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Inertial motions of a rigid body with a cavity filled with a viscous liquid

Mouvement inertiel d'un corps rigide avec une cavité remplie d'un fluide visqueux

Giovanni P. Galdi, Giusy Mazzone *, Paolo Zunino

Department of Mechanical Engineering and Materials Science, University of Pittsburgh, USA

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ABSTRACT

In this note we announce a number of analytical and numerical results related to the motion of a system $\mathcal{S}$ constituted by a rigid body with a cavity that is completely filled with a Navier–Stokes liquid, and that moves in absence of external forces (inertial motions). Our investigation shows, in particular, that the ultimate motion of $\mathcal{S}$ about its center of mass is a permanent rotation, thus proving a longstanding conjecture of N.Ye. Zhukovskii. We also present other interesting features of inertial motions that are emphasized by our numerical tests, but that still lack a rigorous mathematical proof.

RÉSUMÉ

Dans cette note, nous décrivons, par une analyse mathématique rigoureuse, le mouvement d'un système $\mathcal{S}$ constitué d'un corps rigide contenant une cavité remplie d'un fluide. Les résultats ont été confirmés par des simulations numériques. Le système $\mathcal{S}$ n'étant soumis à aucune force extérieure et le fluide y étant contenu répondant aux équations de Navier–Stokes, nous démontrons que le mouvement de ce corps rigide est inertiel. Notre analyse montre que le mouvement du système $\mathcal{S}$ par rapport à son centre de masse est une rotation constante, ce qui démontre totalement la conjecture de N.Ye. Zhukovskii. D'autres propriétés du système sont aussi examinées et décrites dans cet article grâce àux résultats de simulations numériques.

Version française abrégée

On considère le déplacement d'un corps $\mathcal{B}$ (rigide ou déformable) en l'absence de *forces extérieures*. Ce type de mouvement est appelé mouvement inertiel et la dynamique de $\mathcal{B}$ est determinée par le mouvement linéaire uniforme de son centre de masse G. Si $\mathcal{B}$ est un corps rigide, son déplacement est gouverné par les équations d'Euler. Ce système d'équations admet des solutions stationnaires, qui décrivent des rotations permanentes autour d'axes privilégiés, donnés par les vecteurs propres du tenseur d'inertie $\boldsymbol{I}$ de $\mathcal{B}$ en G. En général, dans le cas d'un corps rigide, la classe des solutions admissibles est appelée *mouvements à la Poinsot* ; voir par example [1, Sects. I.8–10]. On suppose maintenant qu'une cavité $\mathcal{C}$ a été creusée

* Corresponding author.
E-mail addresses: galdi@pitt.edu (G.P. Galdi), gim20@pitt.edu (G. Mazzone), paz13@pitt.edu (P. Zunino).

