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Construction of a bipotential representing a linear non-associated constitutive law

Construction d'un bipotentiel représentant une loi linéaire non associée

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ABSTRACT

We consider a material whose stress–strain relation is linear but not symmetrical. As the research of a potential is futile, we attempt to represent the constitutive law by a bipotential. Fitzpatrick method leads us to construct a suitable increasing sequence of bipotentials. The technique is exemplified on coaxial constitutive laws.

RÉSUMÉ

Nous considérons un matériau dont la relation contraintes–déformations est linéaire, mais pas symétrique. La recherche d'un potentiel étant vaine, nous tentons de représenter la loi de comportement par un bipotentiel. La méthode de Fitzpatrick conduit à construire une suite croissante appropriée de bipotentiels. La technique est illustrée sur l'exemple des lois coaxiales.

1. Introduction

Let us consider a material whose constitutive law connecting the stress tensor y to the strain tensor x is linear: $y = Ax$. If the tensor A (of order 4) is symmetric and positive definite, we say that the material is elastic, and that A is its stiffness tensor. Therefore, the material belongs to the class of Generalized Standard Materials (GSM) [1–3]. It can be characterized [4,5] by the convex potential $\varphi(x) = \frac{1}{2}\,\mathrm{tr}[x(Ax)]$, where tr denotes the trace. The inverse constitutive law is characterized by the convex conjugate potential $\varphi^*(y) = \frac{1}{2}\,\mathrm{tr}[y(A^{-1}y)]$. We can remark that the behaviour of this material is described either by one of the three following laws [3,5]:

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(i) $y = D\varphi(x)$
(ii) $x = D\varphi^*(y)$
(iii) $\varphi(x) + \varphi^*(y) = \mathrm{tr}(xy)$

wherein the symbol D denotes the Fréchet derivative. In a configuration where the tensor A is not symmetric, we cannot determine either a potential or a conjugate. To overcome such a handicap, Géry de Saxcé [6–8] proposed to replace the sum $\phi(x) + \phi^*(y)$ of the two potentials with a biconvex function $b(x, y)$ called bipotential. The behaviour is then described indifferently by one of the three equivalent implicit laws:

(i) $y \in \partial_x b(x, y)$
(ii) $x \in \partial_y b(x, y)$
(iii) $b(x, y) = \mathrm{tr}(xy)$

wherein the symbol ∂_x (respectively ∂_y) denotes the partial subdifferential with respect to x (respectively y). Materials whose constitutive law can be represented by a bipotential are called Implicit Standard Materials (ISM).

The aim of this work is to build a bipotential capable of representing non-symmetric linear laws. In cases where the monotonicity of a law is ensured, convex analysis proposes the construction of a sequence called Fitzpatrick functions [9]. Each one of these functions reveals to be a bipotential within the meaning of Géry de Saxcé. We plan to apply this result in the special case of linear coaxial constitutive laws for which the stress and strain tensors have the same eigenvectors.

2. Fitzpatrick's sequence of a linear law

The construction of the Fitzpatrick sequence for the law $y = Ax$ is possible if the single valued operator A is monotone. In this linear case, the monotonicity condition reduces [9–11] to the positive definiteness of the symmetric part S of A. Agreeing that $y_i = Ax_i$ for $i = 1$ to $n-1$, $x_n = x$, $y_n = y$, $x_{n+1} = x_1$, and $y_{n+1} = y_1$, the nth Fitzpatrick function is defined [9] as:

$$F_{A,n}(x, y) = \mathrm{tr}(xy) + \sup_{(x_1, x_2, \ldots, x_{n-1})} \sum_{i=1}^{n} \mathrm{tr}\big[(x_{i+1} - x_i) y_i\big] \tag{1}$$

Estimation [11] of the above supremum leads to:

$$F_{A,n}(x, y) = \mathrm{tr}(xy) + \frac{1}{4} \mathrm{tr}\big[(y - Ax) H_n^{-1} (y - Ax)\big] \tag{2}$$

where the positive definite symmetric tensor H_n is obtained from the tensor $H_2 = S$ by iterating the recursive formula:

$$H_{k+1} = S - \frac{1}{4} A^T H_k^{-1} A \tag{3}$$

from $k = 2$ to $k = n-1$. Actually, the monotonicity of A allows solely the construction of the function $F_{A,2}(x, y)$ originally proposed by Fitzpatrick [12]. The construction of the nth Fitzpatrick function is only ensured if the operator A satisfies [9] the inequality:

$$\sum_{i=1}^{n} \mathrm{tr}\big[(x_{i+1} - x_i) y_i\big] \leqslant 0 \tag{4}$$

once $y_i = Ax_i$ for $i = 1$ to n, $x_{n+1} = x_1$ and $y_{n+1} = y_1$. Under this condition, we say that the constitutive law $y = Ax$ is n-monotone [9] and also that the material is n-monotone [11]. The 2-monotonicity coincides with the classical monotonicity. The n-monotonicity involves all k-monotonicity for k lower than n. The Fitzpatrick sequence is increasing and bounded from below by the pairing $\mathrm{tr}(xy)$. The lower bound $\mathrm{tr}(xy)$ is achieved when the constitutive law $y = Ax$ is satisfied. Each function of the Fitzpatrick sequence is therefore a bipotential. When the tensor A is symmetric and positive definite (as in linear elasticity), all the n-monotonicity conditions are satisfied (A is referred to as cyclically monotone [5,9]) and the Fitzpatrick sequence admits [9] a pointwise limit:

$$F_{A,\infty}(x, y) = \phi(x) + \phi^*(y) \tag{5}$$

which is the sum of potential and conjugate.

