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Love–Bishop rod solution based on strain gradient elasticity theory

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ABSTRACT

In the present work, the propagation of longitudinal stress waves is investigated with a strain gradient elasticity theory given by Lam et al. In principle, the analysis of wave motion is based on the Love rod model including the lateral deformation effects, but in the same time is also taken into account the shear strain effects with Bishop's correction. By applying Hamilton's principle, a general explicit strain gradient elasticity solution is developed for the longitudinal stress waves, and it is compared with the special solutions based on the modified couple stress and classical theories. This work gives useful information with regard to the meaning of the three scale parameters in the strain gradient elasticity theory used here.

1. Introduction

Nowadays, the predictions of the mechanical behaviour of microstructures have attracted great scientific interest due to recent technological developments. The size-dependent behaviour of the different structural elements (e.g., beam, rod, bar, plate, shell) having extremely small overall dimensions, can be investigated with the aid of higher-order theories of linear elasticity. These theories, due to the existence of internal length scale parameters which reflect the microstructural features of nonlocal stress, are different from classical ones. The generalization of the elasticity theory using the higher gradients of the displacement field was started firstly by Cosserat and Cosserat [1]. A more systematic treatment in this topic was presented by Truesdell and Toupin [2]. A fundamental higher-order gradient theory of the linear elasticity, including the first and the second derivatives of the strain tensor, was developed by Mindlin [3]. Fleck and Hutchinson [4–6] proposed a higher-order gradient theory of plasticity by considering only the first derivative of the strain tensor. Based on the higher-order stress theory [3], Lam et al. [7] proposed a strain gradient elasticity theory which reduces the number of independent elastic length-scale parameters from five to three. This theory has been applied to the different problems during the last three years. The static and dynamical models were developed for the Bernoulli–Euler beam [8] and the Timoshenko beam [9] by using both the basic equations of the strain gradient elasticity theory and variation principles. The size effect on the critical buckling of axially loaded micro-scaled Bernoulli–Euler beams was investigated [10] by using both the basic equations of the strain gradient elasticity theory and variational principles. The size effect of microtubules was investigated [11] via the strain gradient elasticity theory for the buckling problem. A nonlinear size-dependent Euler–Bernoulli beam model was developed [12], based on the strain gradient elasticity theory. A closed-form analytical solution was developed [13] for

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the free torsional vibrations of the strain gradient bars. A size-dependent Kirchhoff micro-plate model was developed [14] based on the strain gradient elasticity theory. A microstructure-dependent Bernoulli–Euler beam model was developed [15] for the vibration and stability of micropipes conveying a fluid by using the strain gradient elasticity theory combined with Hamilton's principle. Recently, the longitudinal free vibration of a micro-scaled bar has been addressed [16] using the strain gradient elasticity theory of Lam et al. However, this analysis is based on the simple vibration theory which neglects lateral effects.

To our knowledge, there is no theoretical investigation with the strain gradient elasticity theory of Lam et al. on the longitudinal stress waves based Love–Bishop rod model. In this work, the propagation of longitudinal stress waves considering the lateral deformation [17] and the shear strain effects [18–22] is investigated using the strain gradient elasticity theory for microbars. The aim of the analysis is to develop a more comprehensive solution based on the strain gradient elasticity theory with three scale parameter for size effects in the micro/nanostructures elements. The present analysis shows that the contribution of the new-added terms containing the lateral and shear effects on the wave propagation is significant. Comparative analysis results provide some new useful findings for the relations between the different scale parameters.

2. Basic equations and formulation

The present formulation is based on the strain gradient elasticity theory proposed by Lam et al. [7]. According to this theory the total strain energy U stored in a deformed, isotropic, linearly elastic body occupying volume v (with a volume element $\mathrm{d}v$), is expressed as

$$U=\iiint\left(\sigma_{ij}\varepsilon_{ij}+p_i\gamma_i+\tau^{(1)}_{ijk}\eta^{(1)}_{ijk}+m^{(s)}_{ij}\chi^{(s)}_{ij}\right)\mathrm{d}v \tag{1}$$

The infinitesimal deformation measures appearing above are defined as:

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}) \tag{2}$$

$$\gamma_i=\varepsilon_{mm,i} \tag{3}$$

$$\eta^{(1)}_{ijk}=\frac{1}{3}(\varepsilon_{jk,i}+\varepsilon_{ki,j}+\varepsilon_{ij,k})-\frac{1}{15}\delta_{ij}(\varepsilon_{mm,k}+2\varepsilon_{mk,m})-\frac{1}{15}\left[\delta_{jk}(\varepsilon_{mm,i}+2\varepsilon_{mi,m})+\delta_{ki}(\varepsilon_{mm,j}+2\varepsilon_{mj,m})\right] \tag{4}$$

$$\chi^{s}_{ij}=\frac{1}{2}(e_{ipq}\varepsilon_{qj,p}+e_{jpq}\varepsilon_{qi,p}) \tag{5}$$

where u_i is the displacement vector, ε_{ij} is the strain tensor, $\varepsilon_{mm,i}$ is the dilatation gradient vector, $\eta^{(1)}_{ijk}$ is the deviatoric stretch gradient tensor, χ^{s}_{ij} is the symmetric rotation gradient tensor, δ_{ij} and e_{ijk} are the Kronecker delta and the alternate tensor, respectively.