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dans $\mathcal{B}$ et a été remplie d'un fluide *visqueux* incompressible $\mathcal{L}$. La dynamique du système couplé $\mathcal{S} \equiv \mathcal{B} \cup \mathcal{L}$ autour de son centre de masse $\hat{G}$ montre des propriétés inattendues pour des temps suffisamment longs. Déjà, en 1885, N.Ye. Zhukovskii avait formulé la conjecture suivante : pourvu que $\mathcal{L}$ obéisse aux équations de Navier–Stokes, *le mouvement de* $\mathcal{S}$ *autour de* $\hat{G}$ *tend vers un mouvement rigide, c'est-à-dire une rotation permanente, quelle que ce soit la taille et la forme de* $\mathcal{C}$*, la viscosité de* $\mathcal{L}$ *et le mouvement intial de* $\mathcal{S}$ [2, p. 152]. Cette conjecture peut être formellement justifiée par les observations suivantes : à cause de la viscosité du fluide, la vitesse de $\mathcal{L}$ relativement à $\mathcal{B}$ doit tendre vers zéro, de façon telle que $\mathcal{S}$ tourne en permanence. Dans ces conditions, le gradient de la pression de $\mathcal{L}$ équilibre les forces centrifuges : $\dot{\boldsymbol{\omega}} \times \boldsymbol{x} + \boldsymbol{\omega} \times \boldsymbol{x} \times \boldsymbol{x} = \nabla p$, où $\boldsymbol{\omega}$ est une vitesse angulaire et p une pression. En appliquant le rotationnel à chaque terme, cela donne $\dot{\boldsymbol{\omega}} = \mathbf{0}$. Motivés par la conjecure de N.Ye. Zhukovskii, nous étudions, par des outils analytiques et numériques, le mouvement d'un corps rigide avec une cavité remplie d'un fluide. Le but de cet article est d'annoncer les résultats principaux de cette étude. Tous les détails, preuves et stratégies d'analyse seront présentés dans un publication en préparation. On suppose que $\mathcal{C}$ est un sous-ensemble ouvert et connexe de $\mathbb{R}^3$. Soit $\mathcal{F}$ un système de coordonnées avec son origine au centre de masse de $\mathcal{S} := \mathcal{B} \cup \mathcal{L}$ et les axes orientés comme les vecteurs propres du tenseur d'inertie $\boldsymbol{I}$ de $\mathcal{S}$. En plus, $\boldsymbol{V} = \boldsymbol{V}(\boldsymbol{x}, t)$ et $\mathsf{p} = \mathsf{p}(\boldsymbol{x}, t)$ représentent la vitesse et la pression du fluide $\mathcal{S}$ de densité ρ et de viscosité cinématique ν. Enfin, $\boldsymbol{\omega}$ est la vitesse angulaire de $\mathcal{B}$ par rapport à $\mathcal{F}$. Donc, les mouvements inertiels de $\mathcal{S}$ autour de son centre de masse sont donnés par le système d'équations (1), où $p = \mathsf{p} + \frac{1}{2}(\boldsymbol{\omega} \times \boldsymbol{x})^2$ (voir, par exemple [3]). Notre objectif est d'étudier le comportement asymptotique des solutions de l'équation (1). Le résultat principal est le suivant :

Théoremè 0.1. *Soit* $\mathcal{B}$ *un corps rigide avec une cavité* C *de classe* C^2 *et soit* $\nu > 0$. *Les conditions initiales sont telles que* :

$$\boldsymbol{V}_0 \in L^2_\sigma(C), \qquad \boldsymbol{\omega}_0 \in \mathbb{R}^3 \quad (\textit{énergie initiale finie})$$

Soit $(\boldsymbol{V}, \boldsymbol{\omega}_\infty)$ *une solution faible de* (1) *au sens de Leray–Hopf avec état initial donné. Alors, les propriétés suivantes sont verifiées* :

- $\lim_{t\to\infty} \|\boldsymbol{V}(t)\|_{H^1_0} = 0$;
- *il existe* $\bar{\boldsymbol{\omega}} = \bar{\boldsymbol{\omega}}(\boldsymbol{V}_0, \boldsymbol{\omega}_0) \in \mathbb{R}^3 - \{\mathbf{0}\}$, *tel que* :

$$\lim_{t\to\infty} \left|\boldsymbol{\omega}(t) - \bar{\boldsymbol{\omega}}\right| = 0$$

- *le vecteur* $\bar{\boldsymbol{\omega}}$ *est un vecteur propre du tenseur d'inertie* $\boldsymbol{I}$;
- $\bar{\boldsymbol{\omega}}$ *est orienté comme le moment cinétique (constant)* $\boldsymbol{K}_G$, *et on a* :

$$\bar{\boldsymbol{\omega}} = \lambda^{-1}\boldsymbol{K}_G$$

où λ *est une valeur propre de* $\boldsymbol{I}$.

Nous allons présenter ici des propriétés du système $\mathcal{S}$ mises en évidence par nos simulations numériques : quelques-unes sont enoncées aussi dans le Théorème 0.1, tandis que les autres n'ont pas encore été démontrées :

(i) pour tous les états initiaux, le mouvement asymptotique de $\mathcal{S}$ est une rotation permanente. Les simulations numériques montrent que l'axe de rotation est le même axe de rotation stable pour $\mathcal{S}$ vu comme un corps rigide, voir Fig. 2 ;
(ii) la dynamique du système $\mathcal{S}$ est très sensible aux changements de viscosité ν. Précisement, pour des petites valeurs de ν, le temps nécessaire à $\mathcal{S}$ pour atteindre la rotation permanente devient plus grand. Cette propriété est mise en évidence dans la Fig. 2 (à droite) ;
(iii) pour des petites valeurs de ν, des phénomènes de type « flip-over » peuvent avoir lieu. En gardant tous les autres paramètres du système fixés, quand la valeur de ν passe en dessous d'un seuil, le système tourne à l'envers, et donc la rotation permanente du système change de signe, afin que le moment cinétique total soit préservé. On observe cette propriété en comparant les images à gauche et au centre de la Fig. 2.

