13} + C_{44})(\kappa_{33}\beta_3 + \lambda_{33}\alpha_3) - C_{33}(\kappa_{33}\beta_1 + \lambda_{33}\alpha_1)\\
&b_1 = (C_{13} + C_{44})(\kappa_{11}\beta_3 + \lambda_{11}\alpha_3) - C_{44}(\kappa_{33}\beta_1 + \lambda_{33}\alpha_1) - C_{33}(\kappa_{11}\beta_1 + \lambda_{11}\alpha_1)\\
&c_1 = -C_{44}(\kappa_{11}\beta_1 + \lambda_{11}\alpha_1), \qquad a_2 = C_{44}(\kappa_{33}\beta_3 + \lambda_{33}\alpha_3)\\
&b_2 = C_{11}(\kappa_{33}\beta_3 + \lambda_{33}\alpha_3) + C_{44}(\kappa_{11}\beta_3 + \lambda_{11}\alpha_3) - (C_{13} + C_{44})(\kappa_{33}\beta_1 + \lambda_{33}\alpha_1)\\
&c_2 = C_{11}(\kappa_{11}\beta_3 + \lambda_{11}\alpha_3) - (C_{13} + C_{44})(\kappa_{11}\beta_1 + \lambda_{11}\alpha_1)
\end{aligned} \tag{5}$$

in which κ_{11} (κ_{33}) and λ_{11} (λ_{33}) are the coefficients of permeability and the thermal conductivity, and where ψ_i ($i = 1,2,3,4$) are the harmonic functions that satisfy the following equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{s_i^2\partial z^2}\right)\psi_i = 0 \quad (i = 1,2,3,4) \tag{6}$$

where $s_3^2 = \kappa_{11}/\kappa_{33}$, $s_4^2 = \lambda_{11}/\lambda_{33}$, s_1^2 and s_2^2 are two roots of the following equation (set $s_1^2 \neq s_2^2$):

$$a_0 s^4 - b_0 s^2 + c_0 = 0 \tag{7}$$

Lekhnitskii [18] proved that the numbers s_1 and s_2 for any transversely isotropic body can be real or complex (with a real part different from zero), but cannot be purely imaginary.

Since the stresses in the bending beam are anti-symmetrical about mid-plane $z = 0$, this induces that u and v are the odd function about z, and w is the even function about z. Using the Lur'e method [8], we have the following solutions of (7):

$$\psi_i = \frac{\sin(z s_i\partial_x)}{s_i\partial_x}g_i(x) \quad (i = 1,2,3,4) \tag{8}$$

in which g_i ($i = 1,2,3,4$) are unknown functions of x, yet to be determined, and $\partial_x = \partial/\partial x$, and:

$$\frac{\sin(zs_i\partial_x)}{s_i\partial_x} = z\left(1 - \frac{1}{3!}s_i^2z^2\partial_x^2 + \frac{1}{5!}s_i^4z^4\partial_x^4 - \cdots\right)$$
$$\cos(zs_i\partial_x) = 1 - \frac{1}{2!}z^2s_i^2\partial_x^2 + \frac{1}{4!}z^4s_i^4\partial_x^4 - \cdots \tag{9}$$

Substituting Eq. (8) into Eqs. (3) and (2), we obtain the displacement field and the stress state:

$$u = -\sum_{i=1}^{4}\frac{\sin(zs_i\partial_x)}{s_i\partial_x}g_i', \qquad w = \sum_{i=1}^{4}\frac{\mu_{i1}}{s_i}\cos(zs_i\partial_x)g_i \tag{10}$$

$$P = -\mu_{32}\frac{\sin(zs_3\partial_x)}{s_3\partial_x}g_3'', \qquad T = -\mu_{43}\frac{\sin(zs_4\partial_x)}{s_4\partial_x}g_4'', \qquad \sigma_{xx} = \sum_{i=1}^{4}s_i^2\omega_i\frac{\sin(zs_i\partial_x)}{s_i\partial_x}g_i''$$
$$\sigma_{zx} = \sum_{i=1}^{4}\omega_i\cos(zs_i\partial_x)g_i', \qquad \sigma_{zz} = -\sum_{i=1}^{4}\omega_i\frac{\sin(zs_i\partial_x)}{s_i\partial_x}g_i'' \tag{11}$$

where:

$$\omega_i = \frac{-C_{11} - C_{13}s_i\mu_{i1} + \alpha_1\mu_{i2} + \beta_1\mu_{i3}}{s_i^2} = C_{44}(\mu_{i1}/s_i - 1)$$
$$= -(-C_{13} - C_{33}s_i\mu_{i1} + \alpha_3\mu_{i2} + \beta_3\mu_{i3}) \quad (i = 1, 2, 3, 4) \tag{12}$$