The corresponding stress measures are defined as:

$$\sigma_{ij}=k\delta_{ij}\varepsilon_{mm}+2\mu\varepsilon'_{ij} \tag{6}$$

$$p_i=2\mu l_0^2\gamma_i \tag{7}$$

$$\tau^{(1)}_{ijk}=2\mu l_1^2\eta^{(1)}_{ijk} \tag{8}$$

$$m^{s}_{ij}=2\mu l_2^2\chi^{s}_{ij} \tag{9}$$

where ε'_{ij} is deviatoric strain defined as $\varepsilon'_{ij}=\varepsilon_{ij}-\frac{1}{3}\delta_{ij}\varepsilon_{mm}$, k and μ are the bulk and shear modules, respectively, and l_0, l_1 and l_2 are additional independent higher-order materials length parameters.

In principle, the present wave motion analysis is based on the Love rod model [17] and in the same time the Bishop correction is adopted [18–22] in the framework of the Love rod model. Thus, the effects of shear strain components are also taken into account while calculating the total strain energy (for both the local and nonlocal components). According to Love rod model including lateral deformation effects, the displacement field is expressed as:

$$u=u(x,t),\qquad v=-\nu y\frac{\partial u}{\partial x},\qquad w=-\nu z\frac{\partial u}{\partial x} \tag{10}$$

where u, v and w are the x, y and z components of the displacement vector, respectively, and ν is the Poisson's ratio. The x axis is taken in the longitudinal direction of the microbar; y and z are the axes at the geometrical centre of the cross-section. By substituting Eq. (10) into Eqs. (2)–(6), all the components of the strain tensor ε_{ij} and the stress tensor σ_{ij} are obtained as:

$$\varepsilon_x=\frac{\partial u}{\partial x},\qquad \varepsilon_y=\frac{\partial v}{\partial y}=-\nu\frac{\partial u}{\partial x},\qquad \varepsilon_z=\frac{\partial w}{\partial z}=-\nu\frac{\partial u}{\partial x},\qquad \gamma_{xy}=2\varepsilon_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=-\nu y\frac{\partial^2 u}{\partial x^2}$$

$$\gamma_{xz} = 2\varepsilon_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = -\nu z \frac{\partial^2 u}{\partial x^2}, \qquad \gamma_{yz} = 2\varepsilon_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0 \tag{11}$$

$$\sigma_{xx} = E\varepsilon_x, \qquad \sigma_{yy} = \sigma_{zz} = 0, \qquad \tau_{xy} = -\frac{E}{2(1+\nu)}\nu y \frac{\partial^2 u}{\partial x^2}, \qquad \tau_{xz} = -\frac{E}{2(1+\nu)}\nu z \frac{\partial^2 u}{\partial x^2}, \qquad \tau_{yz} = 0 \tag{12}$$

where σ_{xx}, σ_{yy} and σ_{zz} are the normal stresses and τ_{xy}, τ_{xz} and τ_{yz} are the shear stresses, γ_{xy}, γ_{xz} and γ_{yz} are the shear strains, and E is the elasticity modulus.

Substituting Eq. (11) in Eqs. (3)–(5), the components of the dilatation gradient vector γ_i $(= \varepsilon_{mm,i})$, the deviatoric stretch gradient tensor $\eta_{ijk}^{(1)}$ and the symmetric rotation gradient tensor $\chi_{ij}^{(s)}$ are obtained, respectively as:

$$\gamma_x = (1-2\nu)\frac{\partial^2 u}{\partial x^2} \tag{13}$$

$$\begin{aligned}
&\eta_{111} = \frac{2}{5}(1+2\nu)\frac{\partial^2 u}{\partial x^2}, \quad \eta_{112} = -\frac{4}{15}\nu y \frac{\partial^3 u}{\partial x^3}, \quad \eta_{113} = -\frac{4}{15}\nu z \frac{\partial^3 u}{\partial x^3}, \quad \eta_{121} = -\frac{4}{15}\nu y \frac{\partial^3 u}{\partial x^3} \\
&\eta_{122} = -\frac{1}{5}(1+2\nu)\frac{\partial^2 u}{\partial x^2}, \quad \eta_{131} = -\frac{4}{15}\nu z \frac{\partial^3 u}{\partial x^3}, \quad \eta_{133} = -\frac{1}{5}(1+2\nu)\frac{\partial^2 u}{\partial x^2} \\
&\eta_{221} = -\frac{1}{5}(1+2\nu)\frac{\partial^2 u}{\partial x^2}, \quad \eta_{222} = \frac{1}{5}\nu y \frac{\partial^3 u}{\partial x^3}, \quad \eta_{223} = \frac{1}{15}\nu z \frac{\partial^3 u}{\partial x^3} \\
&\eta_{211} = -\frac{4}{15}\nu y \frac{\partial^3 u}{\partial x^3}, \quad \eta_{212} = -\frac{1}{5}(1+2\nu)\frac{\partial^2 u}{\partial x^2}, \quad \eta_{232} = \frac{1}{15}\nu z \frac{\partial^3 u}{\partial x^3}, \quad \eta_{233} = \frac{1}{15}\nu y \frac{\partial^3 u}{\partial x^3} \\
&\eta_{311} = -\frac{4}{15}\nu z \frac{\partial^3 u}{\partial x^3}, \quad \eta_{313} = -\frac{1}{5}(1+2\nu)\frac{\partial^2 u}{\partial x^2}, \quad \eta_{322} = \frac{1}{15}\nu z \frac{\partial^3 u}{\partial x^3}, \quad \eta_{323} = \frac{1}{15}\nu y \frac{\partial^3 u}{\partial x^3} \\
&\eta_{332} = \frac{1}{15}\nu y \frac{\partial^3 u}{\partial x^3}, \quad \eta_{331} = -\frac{1}{5}(1+2\nu)\frac{\partial^2 u}{\partial x^2}, \quad \eta_{333} = \frac{1}{5}\nu z \frac{\partial^3 u}{\partial x^3}
\end{aligned} \tag{14}$$

$$\chi_{12} = \frac{1}{4}\nu z \frac{\partial^3 u}{\partial x^3}, \qquad \chi_{13} = -\frac{1}{4}\nu y \frac{\partial^3 u}{\partial x^3} \tag{15}$$

By using the relevant equations above, the terms appearing in the total strain energy expression (1) are obtained as:

$$\sigma_{ij}\varepsilon_{ij} \to E\left(\frac{\partial u}{\partial x}\right)^2 + \frac{E\nu^2(y^2+z^2)}{2(1+\nu)}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 \tag{16}$$

$$p_i\gamma_i \to 2\mu l_0^2(1-2\nu)^2\left(\frac{\partial^2 u}{\partial x^2}\right)^2 \tag{17}$$

$$\tau_{ijk}^{(1)}\eta_{ijk}^{(1)} \to 2\mu l_1^2\left[\frac{2}{5}(1+2\nu)^2\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{4}{15}\nu^2(y^2+z^2)\left(\frac{\partial^3 u}{\partial x^3}\right)^2\right] \tag{18}$$

$$m_{ij}^{(s)}\chi_{ij}^{(s)} \to \frac{1}{4}\mu l_2^2\nu^2(y^2+z^2)\left(\frac{\partial^3 u}{\partial x^3}\right)^2 \tag{19}$$

3. Hamilton's principle and the governing equation of wave motion

Inserting Eqs. (16)–(19) into Eq. (1), the total elastic strain energy U is obtained as:

$$\begin{aligned}
U = \frac{1}{2}\int_0^L \Bigg\{ & AE\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\nu^2 EI_\mathrm{p}}{2(1+\nu)}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{AE(1-2\nu)^2}{(1+\nu)}l_0^2\left(\frac{\partial^2 u}{\partial x^2}\right)^2 \\
& + \frac{2El_1^2}{5(1+\nu)}\left[A(1+2\nu)^2\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{2}{3}\nu^2 I_\mathrm{p}\left(\frac{\partial^3 u}{\partial x^3}\right)^2\right] + \frac{E\nu^2 I_\mathrm{p} l_2^2}{8(1+\nu)}\left(\frac{\partial^3 u}{\partial x^3}\right)^2\Bigg\}\mathrm{d}x
\end{aligned} \tag{20}$$

where A is the perpendicular cross section of the bar, and I_p is the second polar moment of area. The kinetic energy T of the bar is given by:

$$T = \frac{1}{2}\rho\int_0^L \left[A\left(\frac{\partial u}{\partial t}\right)^2 + \nu^2 I_\mathrm{p}\left(\frac{\partial^2 u}{\partial x \partial t}\right)^2\right]\mathrm{d}x \tag{21}$$

where ρ is the mass density.