1. Introduction

Consider a body $\mathcal{B}$ (rigid or deformable) freely moving in the physical space. The motion of $\mathcal{B}$ is then called inertial, if there are no *external* forces acting on $\mathcal{B}$. In such a case, the center of mass G of $\mathcal{B}$ will move by uniform and rectilinear motion, so that the relevant dynamics of $\mathcal{B}$ is reduced to the study of its motion about G.

As is well known, if the body $\mathcal{B}$ is *rigid*, inertial motions about the center of mass are governed by the Euler equations that express conservation of $\boldsymbol{K}_G$ — the total angular momentum with respect to G, in the central frame of inertia. The latter is characterized by having its origin in G and its axes parallel to three orthogonal eigenvectors $\boldsymbol{e}_i$, $i = 1, 2, 3$, of the inertial tensor $\boldsymbol{I}$ of $\mathcal{B}$ evaluated with respect to G. An important feature is that time-independent motions (*permanent rotations*) may occur if and only if the (constant) angular velocity $\boldsymbol{\omega}$ of $\mathcal{B}$ is directed along one of the $\boldsymbol{e}_i$. On the other hand, time-dependent motions may be very complicated depending on the "symmetry" of the distribution of mass of $\mathcal{B}$ that, in mathematical terms, is defined by the properties of the eigenvalues, A, B, and C, of $\boldsymbol{I}$, corresponding to $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$, respectively. For example, if $A \neq B = C$, then the most general motion of $\mathcal{B}$ about G is a *regular precession*, where $\mathcal{B}$ rotates

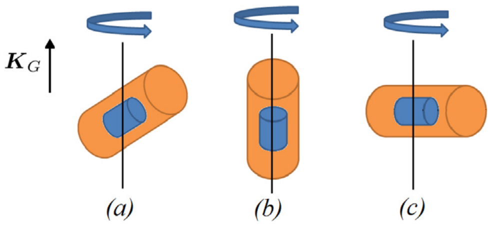

Fig. 1. $\mathcal{S}$ is a cylindrical shell (orange) with the cavity filled with viscous liquid (blue). (a) Configuration of $\mathcal{S}$ at time $t = 0$. At time $t = \infty$, $\mathcal{S}$ will perform a rigid rotation either as in (b) or in (c). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

uniformly around the axis $\boldsymbol{e}_1$, while the latter rotates uniformly around the direction of $\boldsymbol{K}_G$. However, if $A \neq B \neq C$, the generic motion can be quite involved and falls in the category of the so-called *motions à la Poinsot*; see, e.g., [1, Sects. I.8–10].

Suppose now that in the body $\mathcal{B}$ we perform a hollow cavity $\mathcal{C}$, and completely fill it up with a *viscous* liquid $\mathcal{L}$. Then, the motions of the *coupled* system $\mathcal{S} \equiv \mathcal{B} \cup \mathcal{L}$ around its center of mass, $\hat{G}$, are expected to show at large times an altogether special behavior. Actually, already in 1885, N.Ye. Zhukovskii formulated a conjecture according to which, under the assumption that $\mathcal{L}$ is a Navier–Stokes liquid, the *motions of $\mathcal{S}$ about $\hat{G}$ will eventually* (*as time goes to infinity*) *be rigid motions and, precisely, permanent rotations, no matter the size and shape of $\mathcal{C}$, the viscosity of $\mathcal{L}$, and the initial movement of $\mathcal{S}$* [2, p. 152]. This intriguing statement can be formally supported by the following simple argument. Due to viscous effects, the velocity of $\mathcal{L}$ relative to $\mathcal{B}$ must eventually vanish, so that $\mathcal{S}$ will eventually move by rigid motion. In such a state, the pressure gradient in $\mathcal{L}$ must balance centrifugal forces, namely, $\dot{\boldsymbol{\omega}} \times \boldsymbol{x} + \boldsymbol{\omega} \times \boldsymbol{x} \times \boldsymbol{x} = \nabla p$, where $\boldsymbol{\omega}$ is angular velocity and p is pressure, which after taking the curl of both sides, implies $\dot{\boldsymbol{\omega}} = \boldsymbol{0}$.