3. Bidimensional coaxial linear laws

The coaxial linear laws for which the stress tensor and the strain tensor have the same eigenvectors read:

$$y = \mathrm{tr}(qx)e + 2\mu x \tag{6}$$

where q is a symmetrical tensor (of order 2) and μ is a scalar (e denotes the identity tensor as in what follows). We split the tensor q in spherical and deviatoric parts:

$$q = \lambda e + h \tag{7}$$

where $\lambda = \frac{1}{2}\operatorname{tr} q$ and h is a deviatoric tensor ($\operatorname{tr} h = 0$). With this decomposition, the coaxial law becomes:

$$y = \lambda(\operatorname{tr} x)e + 2\mu x + \operatorname{tr}(hx)e \tag{8}$$

If the tensor q is spherical ($h = 0$), this law reduces to the classical isotropic Hooke's law. Note that $\operatorname{tr}(hx)$ involves only the deviatoric part x_d of x:

$$y = (\lambda + \mu)(\operatorname{tr} x)e + 2\mu x_d + \operatorname{tr}(hx_d)e \tag{9}$$

As it is well known, the strict monotonicity of the 2D Hooke law is ensured by the two inequality conditions:

$$\mu > 0 \quad \text{and} \quad \lambda + \mu > 0 \tag{10}$$

The monotonicity of the coaxial law demands (see Section 4.2 below) the additional condition:

$$\operatorname{tr}(h^2) \leqslant 8(\lambda + \mu)\mu \tag{11}$$

The isotropic Hooke law is described by the classical potential:

$$\varphi(x) = \frac{1}{2}\lambda(\operatorname{tr} x)^2 + \mu \operatorname{tr}(x^2) \tag{12}$$

The coaxial law does not admit a potential. The obstruction is the term $\operatorname{tr}(hx_d)e$ in (9), which introduces a lack of symmetry of the tensor A.

4. Fitzpatrick's sequence of a coaxial law

4.1. Choice of a basis in the strain linear space

The 2D strain tensors can be regarded as elements of the 3D linear space E of 2×2 symmetrical matrices. In this space, we choose an orthonormal basis $(d, \frac{h}{\|h\|}, \frac{e}{\sqrt{2}})$ constituted of a unitary deviatoric tensor d orthogonal to h ($\operatorname{tr}(hd) = 0$ and $\operatorname{tr}(d^2) = 1$), the deviator h normalized to 1 (dividing by $\|h\| = [\operatorname{tr}(h^2)]^{\frac{1}{2}}$), and the identity tensor e also normalized to 1 (dividing by $\sqrt{2}$). By duality, we also identify the space of stress tensors to E. In the above chosen basis, the tensor A is identified to the 3×3 matrix:

$$A = \begin{pmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & \sqrt{2}\|h\| & 2(\lambda + \mu) \end{pmatrix} \tag{13}$$

where the lack of symmetry, due to the non-nullity of h, appears clearly. In the appropriate basis, the matrix A takes the shape of a block matrix (two null blocks, a scalar block, and a lower triangular block):

$$A = \begin{pmatrix} 2\mu & 0 \\ 0 & a \end{pmatrix} \tag{14}$$

The lower 2×2 triangular block:

$$a = \begin{pmatrix} 2\mu & 0 \\ \sqrt{2}\|h\| & 2(\lambda + \mu) \end{pmatrix} \tag{15}$$

is not symmetrical. Its symmetric part is:

$$s = \begin{pmatrix} 2\mu & \frac{\sqrt{2}}{2}\|h\| \\ \frac{\sqrt{2}}{2}\|h\| & 2(\lambda + \mu) \end{pmatrix} \tag{16}$$

4.2. Monotonicity of a coaxial law

According to the previous notations, the symmetrical part S of A is the block matrix:

$$S=\begin{pmatrix}2\mu & 0\\ 0 & s\end{pmatrix}\tag{17}$$

This 3×3 matrix is positive if the scalar μ and the 2×2 matrix s are positive. The latter condition requires both positivity of the scalar $\lambda+\mu$ (as for Hooke's Law) and of the determinant of s:

$$\det s=4\mu(\lambda+\mu)-\frac{1}{2}\|h\|^2\tag{18}$$

as announced before (Section 3).