3. The decomposed theorem under homogeneous boundary condition

Suppose that two faces, $z = \pm t$ are free of tractions. The boundary conditions on the upper and lower surfaces of the beam are:

$$z = \pm t, \qquad \sigma_{zz} = 0, \qquad \sigma_{zx} = 0, \qquad T = 0, \qquad P = 0 \tag{13}$$

Inserting Eq. (13) into Eq. (11), we can gain:

$$\frac{\sin(ts_3\partial_x)}{s_3\partial_x}g_3'' = 0, \qquad \frac{\sin(ts_4\partial_x)}{s_4\partial_x}g_4'' = 0, \qquad \sum_{i=1}^{4}\omega_i\frac{\sin(ts_i\partial_x)}{s_i\partial_x}g_i'' = 0, \qquad \sum_{i=1}^{4}\omega_i\cos(ts_i\partial_x)g_i' = 0 \tag{14}$$

Taking the operator $\frac{\sin(ts_3\partial_x)}{s_3}\frac{\sin(ts_4\partial_x)}{s_4\partial_x}$ on both sides of the fourth equation of (14), one obtains:

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}\begin{pmatrix} g_1' \\ g_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{15}$$
$$L_{1i} = \omega_i\frac{\sin(ts_i\partial_x)}{s_i}, \qquad L_{2i} = \omega_i\frac{\sin(ts_3\partial_x)}{s_3}\frac{\sin(ts_4\partial_x)}{s_4\partial_x}\cos(ts_i\partial_x) \tag{16}$$

According to the Lur'e method [8], the solutions of Eq. (15) have the following form:

$$g_1' = L_{22}\xi_1 - L_{12}\xi_2, \qquad g_2' = -L_{21}\xi_1 + L_{11}\xi_2 \tag{17}$$

and, ξ_i $(i = 1, 2)$ satisfy:

$$\frac{\omega_1\omega_2(s_1^2 - s_2^2)}{2s_1s_2}\Re\partial_x^4\xi_i = 0$$
$$\Re = \frac{1}{\partial_x^2}\frac{\sin(ts_3\partial_x)}{s_3\partial_x}\frac{\sin(ts_4\partial_x)}{s_4\partial_x}\left\{\frac{\sin[(s_1 - s_2)t\partial_x]}{(s_1 - s_2)\partial_x} - \frac{\sin[(s_1 + s_2)t\partial_x]}{(s_1 + s_2)\partial_x}\right\} \tag{18}$$

According to the theorem of the Appendix in Ref. [19], we can obtain $\xi_i = \xi_i^{(1)} + \xi_i^{(2)}$, and:

$$\partial_x^4\xi_i^{(1)} = 0 \tag{19}$$
$$\Re\xi_i^{(2)} = 0 \tag{20}$$

Under homogeneous boundary conditions, the bending general solution of the transversely isotropic thermoporoelastic beam consists of the general solutions of four governing differential equations: the 4-order equation (19), the transcendental equation (20), the pore pressure equation (the first equation of (14)) and the thermal equation (the second equation of (14)).

In the following four sections, we will discuss these four governing differential equations, and four stress states will be obtained.

3.1. 4-Order equation and the interior state

By using Eqs. (19) and (17), and letting:

$$\eta = \frac{\sin(ts_3\partial_x)}{s_3\partial_x}\frac{\sin(ts_4\partial_x)}{s_4\partial_x}\xi_1^{(1)} \tag{21}$$

we can get the interior state:

$$\sigma_{zz}^{(I)} = 0, \qquad \sigma_{xx}^{(I)} = \omega_1\omega_2\left(s_1^2 - s_2^2\right)z\partial_x^2\eta, \qquad \sigma_{zx}^{(I)} = \frac{1}{2}\omega_1\omega_2\left(s_1^2 - s_2^2\right)\left(t^2 - z^2\right)\partial_x^3\eta \tag{22}$$

It is easy to know that η satisfies the 4-order equation $\partial_x^4\eta = 0$.