Also motivated by the above conjecture, we have investigated the motions of a rigid body with a cavity filled by a viscous liquid about its center of mass, from both numerical and analytical viewpoints. In fact, the aim of this note is just to announce our main findings, leaving the associated proofs and strategies as object of a forthcoming paper.

More specifically, our analytical results (supported by appropriate numerical tests) show that under the assumption that $\mathcal{C}$ is of class C^2, every motion of $\mathcal{S}$ that initially has a finite kinetic energy, and at each successive instant of time possesses as little regularity as to belong to a very general class of weak solutions, must become as time goes to infinity a constant rotation. This rotation is directed along one of the eigenvectors $\hat{\boldsymbol{e}}_i$ of the inertia tensor of the *whole* system $\mathcal{S}$, $\hat{\boldsymbol{I}}$, which must be in turn aligned with the (constant) total angular momentum $\boldsymbol{K}_G$. The case when $\mathcal{B}$ is a cylinder-like shell with constant density, is sketched in Fig. 1.

Unless the eigenvalues of the inertia tensor of $\mathcal{S}$ are all equal, our mathematical analysis leaves out the outstanding problem of which among the axes $\hat{\boldsymbol{e}}_i$, around which the "final" rotation occurs, will eventually align with $\boldsymbol{K}_G$. An important and, by and large, reasonable hint in this direction comes from our numerical simulation that shows that the specific axis is one along which permanent rotations of $\mathcal{S}$ (as a whole rigid body) are stable. Thus, for example, in the case of the cylindrical shell of Fig. 1, the configuration (b) is the one eventually attained. Our numerical investigation provides further important information about the "final" state s_∞ that should be ultimately backed up by mathematical analysis. Among others, particularly significant is the study of the effect of the viscosity, ν, on the attainability of s_∞. In fact, on the one hand, decreasing ν is seen to generate a more complicated dynamics on finite times, and consequently a delay, as expected, in the achievement of the permanent rotation. On the other hand, numerical tests also show that reducing ν may generate, in certain cases, a flip-over of the axis of the "final" permanent rotation. Deeper and more detailed investigation of these aspects will be the object of future work.

2. Formulation of the problem

We denote by $\mathcal{C}$ the cavity contained in the body $\mathcal{B}$ and completely filled with a Navier–Stokes liquid $\mathcal{L}$. We assume that $\mathcal{C}$ is an open and connected bounded set of $\mathbb{R}^3$. Moreover, let $\mathcal{F}$ be the frame with the origin in the center of mass of the coupled system $\mathcal{S} := \mathcal{B} \cup \mathcal{L}$, and axes directed along three orthogonal eigenvectors of the inertia tensor $\boldsymbol{I}$ of $\mathcal{S}$ with respect to its center of mass. Further, we denote by $\boldsymbol{V} = \boldsymbol{V}(\boldsymbol{x}, t)$ and $\mathsf{p} = \mathsf{p}(\boldsymbol{x}, t)$ velocity and pressure field, respectively, of $\mathcal{L}$ referred to $\mathcal{F}$, and by ρ and ν its density and (positive) coefficient of kinematic viscosity. Finally, $\boldsymbol{\omega}$ denotes the angular velocity of $\mathcal{B}$, again referred to $\mathcal{F}$.

The inertial motions of $\mathcal{S}$ around its center of mass are then governed by the following set of equations where $p = \mathsf{p} + \frac{1}{2}(\boldsymbol{\omega} \times \boldsymbol{x})^2$ (see, e.g., [3]):

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{V}}{\partial t} + \dot{\boldsymbol{\omega}} \times \boldsymbol{x} + 2\boldsymbol{\omega} \times \boldsymbol{V} + \boldsymbol{V} \cdot \nabla \boldsymbol{V} + \nabla p - \nu \Delta \boldsymbol{V} = 0 \\ &\operatorname{div} \boldsymbol{V} = 0 \end{aligned}\right\} \quad \text{in } \mathcal{C} \times (0, \infty)$$

$$\dot{\boldsymbol{A}} + \boldsymbol{\omega} \times \boldsymbol{A} = \boldsymbol{0}, \quad \boldsymbol{A} := \rho \int_{\mathcal{C}} \boldsymbol{x} \times \boldsymbol{V} \, \mathrm{d}V + \boldsymbol{I} \cdot \boldsymbol{\omega}, \quad \text{in } (0, \infty) \tag{1}$$

The first two equations in (1) are the Navier–Stokes equations in the rotating frame $\mathcal{F}$, and represent the "dissipative" component of $\mathcal{S}$, while the third equation is the "conservative" component and describes the conservation of the total angular momentum of $\mathcal{S}$, again with respect to $\mathcal{F}$.