4.3. Strict *n*-monotonicity of a coaxial law

The skew symmetric part of a 2×2 matrix is necessarily proportional to the matrix:

$$J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\tag{19}$$

Thus, the matrix a splits up into $a=s+\frac{\sqrt{2}}{2}\|h\|J$ and its transpose into $a^{\mathrm{T}}=s-\frac{\sqrt{2}}{2}\|h\|J$. Note that the symmetrical matrix $a^{\mathrm{T}}s^{-1}a$ is proportional to s:

$$a^{\mathrm{T}}s^{-1}a=\left(1+\frac{\|h\|^2}{2\det s}\right)s\tag{20}$$

The strict 2-monotonicity being prior to all others, necessarily the diagonal of A has to be non-negative:

$$\lambda+\mu>0\quad\text{and}\quad\mu>0\tag{21}$$

The block structure already noticed (14) of the matrix A reduces the study of its n-monotonicity to that of the matrix a. In the case of a 2×2 matrix as a, the n-monotony is governed by the ratio between the multiplicative factor of J in the skew symmetric part s and the square root of the determinant (necessarily non-negative) of the symmetric part s (necessarily positive definite). More precisely, we define an angle θ included between 0 and $\frac{\pi}{2}$ by the equality:

$$\frac{\sqrt{2}}{2}\|h\|=\sqrt{\det s}\tan\theta\tag{22}$$

Then the n-monotonicity condition is [11]:

$$n\theta\leqslant\pi\tag{23}$$

With the notation (22), property (20) can be rewritten as:

$$a^{\mathrm{T}}s^{-1}a=\left(1+\tan^2\theta\right)s=\frac{1}{\cos^2\theta}s\tag{24}$$

4.4. Determination of the matrix sequence H_k

The matrix $H_2=S$ having the block structure (17), the relation (24) being satisfied, we solve the recurrence relation (3) by conjecturing the following block structure for the matrices H_k:

$$H_k=\begin{pmatrix}2\mu\alpha_k & 0\\ 0 & \frac{1}{2}\beta_k s\end{pmatrix}\tag{25}$$

The two introduced numerical sequences α_k and β_k have initial values $\alpha_2=1$ and $\beta_2=2$. Relation (3) splits into 2 homographic recurrences:

$$\alpha_{k+1}=1-\frac{1}{4\alpha_k}\quad\text{and}\quad\beta_{k+1}=2-\frac{1}{\cos^2\theta\beta_k}\tag{26}$$

The solutions are [11]: $\alpha_k=\frac{1}{2}\frac{k}{k-1}$ and $\beta_k=\frac{\sin(k\theta)}{\sin((k-1)\theta)}\frac{1}{\cos\theta}$. Introducing the variable $X=\cos\theta$ and the Chebyshev polynomials of second kind $U_k(X)=\frac{\sin(k+1)\theta}{\sin\theta}$, we can express β_k as a function of X:

$$\beta_k=\frac{1}{X}\frac{U_{k-1}(X)}{U_{k-2}(X)}\tag{27}$$

4.5. nth Fitzpatrick function of a coaxial law

The general expression (2) of the nth Fitzpatrick function of a linear law specializes with:

$$H_n = \begin{pmatrix} \frac{n}{n-1}\mu & 0 \\ 0 & \frac{\sin(n\theta)}{2\cos\theta\sin((n-1)\theta)} s \end{pmatrix} \tag{28}$$

whose inverse is:

$$H_n^{-1} = \begin{pmatrix} \frac{n-1}{n}\frac{1}{\mu} & 0 \\ 0 & -\frac{\sin((n-1)\theta)}{2\cos\theta\sin(n\theta)}\frac{1}{\mu(\lambda+\mu)} J s J \end{pmatrix} \tag{29}$$

5. Bipotential of monotonic coaxial law

From $k = 2$ to $k = n$, any functions $F_{A,k}(x, y)$ of the finite Fitzpatrick sequence represent the n-monotone coaxial law. Guided by the cyclically monotone case (end of Section 2), we propose to select as best bipotential the largest one: $b(x, y) = F_{A,k}(x, y)$.

6. Discussion

Coaxial laws are non-associated constitutive laws [8,10,11,13–15]. They respect all the principles originally enacted by Robert Hooke to model the behaviour of materials called 'elastic': linearity, monotonicity and coaxiality (neither isotropy, nor existence of a potential are required). We propose to revisit the Hooke modelling by taking into account the four characteristic parameters: λ, μ and two independent coefficients of the deviator h. Displacement and stress fields will be obtained as extrema of a bifunctional [7] generated by the bipotential defined in Section 5. The numerical implementation will be based on Uzawa-type algorithms.

7. Conclusion

In two dimensions, after developing the matrix of a linear n-monotone coaxial law in a suitable orthonormal basis (involving a deviator entering in the constitutive law), we were able to exhibit a bipotential by applying the Fitzpatrick method issued from convex analysis. The analysis of the relationship between the bipotential of Géry de Saxcé representing the non-associated constitutive laws and the Fitzpatrick functions representing maximal monotone operators proved to be relevant. Therefore, we recommend the application of this method to find the best bipotential representing the behaviour of a given implicit standard material.

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