3.2. Transcendental equation and transcendental states

By using Eqs. (20) and (17), and letting

$$\frac{\phi}{\omega_1\omega_2} = -\frac{\sin(ts_3\partial_x)}{s_3\partial_x}\frac{\sin(ts_4\partial_x)}{s_4\partial_x}\left[\cos(ts_2\partial_x)\frac{\sin(zs_1\partial_x)}{s_1\partial_x} - \cos(ts_1\partial_x)\frac{\sin(zs_2\partial_x)}{s_2\partial_x}\right]\frac{\xi_1^{(2)}}{\partial_x^2} \tag{23}$$

we can get the transcendental state:

$$\sigma_{xx}^{(TR)} = \frac{\partial^4\phi}{\partial x^2\partial z^2}, \qquad \sigma_{zx}^{(TR)} = -\frac{\partial^4\phi}{\partial x^3\partial z}, \qquad \sigma_{zz}^{(TR)} = \frac{\partial^4\phi}{\partial x^4} \tag{24}$$

It is easy to know that ϕ satisfies:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{1}{s_1^2}\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{1}{s_2^2}\frac{\partial^2}{\partial z^2}\right)\phi = 0, \quad \phi = 0, \qquad \partial\phi/\partial z = 0 \quad (z = \pm t) \tag{25}$$

3.3. Pore pressure equation and pore pressure states

Let:

$$g_1 = g_2 = 0, \qquad \frac{\sin(ts_3\partial_x)}{s_3\partial_x}g_3'' = 0, \qquad \varphi_3 = -\frac{\sin(zs_3\partial_x)}{s_3\partial_x}g_3 \tag{26}$$

It is easy to obtain the pore pressure states:

$$P = \mu_{32}\frac{\partial^2\varphi_3}{\partial x^2}, \qquad \sigma_{xx}^{(P)} = \omega_3\frac{\partial^2\varphi_3}{\partial z^2}, \qquad \sigma_{zz}^{(P)} = \omega_3\frac{\partial^2\varphi_3}{\partial x^2}, \qquad \sigma_{zx}^{(P)} = -\omega_3\frac{\partial^2\varphi_3}{\partial x\partial z} \tag{27}$$

It is easy to know that φ_3 satisfies:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{1}{s_3^2}\frac{\partial^2}{\partial z^2}\right)\varphi_3 = 0, \qquad \partial^2\varphi_3/\partial x^2 = 0 \quad (z = \pm t) \tag{28}$$

3.4. Thermal equation and thermal states

Let:

$$g_1 = g_2 = 0, \qquad \frac{\sin(ts_4\partial_x)}{s_4\partial_x}g_4'' = 0, \qquad \varphi_4 = -\frac{\sin(zs_4\partial_x)}{s_4\partial_x}g_4 \tag{29}$$

It is easy to obtain the thermal states:

$$T = \mu_{43}\frac{\partial^2\varphi_3}{\partial x^2}, \qquad \sigma_{xx}^{(T)} = \omega_4\frac{\partial^2\varphi_4}{\partial z^2}, \qquad \sigma_{zz}^{(T)} = \omega_4\frac{\partial^2\varphi_4}{\partial x^2}, \qquad \sigma_{zx}^{(T)} = -\omega_4\frac{\partial^2\varphi_4}{\partial x\partial z} \tag{30}$$

It is easy to know that φ_4 satisfies:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{1}{s_4^2}\frac{\partial^2}{\partial z^2}\right)\varphi_4 = 0, \qquad \partial^2\varphi_4/\partial x^2 = 0 \quad (z = \pm t) \tag{31}$$

3.5. The decomposed theorem

In a homogeneous, transversely isotropic plate occupying the volume Ω, suppose that σ_{ij} satisfy the following conditions:

(1) σ_{ij} in Ω satisfies the equilibrium equations without body force and compatibility equations;
(2) σ_{ij} are anti-symmetrical about mid-plane $z=0$;
(3) $\sigma_{zz}=0$, $\sigma_{zx}=0$, $T=0$, $P=0$ on the faces $z=\pm t$ $(x\in D)$.