By applying Hamilton's principle, the governing equation of wave motion is obtained in the following form:

$$\frac{\partial^2 u}{\partial x^2}-\frac{\rho}{E}\frac{\partial^2 u}{\partial t^2}-\frac{1}{2(1+\nu)}\nu^2 r_0^2\frac{\partial^4 u}{\partial x^4}-\frac{(1-2\nu)^2}{(1+\nu)}l_0^2\frac{\partial^4 u}{\partial x^4}-\frac{2}{5}\frac{(1+2\nu)^2 l_1^2}{(1+\nu)}\frac{\partial^4 u}{\partial x^4}+\frac{\rho}{E}\nu^2 r_0^2\frac{\partial^4 u}{\partial x^2\partial t^2}$$
$$+\frac{1}{8(1+\nu)}\nu^2 r_0^2 l_2^2\frac{\partial^6 u}{\partial x^6}+\frac{4}{15}\frac{1}{(1+\nu)}\nu^2 r_0^2 l_1^2\frac{\partial^6 u}{\partial x^6}=0 \tag{22}$$

where $r_0^2=\frac{I_p}{A}$.

Neglecting the contributions of the shear strains components on the total strain energy, the general solution reduces to the Love solution. In the degenerate case, the existing components of the stretch gradient tensor $\eta_{ijk}^{(1)}$ are obtained as:

$$\eta_{111}=\frac{2}{5}(1+\nu)\frac{\partial^2 u}{\partial x^2},\qquad \eta_{122}=-\frac{1}{5}(1+\nu)\frac{\partial^2 u}{\partial x^2},\qquad \eta_{133}=-\frac{1}{5}(1+\nu)\frac{\partial^2 u}{\partial x^2}$$
$$\eta_{221}=-\frac{1}{5}(1+\nu)\frac{\partial^2 u}{\partial x^2},\qquad \eta_{212}=-\frac{1}{5}(1+\nu)\frac{\partial^2 u}{\partial x^2},\qquad \eta_{313}=-\frac{1}{5}(1+\nu)\frac{\partial^2 u}{\partial x^2}$$
$$\eta_{331}=-\frac{1}{5}(1+\nu)\frac{\partial^2 u}{\partial x^2} \tag{23}$$

and the terms appearing in the total strain energy expression (1) become:

$$\sigma_{ij}\varepsilon_{ij}\to E\left(\frac{\partial u}{\partial x}\right)^2 \tag{24}$$

$$p_i\gamma_i\to 2(1-2\nu)^2\mu l_0^2\left(\frac{\partial^2 u}{\partial x^2}\right)^2 \tag{25}$$

$$\tau_{ijk}^{(1)}\eta_{ijk}^{(1)}\to\frac{4}{5}(1+\nu)^2\mu l_1^2\left(\frac{\partial^2 u}{\partial x^2}\right)^2 \tag{26}$$

In the result, the governing equation of the longitudinal stress waves is found as:

$$\frac{\partial^2 u}{\partial x^2}-\frac{\rho}{E}\frac{\partial^2 u}{\partial t^2}-\frac{(1-2\nu)^2}{1+\nu}l_0^2\frac{\partial^4 u}{\partial x^4}-\frac{2}{5}(1+\nu)l_1^2\frac{\partial^4 u}{\partial x^4}+\frac{\rho}{E}\nu^2 r_0^2\frac{\partial^4 u}{\partial x^2\partial t^2}=0 \tag{27}$$

As expected, the scale parameter l_2 disappears in the Love model. It must be noted that for a finite bar, such a model has to be completed with higher-order boundary conditions that can be obtained from the application of variational principles.

Neglecting the contributions of lateral deformations and shear strains components on the energy expressions, the general solution reduces to the one-dimensional rod (i.e. $u=u(x,t)$, $v=0$ and $w=0$) solution. In this degenerate case, the existing components of the stretch gradient tensor $\eta_{ijk}^{(1)}$ are obtained as:

$$\eta_{111}=\frac{2}{5}\frac{\partial^2 u}{\partial x^2},\qquad \eta_{122}=-\frac{1}{5}\frac{\partial^2 u}{\partial x^2},\qquad \eta_{133}=-\frac{1}{5}\frac{\partial^2 u}{\partial x^2},\qquad \eta_{221}=-\frac{1}{5}\frac{\partial^2 u}{\partial x^2},\qquad \eta_{212}=-\frac{1}{5}\frac{\partial^2 u}{\partial x^2}$$
$$\eta_{313}=-\frac{1}{5}\frac{\partial^2 u}{\partial x^2},\qquad \eta_{331}=-\frac{1}{5}\frac{\partial^2 u}{\partial x^2} \tag{28}$$

and the terms appearing in the total strain energy expression (1) become:

$$\sigma_{ij}\varepsilon_{ij}\to E\left(\frac{\partial u}{\partial x}\right)^2 \tag{29}$$

$$p_i\gamma_i\to 2\mu l_0^2\left(\frac{\partial^2 u}{\partial x^2}\right)^2 \tag{30}$$

$$\tau_{ijk}^{(1)}\eta_{ijk}^{(1)}\to\frac{4}{5}\mu l_1^2\left(\frac{\partial^2 u}{\partial x^2}\right)^2 \tag{31}$$

In the result, the governing equation of the longitudinal stress waves writes:

$$\frac{\partial^2 u}{\partial x^2}-\frac{\rho}{E}\frac{\partial^2 u}{\partial t^2}-\frac{l_0^2}{1+\nu}\frac{\partial^4 u}{\partial x^4}-\frac{2l_1^2}{5(1+\nu)}\frac{\partial^4 u}{\partial x^4}=0 \tag{32}$$

The above equation is identical to that given in [16]. As is clearly seen, in this particular solution, the scale parameter l_2 disappears.