With the idea in mind that $\mathcal{S}$ will eventually move by rigid motion, it is convenient to introduce a new variable, the *"final" angular velocity*, as follows:

$$\boldsymbol{\omega}_\infty := \boldsymbol{I}^{-1} \cdot \boldsymbol{A}(t) \tag{2}$$

Then, (1) can be equivalently rewritten in the following form:

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{V}}{\partial t} + (\dot{\omega}_\infty + \dot{\boldsymbol{a}}) \times \boldsymbol{x} + 2(\boldsymbol{\omega} + \boldsymbol{a}) \times \boldsymbol{V} + \boldsymbol{V} \cdot \nabla \boldsymbol{V} + \nabla p - \nu \Delta \boldsymbol{V} = 0 \\ &\operatorname{div} \boldsymbol{V} = 0 \end{aligned}\right\} \quad \text{in } \mathcal{C} \times (0, \infty)$$
$$\boldsymbol{I} \cdot \dot{\boldsymbol{\omega}}_\infty + (\boldsymbol{\omega}_\infty + \boldsymbol{a}) \times \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty = \boldsymbol{0}, \quad \text{in } (0, \infty) \tag{3}$$

where

$$\boldsymbol{a} := -\rho \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \boldsymbol{x} \times \boldsymbol{V} \right)$$

We endow (3) with the following no-slip boundary condition:

$$\boldsymbol{V} = \boldsymbol{0}, \quad \text{on } \partial \mathcal{C} \tag{4}$$

and initial conditions

$$\boldsymbol{V}(\boldsymbol{x}, 0) = \boldsymbol{V}_0(\boldsymbol{x}), \qquad \boldsymbol{\omega}_\infty(0) = \boldsymbol{\omega}_{\infty 0} \tag{5}$$

where $\boldsymbol{V}_0(\boldsymbol{x})$ and $\boldsymbol{\omega}_{\infty 0}$ are prescribed quantities.

Our objective is to investigate the asymptotic behavior of solutions to the problem (3)–(5).

3. Analytical results

The asymptotic behavior is analyzed in the class of *weak solutions* à la Leray–Hopf, that we are going to define next.[1]

Definition 3.1. We say that $(\boldsymbol{V}, \boldsymbol{\omega}_\infty)$ is a weak solution to problem (3)–(5) if:

(i) $\boldsymbol{V} \in C_w(0, \infty; L^2_\sigma(\mathcal{C})) \cap L^\infty(0, \infty; L^2_\sigma(\mathcal{C})) \cap L^2(0, \infty; H^1_0(\mathcal{C}))$;
(ii) $\boldsymbol{\omega}_\infty \in W^{1,\infty}(0, \infty)$;
(iii) $\|\boldsymbol{V}(t) - \boldsymbol{V}_0\|_2 \to 0$, $|\boldsymbol{\omega}_\infty(t) - \boldsymbol{\omega}_0| \to 0$, as $t \to 0$;
(iv) $(\boldsymbol{V}, \boldsymbol{\omega}_\infty)$ satisfies the *Strong Energy Inequality*:

$$\begin{aligned} &\|\boldsymbol{V}(t) - \boldsymbol{a}(t) \times \boldsymbol{x}\|_2^2 + \boldsymbol{a}(t) \cdot \boldsymbol{I} \cdot \boldsymbol{a}(t) + \boldsymbol{\omega}_\infty(t) \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty(t) + 2\nu \int_s^t \|\nabla \boldsymbol{V}(\tau)\|_2^2 \\ &\quad \leqslant \|\boldsymbol{V}(s) - \boldsymbol{a}(s) \times \boldsymbol{x}\|_2^2 + \boldsymbol{a}(s) \cdot \boldsymbol{I} \cdot \boldsymbol{a}(s) + \boldsymbol{\omega}_\infty(s) \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty(s) \end{aligned}$$

for all $t \geqslant s \geqslant 0$ and a.a. $s \geqslant 0$, including $s = 0$;
(v) $(\boldsymbol{V}, \boldsymbol{\omega}_\infty)$ satisfies (3)–(5) in the sense of distributions.