Then there exists an interior state $\sigma_{ij}^{(I)}$, a transcendental state $\sigma_{ij}^{(TR)}$, a pore pressure state $\sigma_{ij}^{(P)}$ and a thermal state $\sigma_{ij}^{(T)}$ in Ω such that $\sigma_{ij}=\sigma_{ij}^{(I)}+\sigma_{ij}^{(TR)}+\sigma_{ij}^{(P)}+\sigma_{ij}^{(T)}$.

4. Approximate equations: under temperature loading

In this section, a beam under an anti-symmetric transverse load is considered. The boundary conditions are given as:

$$z=\pm t,\qquad \sigma_{zx}=0,\qquad \sigma_{zz}=\pm q,\qquad P=0,\qquad T=\pm\theta \tag{32}$$

Inserting Eq. (32) into Eq. (11), we can gain:

$$\begin{gathered}\sum_{i=1}^{4}\left[\omega_i\cos(ts_i\partial_x)g_i'\right]=0,\qquad \sum_{i=1}^{4}\left[\omega_i\frac{\sin(ts_i\partial_x)}{s_i\partial_x}g_i''\right]=-q,\qquad \frac{\sin(ts_3\partial_x)}{s_3\partial_x}g_3''=0\\ \frac{\sin(ts_4\partial_x)}{s_4\partial_x}g_4''=-\frac{\theta}{\mu_{43}}\end{gathered} \tag{33}$$

Taking the operator $\frac{\sin(ts_3\partial_x)}{s_3}\frac{\sin(ts_4\partial_x)}{s_4\partial_x}$ on both sides of the first equation of (33), and using the third and fourth equations of (33), one obtains:

$$\begin{pmatrix}L_{11} & L_{12}\\ L_{21} & L_{22}\end{pmatrix}\begin{pmatrix}g_1'\\ g_2'\end{pmatrix}=\begin{pmatrix}h_1\\ h_2\end{pmatrix} \tag{34}$$

where:

$$h_1=\frac{\omega_4\theta}{\mu_{43}}-q,\qquad h_2=\frac{\omega_4\sin(ts_3\partial_x)}{\mu_{43}s_3\partial_x}\cos(ts_4\partial_x)\theta \tag{35}$$

Taking the operator L_{22}/∂_x on both sides of the first equation of (34), and using the second equation of (34), one obtains:

$$\frac{(s_1^2-s_2^2)\omega_1\omega_2}{2s_1s_2}\Re\partial_x^4 g_1=(L_{22}h_1-L_{12}h_2)/\partial_x \tag{36}$$

Taking the operator L_{21}/∂_x on both sides of the first equation of (34), and using the second equation of (34), one obtains:

$$\frac{(s_1^2-s_2^2)\omega_1\omega_2}{2s_1s_2}\Re\partial_x^4 g_2=(-L_{21}h_1+L_{11}h_2)/\partial_x \tag{37}$$

These equations are of infinite order, so they are not applicable in most cases. In the following, we will make certain simplifications to develop an approximate thermoporoelastic beam theory. Using the Taylor series of the trigonometric functions in Eq. (8), then omitting all the terms associated with t^4 and higher orders, we obtain the following results:

$$\begin{aligned}g_1''&=-\frac{3}{\omega_1(s_1^2-s_2^2)t^3\partial_x^2}\left[1+\left(\frac{s_1^2}{10}-\frac{2s_2^2}{5}\right)t^2\partial_x^2\right]q-\frac{\omega_4(s_2^2-s_4^2)}{\mu_{43}\omega_1(s_1^2-s_2^2)t}\left[1+\left(\frac{s_1^2}{10}+\frac{s_4^2}{15}\right)t^2\partial_x^2\right]\theta\\ g_2''&=\frac{3}{\omega_2(s_1^2-s_2^2)t^3\partial_x^2}\left[1+\left(\frac{s_2^2}{10}-\frac{2s_1^2}{5}\right)t^2\partial_x^2\right]q+\frac{\omega_4(s_1^2-s_4^2)}{\mu_{43}\omega_2(s_1^2-s_2^2)t}\left[1+\left(\frac{s_2^2}{10}+\frac{s_4^2}{15}\right)t^2\partial_x^2\right]\theta\\ g_4''&=-\left(1+\frac{1}{6}s_4^2t^2\partial_x^2\right)\frac{\theta}{\mu_{43}t}\end{aligned} \tag{38}$$