4. General dispersion relation and numerical examples

A harmonic longitudinal wave propagating along the axial direction can be expressed in complex form as:

$$u = \tilde{U} e^{ik(x-ct)} \tag{33}$$

where $\tilde{U}$ is the wave amplitude, k is the wave number, c is the phase velocity, and t is the time. Inserting Eq. (33) into Eq. (22), the corresponding general solution for a rod with radius a is obtained as:

$$c^* = \sqrt{\frac{1 + \frac{1}{1+\nu}[\frac{1}{2}\nu^2 r_0^2 + (1-2\nu)^2 l_0^2 + \frac{2}{5}(1+2\nu)^2 l_1^2]k^2 + \frac{\nu^2 r_0^2}{4(1+\nu)}(\frac{16}{15}l_1^2 + \frac{1}{2}l_2^2)k^4}{1 + \nu^2 r_0^2 k^2}} \tag{34}$$

where c^* $(= \frac{c}{c_0})$ is the dimensionless phase velocity, $c_0^2 = \frac{E}{\rho}$ and, for a rod with radius a, $r_0^2 = a^2/2$.

In the above equation, the length scale parameters l_0, l_1, and l_2 reflect the microstructure dilatation gradient, the microstructure stretch gradient and the microstructure rotation gradient, respectively. When the length scale parameters l_0 and l_1 vanish, the strain gradient elasticity theory reduces to the modified couple stress theory [23]. Thus, for $l_0 = l_1 = 0$, Eq. (34) gives the modified couple stress theory solution [21]. Hence, the present strain gradient elasticity theory may be regarded as the wider form of the modified couple stress theory.

Inserting Eq. (33) into Eq. (27), the degenerate dispersion relation for the Love rod model is obtained as follows:

$$c^* = \sqrt{\frac{1 + [\frac{(1-2\nu)^2 l_0^2}{1+\nu} + \frac{2(1+\nu) l_1^2}{5}]k^2}{1 + \nu^2 r_0^2 k^2}} \tag{35}$$

In the limit case, for $k \to \infty$, the dispersion relation (35) is reduced in the following form:

$$c^* = \frac{1}{\nu r_0}\sqrt{\frac{(1-2\nu)^2 l_0^2}{1+\nu} + \frac{2(1+\nu) l_1^2}{5}} \tag{36}$$

In the limit case, i.e. $\nu \to 0.5$, the scale parameter l_0 disappears for the Love rod model.

Expression (36) is called the Love dispersion relation with short wavelengths [24]. On the other hand, the Love dispersion relation based on Aifantis' gradient elasticity theory is given as [25]:

$$c^* = \sqrt{\frac{1 + l_s^2 k^2}{1 + \nu^2 r_0^2 k^2}} \tag{37}$$

where l_s is the scale parameter in the gradient elasticity model. In the limit case, for $k \to \infty$: the relation (37) reduces to the following form:

$$c^* = \frac{l_s}{\nu r_0} \tag{38}$$

Equating asymptotic phase velocities (36) and (38), the relation between the scale parameters in the gradient and present elasticity models can be helpful for the estimation:

$$l_s = \sqrt{\frac{(1-2\nu)^2 l_0^2}{1+\nu} + \frac{2(1+\nu) l_1^2}{5}} \tag{39}$$

Inserting Eq. (33) into Eq. (32), the dispersion relation reduces to the one-dimensional rod model as follows:

$$c^* = \sqrt{1 + \frac{(l_0 k)^2}{1+\nu} + \frac{2}{5}\frac{(l_1 k)^2}{(1+\nu)}} \tag{40}$$

Contrary to the Love rod model, the scale parameter l_0 is also present, for the limit case of Poisson's ratio (i.e., $\nu \to 0.5$). On other hand, the one-dimensional rod dispersion relation based on Aifantis' gradient elasticity theory is given by [26–28]:

$$c^* = \sqrt{1 + l_s^2 k^2} \tag{41}$$

In the both solutions, the lateral effects have been neglected. If we compare Eqs. (40) and (41), we can see that a special strain gradient elasticity solution (40) containing Poisson's ratio is more reasonable physically than Aifantis' solution (41).