We can prove the following.

Lemma 3.2. *For any initial data $\boldsymbol{V}_0 \in L^2_\sigma(\mathcal{C})$, $\boldsymbol{\omega}_0 \in \mathbb{R}^3$, there exists at least one weak solution.*

Our main result is stated next.

[1] Our notation is standard. In particular, L^2_σ is the subspace of L^2 of solenoidal vector functions with normal component vanishing at the boundary, H^1_0, H^2, $W^{1,\infty}$ are usual Sobolev spaces, $\|\cdot\|_2$ and $|\cdot|$ denote L^2 and Euclidean norm, respectively, etc.

Theorem 3.3. *Let $\mathcal{B}$ be a body with a cavity C of class C^2, and $\nu > 0$. Moreover, let*

$$\boldsymbol{V}_0 \in L^2_\sigma(C), \qquad \boldsymbol{\omega}_0 \in \mathbb{R}^3 \quad (\text{finite initial energy})$$

be arbitrarily given, and $(\boldsymbol{V}, \boldsymbol{\omega}_\infty)$ be a corresponding weak solution.
Then, the following properties hold:

- $\lim_{t\to\infty} \|\boldsymbol{V}(t)\|_{H^1_0} = 0$;
- *there is $\bar{\boldsymbol{\omega}} = \bar{\boldsymbol{\omega}}(\boldsymbol{V}_0, \boldsymbol{\omega}_0) \in \mathbb{R}^3 - \{\mathbf{0}\}$, such that*

$$\lim_{t\to\infty} \left|\boldsymbol{\omega}_\infty(t) - \bar{\boldsymbol{\omega}}\right| = \lim_{t\to\infty} \left|\boldsymbol{\omega}(t) - \bar{\boldsymbol{\omega}}\right| = 0$$

- *the vector $\bar{\boldsymbol{\omega}}$ is an eigenvector of the inertia tensor $\boldsymbol{I}$*;
- *$\bar{\boldsymbol{\omega}}$ is directed along the initial (constant) angular momentum $\boldsymbol{K}_G$, and we have: $\bar{\boldsymbol{\omega}} = \lambda^{-1}\boldsymbol{K}_G$, where λ is an eigenvalue of $\boldsymbol{I}$.*

Remark 1. The Zhukovskii conjecture is proved to be true for any body $\mathcal{B}$ and any viscosity $\nu > 0$, provided that only the initial kinetic energy of the system is finite and the cavity is of class C^2.

Remark 2. In general, the rate at which the system tends to the constant rigid motion is not known. However, if all (three) eigenvalues of $\boldsymbol{I}$ coincide, then the rate is exponential.

Remark 3. The relative velocity $\boldsymbol{V}$ is shown to vanish, as $t \to \infty$, in the H^1_0-norm. Pointwise decay appears more difficult to establish.

Remark 4. It is an *open question* to ascertain which permanent rotation is indeed attained.

The general idea in the proof of Theorem 3.3 is to employ tools from the dynamical system theory. In their classical use, these tools require the uniqueness property that, however is not guaranteed in the class of weak solutions. Nevertheless, one can show that uniqueness only for large times would suffice. Then, the question arises of whether every weak solution becomes “strong” (and therefore unique) for sufficiently large times. In the case of the Navier–Stokes equations, this is true (and well-known) because every weak solution becomes “small” for large times. The same property is not obvious in the case at hand, due to the presence of an, in general, large “conservative” component (conservation of the total angular momentum). One may guess the property to be true for “small” total angular momentum, but what for *arbitrarily large angular momentum*?

However, we can show the following result that ensures that every weak solution becomes “strong” for large times and, therefore, unique, without imposing restrictions on the size of the data.

Lemma 3.4. *Let $(\boldsymbol{V}, \boldsymbol{\omega}_\infty)$ be a weak solution corresponding to initial data $(\boldsymbol{V}_0, \boldsymbol{\omega}_0)$. Then, there exists $t^* = t^*(\boldsymbol{V}_0, \boldsymbol{\omega}_0)$ such that*:

$$\boldsymbol{V} \in C\big(t^*, \infty, H^1(C)\big) \cap L^2\big(t^*, T; H^2(C)\big), \quad \frac{\partial \boldsymbol{V}}{\partial t}, \quad \nabla p \in L^2\big(t^*, T; L^2(C)\big)$$
$$\boldsymbol{\omega}_\infty \in H^2(t^*, T), \quad \text{for all } T > t^*$$
$$\text{Moreover} \quad \lim_{t\to\infty} \big\|\nabla \boldsymbol{V}(t)\big\|_2 = 0$$