Inserting Eq. (38) into Eqs. (10) and (11), then dropping all the terms associated with t^4 and higher orders, we can obtain the displacement field and the stress states:

$$u=\frac{3z}{(s_1^2-s_2^2)\omega_1\omega_2 t^3\partial_x^3}\left[\omega_2-\omega_1+\left(\frac{t^2}{10}-\frac{z^2}{6}\right)(\omega_2 s_1^2-\omega_1 s_2^2)\partial_x^2+\frac{2}{5}(\omega_1 s_1^2-\omega_2 s_2^2)\right]q$$
$$+\frac{z\omega_4}{\mu_{43}(s_1^2-s_2^2)\omega_1\omega_2 t\partial_x}\left\{\left[\omega_2 s_2^2-\omega_1 s_1^2-s_4^2(\omega_2-\omega_1)\right]\left(1+\frac{1}{15}s_4^2t^2\partial_x^2\right)\right.$$
$$\left.+\left[(\omega_2-\omega_1)s_1^2s_2^2-s_4^2(\omega_2 s_1^2-\omega_1 s_2^2)\right]\left(\frac{t^2}{10}-\frac{z^2}{6}\right)\partial_x^2\right\}\theta+\left[1+\frac{1}{6}(t^2-z^2)s_4^2\partial_x^2\right]\frac{z\theta}{\mu_{43}t\partial_x}$$
$$w=\frac{3}{At^3\partial_x^4}\left[B+C\left(\frac{t^2}{10}-\frac{z^2}{2}\right)\partial_x^2-\frac{2}{5}Dt^2\partial_x^2\right]q-\left[1+\left(\frac{t^2}{6}-\frac{z^2}{2}\right)s_4^2\partial_x^2\right]\frac{\mu_{41}\theta}{\mu_{43}s_4t\partial_x^2}$$
$$+\frac{\omega_4}{\mu_{43}At\partial_x^2}\left[(D-Bs_4^2)\left(1+\frac{1}{15}s_4^2t^2\partial_x^2\right)+(s_1^2s_2^2B-s_4^2C)\left(\frac{t^2}{10}-\frac{z^2}{2}\right)\partial_x^2\right]\theta \tag{39}$$

where:

$$A=s_1s_2(s_1^2-s_2^2)\omega_1\omega_2,\qquad B=\mu_{21}\omega_1s_1-\mu_{11}\omega_2s_2$$
$$C=\mu_{21}\omega_1s_1s_2^2-\mu_{11}\omega_2s_2s_1^2,\qquad D=\mu_{21}\omega_1s_1^3-\mu_{11}\omega_2s_2^3 \tag{40}$$
$$\sigma_{xx}=-\left[1+\left(\frac{t^2}{10}-\frac{z^2}{6}\right)(s_1^2+s_2^2)\partial_x^2\right]\frac{3zq}{t^3\partial_x^2}+\left[(s_1^2+s_2^2-1)s_4^2-s_1^2s_2^2\right]\left(\frac{t^2}{10}-\frac{z^2}{6}\right)\partial_x^2\frac{z\omega_4\theta}{\mu_{43}t}$$
$$\sigma_{zx}=\frac{3(z^2-t^2)q}{2t^3\partial_x},\qquad \sigma_{zz}=\frac{(3t^2-z^2)zq}{2t^3} \tag{41}$$

5. Approximate equations: under pore pressure loading

In this section, a beam under anti-symmetric transverse load is considered. The boundary conditions are given as:

$$z=\pm t,\qquad \sigma_{zx}=0,\qquad \sigma_{zz}=\pm q,\qquad P=\pm p_0,\qquad T=0 \tag{42}$$