Some numerical examples are presented in Figs. 1–6. In Fig. 1, the present general solution based on the strain gradient elasticity theory proposed by Lam et al., and the degenerate solution [21] based on the modified couple stress theory are compared, where c^*_{sge} and c^*_{mc} denote the dimensionless phase velocities for the strain gradient elasticity and the modified

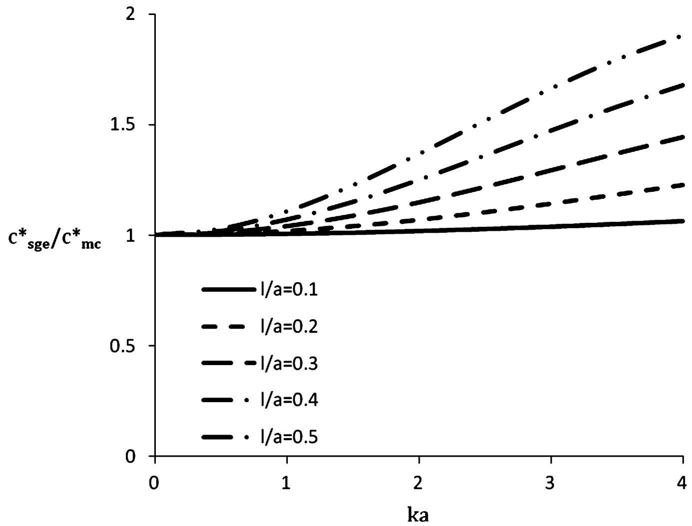

Fig. 1. Comparison of dispersion curves for strain gradient elasticity and modified couple stress theories.

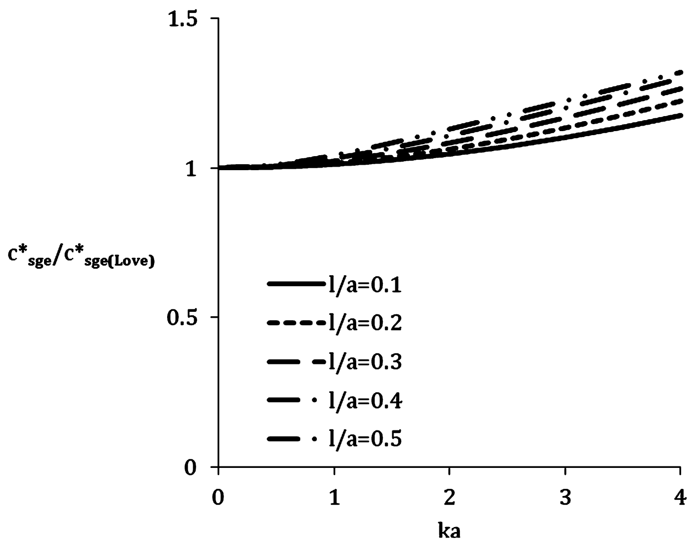

Fig. 2. Comparison of dispersion curves for the present general solution and the degenerate Love solution.

couple stress theories, respectively, k_a is the dimensionless wave number and l/a is the dimensionless material length scale parameter. Fig. 1 shows that except the small values of wave numbers, as the length scale parameters ($l_0 = l_1 = l_2 = l$) increase, the difference between the results of these two theories becomes significant. However, it can be easily seen that by with increasing the radius of rod, the difference decreases.

In Fig. 2, the present general solution and its degenerate nonlocal Love rod solution are compared, where c^*_{sge} and $c^*_{\mathrm{sge(Love)}}$ denote the dimensionless phase velocities for the present general solution and the degenerate Love solution. Fig. 2 shows that by increasing the wave number, the difference between these two solutions increases. This difference is more significant for the high values of the length scale parameters. It can be concluded from Fig. 2 that the effect of the shear strain on the phase velocity is more significant for the high values of the wave number and of the material length scale parameter. Since there is no sufficient information about the length scale parameters, in the open literature, for only simple comparison purposes, they are taken equal to each other in the numerical calculations, as in Figs. 1 and 2. In Fig. 3, the Love rod solution based on the strain gradient elasticity theory and the classical Love solution are compared for the different ratios of the length parameters (l_0 and l_1), where $m = \frac{l_0}{l_1}$ and l_1 is taken to be l, $c^*_{\mathrm{sge}(L)}$ and $c^*_{\mathrm{classic}(L)}$ denote the dimensionless phase velocities of the gradient Love solution and of the classical Love solution, respectively. Fig. 3 shows that, for $m > 1$ (i.e. if the microstructure dilatation gradient is greater than the microstructure stretch gradient), the difference between these two solutions becomes more significant. Furthermore, it must be noticed that the comparative results are independent from the rod radius. In Fig. 4, the propagation of the longitudinal Love stress waves with short wave lengths are shown for the