4. Numerical tests

A numerical solver has been developed to simulate the long time behavior of the system. The algorithm is characterized by the combination of time stepping schemes with the finite-element method for spatial discretization. Since the conservation of angular momentum is the fundamental governing principle for the system, the family of *geometric integrators* is particularly adapted to approximate such problem [4], because they preserve its invariants. For the particular case addressed here, we start by integrating the body dynamics with the θ-method. In particular, we adopt $\theta = \frac{1}{2}$, because it provides good properties as a geometric integrator. For the Navier–Stokes equations, we apply the implicit Euler time advancing scheme, $\theta = 1$. At each time step, we use sub-iterations to uncouple the solution of the discrete body and fluid problems and to linearize the corresponding equations. For the spatial approximation of the fluid equations, we exploit the finite-element method. We address the saddle point formulation of the problem in terms of velocity and pressure variables. In order to achieve a stable discretization of the divergence-free constraint, we use *inf–sup* stable mixed finite elements, such as the $\mathbb{P}^2 - \mathbb{P}^1$ approximation of the velocity and pressure fields, respectively, see [5]. Since our numerical tests involve relatively simple geometrical configurations, moderately refined computational grids will be applied. The system of algebraic equations arising from this method is solved by means of direct techniques, which are convenient since the number of degrees of freedom is not excessively large.

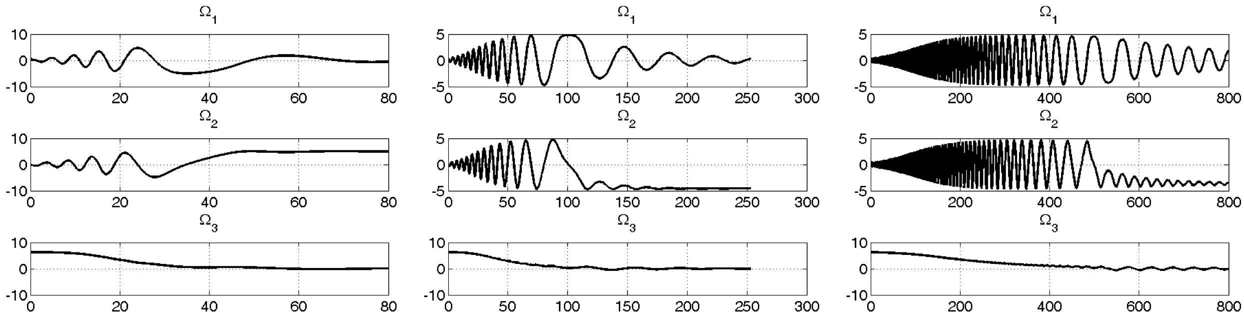

Fig. 2. Numerical simulations for a system characterized by $A = 6.735$, $B = 6.762$, $C = 5.543$ for decreasing kinematic viscosity $\nu = 0.1, 0.01, 0.001$ from left to right.

Numerical simulation emphasizes a number of important features of the motion of the body–liquid system $\mathcal{S}$ around its center of mass. We have selected those we think are of particular significance. Some of these properties have been established also analytically, in the sense specified in Theorem 3.3. However, several others still lack a rigorous mathematical proof that is currently beyond our reach.

(i) Regardless of the initial conditions, $\mathcal{S}$ will eventually move as a unique rigid body and will execute a permanent rotation. The permanent rotation that is found numerically is the one that is stable when $\mathcal{S}$ is viewed as a whole rigid body. This property is verified in all simulations in Fig. 2.

(ii) The dynamics can be very complicated depending on the magnitude of the coefficient of kinematic viscosity ν. In particular, the smaller is ν, the larger is the time employed by $\mathcal{S}$ to reach the (rigid) permanent rotation. This property is visualized in Fig. 2, right panel.

(iii) Reducing the magnitude of ν may generate a “flip-over” phenomenon. In other words, all other data being the same, numerical tests show that if ν drops below a certain value, the axis around which the permanent rotation occurs may orient itself in the opposite way, so that the angular velocity changes sign in order to keep the total angular momentum constant. This property is observed by the comparison of the left and middle panels in Fig. 2.

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