Inserting Eq. (42) into Eq. (11), we can gain:

$$\sum_{i=1}^{4}\left[\omega_i\cos(ts_i\partial_x)g_i'\right]=0,\qquad \sum_{i=1}^{4}\left[\omega_i\frac{\sin(ts_i\partial_x)}{s_i\partial_x}g_i''\right]=-q,\qquad \frac{\sin(ts_3\partial_x)}{s_3\partial_x}g_3''=-\frac{p_0}{\mu_{32}}$$
$$\frac{\sin(ts_4\partial_x)}{s_4\partial_x}g_4''=0 \tag{43}$$

Taking the operator $\frac{\sin(ts_3\partial_x)}{s_3}\frac{\sin(ts_4\partial_x)}{s_4\partial_x}$ on both sides of the first equation of (43), and using the third and fourth equations of (43), one obtains:

$$\begin{pmatrix}L_{11}&L_{12}\\L_{21}&L_{22}\end{pmatrix}\begin{pmatrix}g_1'\\g_2'\end{pmatrix}=\begin{pmatrix}f_1\\f_2\end{pmatrix} \tag{44}$$

where:

$$f_1=\frac{\omega_3p_0}{\mu_{32}}-q,\qquad f_2=\frac{\omega_3\sin(ts_4\partial_x)}{\mu_{32}s_4\partial_x}\cos(ts_3\partial_x)p_0 \tag{45}$$

Taking the operator L_{22}/∂_x on both sides of the first equation of (44), and using the second equation of (44), one obtains:

$$\frac{(s_1^2-s_2^2)\omega_1\omega_2}{2s_1s_2}\Re\partial_x^4g_1=(L_{22}f_1-L_{12}f_2)/\partial_x \tag{46}$$

Taking the operator L_{21}/∂_x on both sides of the first equation of (44), and using the second equation of (44), one obtains:

$$\frac{(s_1^2-s_2^2)\omega_1\omega_2}{2s_1s_2}\Re\partial_x^4g_2=(-L_{21}f_1+L_{11}f_2)/\partial_x \tag{47}$$

Because these equations are of infinite order, they are not applicable in most cases. In the following, we will try to develop an approximate thermoporoelastic beam theory. Using the Taylor series of the trigonometric functions in Eq. (8), then dropping all the terms associated with t^4 and higher orders, the results turn out to be:

$$g_1'' = -\frac{3}{\omega_1(s_1^2 - s_2^2)t^3\partial_x^2}\left[1 + \left(\frac{s_1^2}{10} - \frac{2s_2^2}{5}\right)t^2\partial_x^2\right]q - \frac{\omega_3(s_2^2 - s_3^2)}{\mu_{32}\omega_1(s_1^2 - s_2^2)t}\left[1 + \left(\frac{s_1^2}{10} + \frac{s_3^2}{15}\right)t^2\partial_x^2\right]p_0$$

$$g_2'' = \frac{3}{\omega_2(s_1^2 - s_2^2)t^3\partial_x^2}\left[1 + \left(\frac{s_2^2}{10} - \frac{2s_1^2}{5}\right)t^2\partial_x^2\right]q + \frac{\omega_3(s_1^2 - s_3^2)}{\mu_{32}\omega_2(s_1^2 - s_2^2)t}\left[1 + \left(\frac{s_2^2}{10} + \frac{s_3^2}{15}\right)t^2\partial_x^2\right]p_0$$

$$g_3'' = -\left(1 + \frac{1}{6}s_3^2t^2\partial_x^2\right)\frac{p_0}{\mu_{32}t} \tag{48}$$