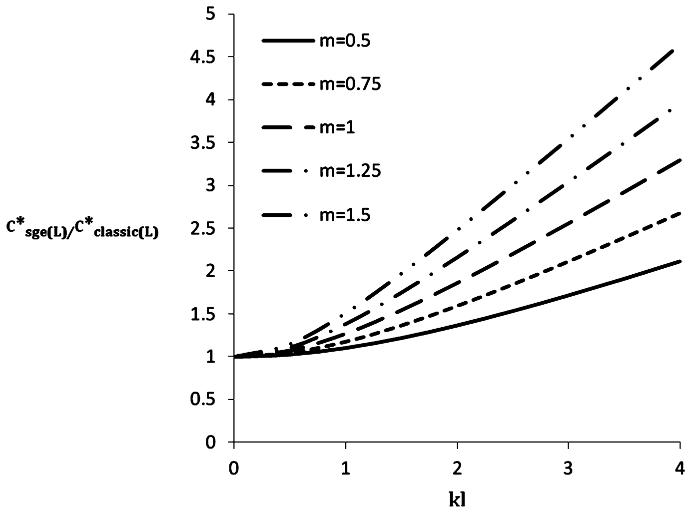

Fig. 3. Comparison of dispersion curves for the present Love solution and the classical Love solution.

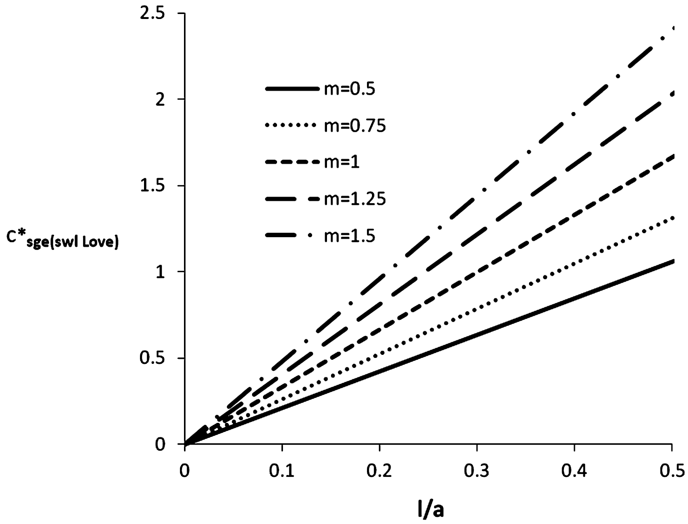

Fig. 4. Distributions of gradient Love stress waves having short wavelengths for the different ratios of material length parameters.

different ratios ($m = \frac{l_0}{l_1}$) of the length parameters (l_0 and l_1), where $c^*_{\text{sge(swl Love)}}$ denotes the dimensionless phase velocities for the gradient Love waves with short wave lengths. Fig. 4 shows that by increasing dimensionless length scale parameter l/a, the wave phase velocity quickly increases linearly. This increase is more significant for $m > 1$. In Fig. 5, the longitudinal wave propagation of the one-dimensional rod model is shown for the different ratios of the length parameters (l_0 and l_1), where $m = \frac{l_1}{l_0}$ and l_0 is taken to be l, $c^*_{\text{sge(rod)}}$ denotes the dimensionless phase velocities for the nonlocal one-dimensional rod solution based on the modified strain gradient elasticity theory. Fig. 5 shows that the wave phase velocity is very sensitive to the length scale parameters and this sensitivity quickly increases for $m > 1$ (i.e. if the microstructure stretch gradient is greater than the microstructure dilatation gradient). Comparing Figs. 3 and 5, it can be seen that the wave propagations for two different models (i.e., Love rod and one-dimensional rod) are more sensitive to the different microstructure features. In Fig. 6a and b, the present and the Aifantis Love stress waves are compared for the different scale ratios $r = \frac{l_1(=l_0)}{l_s}$ and Poisson ratios $\nu = 1/3$ and $1/2$. Increasing Poisson's ratios and the wave numbers, the predictive results are seen closer to each other for around $r = 1$.

5. Conclusion

In the present work, a general solution based on the strain gradient elasticity theory [7] for the nonlocal longitudinal stress waves is presented. Both lateral deformation and the shear strain component effects are taken into account in this wave motion analysis based on the classical Love–Bishop approach [18–22]. Therefore, the general solution presented here