Inserting Eq. (48) into Eqs. (10) and (11), we can obtain the displacement field and the stress states:

$$\begin{aligned} u = {} & \frac{3z}{(s_1^2 - s_2^2)\omega_1\omega_2t^3\partial_x^3}\left[\omega_2 - \omega_1 + \left(\frac{t^2}{10} - \frac{z^2}{6}\right)(\omega_2s_1^2 - \omega_1s_2^2)\partial_x^2 + \frac{2}{5}(\omega_1s_1^2 - \omega_2s_2^2)\right]q \\ & + \frac{z\omega_3}{\mu_{32}(s_1^2 - s_2^2)\omega_1\omega_2t\partial_x}\left\{\left[\omega_2s_2^2 - \omega_1s_1^2 - s_3^2(\omega_2 - \omega_1)\right]\left(1 + \frac{1}{15}s_3^2t^2\partial_x^2\right)\right. \\ & \left. + \left[(\omega_2 - \omega_1)s_1^2s_2^2 - s_3^2(\omega_2s_1^2 - \omega_1s_2^2)\right]\left(\frac{t^2}{10} - \frac{z^2}{6}\right)\partial_x^2\right\}p_0 + \left[1 + \frac{1}{6}(t^2 - z^2)s_3^2\partial_x^2\right]\frac{zp_0}{\mu_{32}t\partial_x} \end{aligned}$$

$$\begin{aligned} w = {} & \frac{3}{At^3\partial_x^4}\left[B + C\left(\frac{t^2}{10} - \frac{z^2}{2}\right)\partial_x^2 - \frac{2}{5}Dt^2\partial_x^2\right]q - \left[1 + \left(\frac{t^2}{6} - \frac{z^2}{2}\right)s_3^2\partial_x^2\right]\frac{\mu_{31}p_0}{\mu_{32}s_3t\partial_x^2} \\ & + \frac{\omega_3}{\mu_{32}At\partial_x^2}\left[(D - Bs_3^2)\left(1 + \frac{1}{15}s_3^2t^2\partial_x^2\right) + (s_1^2s_2^2B - s_3^2C)\left(\frac{t^2}{10} - \frac{z^2}{2}\right)\partial_x^2\right]p_0 \end{aligned} \tag{49}$$

$$\sigma_{xx} = -\left[1 + \left(\frac{t^2}{10} - \frac{z^2}{6}\right)(s_1^2 + s_2^2)\partial_x^2\right]\frac{3zq}{t^3\partial_x^2} + \left[(s_1^2 + s_2^2 - 1)s_3^2 - s_1^2s_2^2\right]\left(\frac{t^2}{10} - \frac{z^2}{6}\right)\partial_x^2\frac{z\omega_3p_0}{\mu_{32}t}$$

$$\sigma_{zx} = \frac{3(z^2 - t^2)q}{2t^3\partial_x}, \qquad \sigma_{zz} = \frac{(3t^2 - z^2)zq}{2t^3} \tag{50}$$

6. Conclusion and discussion

By using the general solution of thermoporoelasticity and the Lur'e method, a refined theory of thermoporoelastic beam is deduced systematically and directly from the linear thermoporoelastic theory without any additional assumptions. The refined theory constructed by Cheng is improved in this paper.

Under homogeneous boundary conditions, a refined equation of the transversely isotropic thermoporoelastic beam is obtained, which is consistent with four governing differential equations: the 4-order equation, the transcendental equation, the pore pressure equation and the thermal equation. Moreover, the decomposed form of a transversely isotropic elasticity beam is given. The interior state, the transcendental state, the pore pressure state, and thermal state can be derived directly from the refined equation.

Under non-homogeneous boundary conditions, the approximate equations and solutions are accurate up to the second-order terms with respect to plate thickness. The refined plate theory can be extended to other well-known elastic and thermoelastic models.

Finally, we discuss the comparison of the presented results with available results of Ref. [20].

Inserting Eq. (55) of [20] into Eq. (16) of Ref. [20], we can obtain:

$$\sigma_z = \frac{z}{2h^3}(3h^2 - 4z^2)q, \qquad \tau_{xz} = \frac{6}{h^3\partial_x}\left(z^2 - \frac{1}{4}h^2\right)q$$

$$\sigma_x = -\frac{12z}{h^3\partial_x^2}\left[1 + \left(\frac{h^2}{40} - \frac{z^2}{6}\right)(s_1^2 + s_2^2)\partial_x^2\right]q \tag{51}$$

The boundary conditions (32) and (41) are transformed into:

$$z = \pm h/2, \qquad \sigma_{zx} = 0, \qquad \sigma_{zz} = \pm q/2, \qquad P = 0, \qquad T = 0 \tag{52}$$

Therefore, the presented results (41) and (50) can be transformed into Eq. (51).