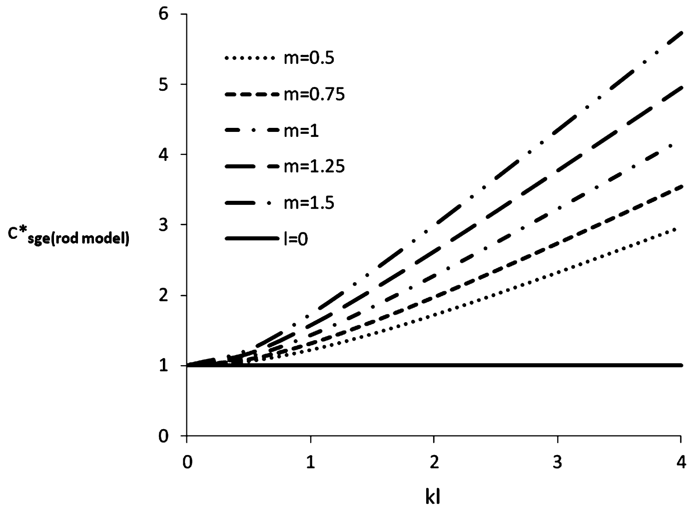

Fig. 5. Distribution of gradient one-dimensional rod stress waves for the different ratios of material length parameters.

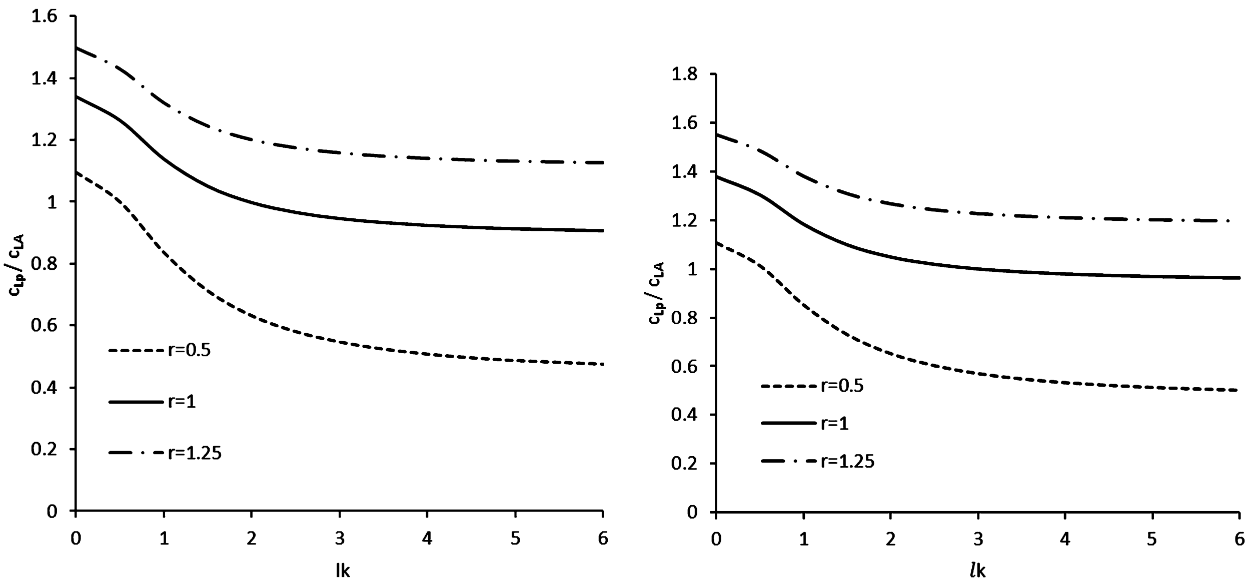

Fig. 6. Comparison of Love rod dispersion curves for the present solution and Aifantis solution, for $\nu = 1/3$ and $\nu \to 1/2$, respectively.

may be regarded as the generalized Love–Bishop solution based on the strain gradient elasticity theory. In the meantime, it should be noted that the recent similar studies [16,21] are particular cases of the present analysis.

It can be seen that Love rod and one-dimensional rod special solutions that contain two length scale parameters can be obtained from the present model. However, it can be easily seen that the size effects for the same rod models (i.e., Love rod and one-dimensional rod models) cannot be explained with the modified couple stress theory [23]. Thus, it can be concluded that the investigation of the size effects using the strain gradient elasticity theory proposed by Lam et al. is more comprehensive and efficient, compared to the modified couple stress theory proposed by Yang et al. [23]. However, on the other hand, as it is known from the literature [28,29], the sign of the gradient terms is significant for dynamical applications. According to this analytical finding explained, the second-order gradient model with negative sign gives stable results for the dispersive properties of heterogeneous materials, but these predictive results are unrealistic physically. On the other hand, the second-order gradient model with a positive sign possesses some merits in predicting dispersive properties of heterogeneous materials, but are unstable, thus cannot be used to study the practical dynamic problems. Considering this fact cannot be said an assertive result about the capability of this gradient elasticity wave equation to capture small length scale effects in the heterogeneous materials. As expected, the obtained results show that the different rod models are affected by different microstructure features. Therefore, more extensive information about the material scale parameters

l_0, l_1 and l_2 should be provided based on molecular simulation and experiment to obtain more accurate results with the strain gradient elasticity theory.

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