We will next discuss the comparison of the presented interior state (22) with the available results of Ref. [20].

Let:

$$\omega_i = C_{44}(1+k_i) \quad (i=1,2) \tag{53}$$

The interior state (32) of Ref. [20] can be found to be:

$$\sigma_x^{(1)} = \frac{\omega_1\omega_2(s_1^2-s_2^2)}{C_{44}(k_1-k_2)} z\big(w^{(1)}\big)'', \qquad \tau_{xz}^{(1)} = \frac{\omega_1\omega_2(s_1^2-s_2^2)}{2C_{44}(k_1-k_2)}\left(\frac{h^2}{4}-z^2\right)\big(w^{(1)}\big)''', \qquad \sigma_z^{(1)} = 0 \tag{54}$$

So they are the same in form.

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References

[1] M.A. Biot, General theory of three dimensional consolidation, J. Appl. Phys. 12 (1940) 155–164.
[2] C.Z. Huang, Y. Xiao, Analytic solution of a two dimensional consolidation problem, Chin. J. Geotechn. Eng. 18 (1996) 47–54.
[3] J.D. Zheng, H.Z. Hu, H.J. Xu, et al., The application of the finite element method of solving Biot's consolidation equations, Appl. Math. Mech. 3 (1982) 793–804.
[4] X.Y. Li, X.W. Li, Theory of thermo-consolidation for saturated porous elastic media, Acta Mech. Solida Sin. 11 (1990) 330–338.
[5] S.L. Chen, J.M. Zhang, An analysis of axisymmetric consolidation for transversely isotropic saturated soils, Chin. J. Geotechn. Eng. 24 (2002) 25–30.
[6] Y.Y. Hu, The generalized solutions of homogeneous Biot's equations of consolidation, Eng. Mech. 23 (2006) 155–159.
[7] S. Cheng, Elasticity theory of plates and a refined theory, J. Appl. Mech. 46 (1979) 644–650.
[8] A.I. Lur'e, Three-Dimensional Problems of the Theory of Elasticity, Interscience Publishers, New York, 1964.
[9] R.D. Gregory, The general form of the three-dimensional elastic field inside an isotropic plate with free faces, J. Elast. 28 (1992) 1–28.
[10] F.Y. Wang, Two-dimensional theories deduced from three-dimensional theory for a transversely isotropic body—I. Plate problems, Int. J. Solids Struct. 26 (1990) 455–470.
[11] W. Wang, C.H. Luo, A refined theory of thermoelastic plates under steady temperature, Eng. Mech. 17 (2000) 111–118, 141.
[12] Y. Gao, B.S. Zhao, The refined theory for a magnetoelastic body—I. Plate problems, Int. J. Appl. Electromagn. Mech. 29 (2009) 1–14.
[13] S.P. Xu, W. Wang, A refined theory of transversely isotropic piezoelectric plates, Acta Mech. 171 (2004) 15–27.
[14] B.S. Zhao, M.Z. Wang, Equivalence of the refined theory and the decomposed theorem of an elastic plate, Appl. Math. Mech. 26 (2005) 486–494.
[15] Y. Gao, B.S. Zhao, The refined theory of thermoelastic rectangular plates, J. Therm. Stresses 30 (2007) 505–520.
[16] Y. Gao, B.S. Zhao, The refined theory of thermoelastic plane problems, J. Therm. Stresses 30 (2007) 1233–1248.
[17] X.Y. Li, W.Q. Chen, H.Y. Wang, General steady-state solutions for transversely isotropic thermoporoelastic media in three dimensions and its application, Eur. J. Mech. A 29 (2010) 317–326.
[18] S.G. Lekhnitskii, Theory of Elasticity of an Anisotropic Body, Mir, Moscow, 1981.
[19] B.S. Zhao, M.Z. Wang, The transcendental equation in the refined theory of transversely isotropic elasticity plate, Acta Sci. Nat. Univ. Pekin. 40 (2004) 37–45.
[20] Y. Gao, B.X. Xu, B.S. Zhao, The refined theory of beams for a transversely isotropic body, Acta Mech. 191 (2007) 109–122.