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Stokes–Darcy coupling for periodically curved interfaces

Sur les conditions aux limites entre l'équation de Stokes et de Darcy pour une interface courbée

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A B S T R A C T

We investigate the boundary condition between a free fluid and a porous medium, where the interface between the two is given as a periodically curved structure. Using a coordinate transformation, we can employ methods of periodic homogenisation to derive effective boundary conditions for the transformed system. In the porous medium, the fluid velocity is given by Darcy's law with a non-constant permeability matrix. In tangential direction as well as for the pressure, a jump appears. Its magnitudes can be calculated with the help of a generalised boundary layer function. The results can be interpreted as a generalised law of Beavers and Joseph for curved interfaces.

R É S U M É

On considère le comportement d'un fluide libre au-dessus d'un milieu poreux avec une interface courbée périodique. Utilisant une transformation des coordonnées, on peut utiliser des méthodes d'homogénéisation périodique pour la dérivation des conditions aux limites. Le comportement du fluide en milieu poreux est donné par une loi de Darcy avec une matrice de perméabilité non constante. Ensuite, on obtient le comportement du fluide à l'interface. Une discontinuité apparaît pour la pression ainsi que pour la vitesse tangentielle. L'amplitude des discontinuités peut être calculée par une fonction de couche limite généralisée. Ainsi, les résultats donnent une loi généralisée de Beavers et Joseph pour des interfaces courbées.

1. Introduction

The interface condition coupling a free flow with a flow in a porous medium is of great interest in mathematical modelling, groundwater flow or soil chemistry, among others. From a physical point of view, the fluid velocity of an incompressible fluid has to be continuous in normal direction to the interface due to mass conservation. However, other conditions are not so obvious due to the different nature of the governing equations: For the free fluid, the Stokes or Navier–Stokes equation is of second order for the velocity and of first order for the pressure, whereas for the Darcy equation in the porous medium the order of the terms is exchanged. By practical experiments, Beavers and Joseph [1] concluded that a jump in

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the effective velocities appears in tangential direction. Using a statistical approach, this condition was verified by Saffman in [2]. However, some parameters in this approach still need to be determined by experiments.

Starting in 1996, Willie Jäger and Andro Mikelić applied the theory of homogenisation to the problem. They first developed a theory of mathematical boundary layers in [3], using these to rigorously derive Saffman's modification of the jump condition:

$$\sqrt{k^\varepsilon}\,(\nabla v_{\mathrm{F}} \nu)\cdot\tau = \alpha v_{\mathrm{F}}\cdot\tau + \mathcal{O}\big(k^\varepsilon\big) \tag{1}$$

in [4], where v_{F} denotes the velocity of the free fluid at the interface; $k^\varepsilon = k\varepsilon^2$ is the (scalar) permeability of the porous medium (where ε denotes its characteristic length), and ν and τ are the unit normal and unit tangential vector, respectively. The slip-coefficient α can be calculated explicitly. They considered a situation that corresponds to the experimental setup of Beavers and Joseph. Later in [5], Mikelić and Marciniak-Czochra extended the results to an arbitrary body force, which gave an additional pressure jump condition. However, all the results above suffer from one drawback: only a planar boundary in the form of a line or a plane is considered. Therefore, the effect of a possible curvature of the interface is not known. Generalisations of the boundary layers in [3] were developed by Maria Neuss-Radu in [6]. However, applications only treat reaction–diffusion systems without flow, and explicit results can only be obtained in the case of a layered medium, see [7].

In [8] we proposed a new approach to consider the case of a non-flat interface by using a coordinate transformation. In this note — using generalised boundary layer functions developed in [9] — we are able to derive boundary conditions of Beavers and Joseph for the case of a periodically curved non-flat interface.

2. Overview of the geometries

In this section we describe the main geometrical settings that are used throughout this work. Let $L, K, h > 0$. Then $\Omega := (0, L)\times(-K, h)$ is a rectangular domain in $\mathbb{R}^2$ (later corresponding to the reference domain) with parts $\Omega_1 := (0, L)\times(0, h)$ (later the reference free fluid domain), $\Omega_2 := (0, L)\times(-K, 0)$ (the reference porous medium) and $\Sigma = (0, L)\times\{0\}$ (later the reference interface). Let $g \in \mathcal{C}^\infty(\mathbb{R})$ be a given function such that $g(y + L) = g(y)$ for all $y \in \mathbb{R}$. We consider g to describe a periodic curved structure in our domain of interest. Define the coordinate transformation:

$$\psi : \Omega \longrightarrow \tilde{\Omega}, \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + g(x_1) \end{pmatrix}$$

such that $\tilde{\Omega} = \psi(\Omega)$, $\tilde{\Omega}_1 := \psi(\Omega_1)$, $\tilde{\Omega}_2 := \psi(\Omega_2)$ and $\tilde{\Sigma} := \psi(\Sigma) = \{(y, g(y)) \mid y \in (0, L)\}$. We are interested in the behaviour of a fluid flowing through the curved channel $\tilde{\Omega}$, where $\tilde{\Omega}_1$ represents a domain with a free fluid flow, and $\tilde{\Omega}_2$ is a porous medium. We are especially interested in the behaviour of the fluid at the curved boundary $\tilde{\Sigma}$. Let $\tilde{\Omega}_S \Subset \tilde{\Omega}_2$ be a given solid inclusion. We will use a sequence of such inclusions to create a porous medium via homogenisation theory.

To do so, define an ε-periodic geometry in Ω_2 by the use of a reference cell $Y := [0, 1]^2$, containing a connected open set Y_{S} (corresponding to the solid part of the cell). Its boundary ∂Y_{S} is assumed to be of class $\mathcal{C}^\infty$ with $\partial Y_{\mathrm{S}} \cap \partial Y = \emptyset$. Let $Y^* := Y\backslash\overline{Y_{\mathrm{S}}}$ be the fluid part of the reference cell.

For given $\varepsilon > 0$ such that $\frac{L}{\varepsilon} \in \mathbb{N}$, let χ be the characteristic function of Y^*, extended by periodicity to the whole $\mathbb{R}^2$. Set $\chi^\varepsilon(x) := \chi(\frac{x}{\varepsilon})$ and define the fluid part of the porous medium as $\Omega_2^\varepsilon = \{x \in \Omega_2 \mid \chi^\varepsilon(x) = 1\}$. The fluid domain is then given by $\Omega^\varepsilon = \Omega_1 \cup \Sigma \cup \Omega_2^\varepsilon$, and the solid part by $\Omega_{\mathrm{S}} = \Omega_2\backslash\Omega_2^\varepsilon$.

In order to obtain the effective fluid behaviour near Σ, we have to define a number of so-called boundary layer problems. To this end, we introduce the following setting: we consider the domain $[0, 1]\times\mathbb{R}$ subdivided as follows: $Z^+ = [0, 1]\times(0, \infty)$ corresponds to the free fluid region, whereas the union of translated reference cells $Z^- = \bigcup_{k=1}^\infty \{Y^* - \binom{0}{k}\}\backslash S$ is considered to be the void space in the porous part. Here $S = [0, 1]\times\{0\}$ denotes the interface between Z^+ and Z^-. Finally, let $Z = Z^+ \cup Z^-$ and $Z_{\mathrm{BL}} = Z^+ \cup S \cup Z^-$ be the fluid domain without and with interface.

3. Fluid behaviour at the interface — main results

For a given body force $\tilde{f} \in L^2(\tilde{\Omega})$, we assume that a mathematical description of the fluid is given by the steady-state Stokes equation with no slip condition on the boundary of the solid inclusion and on the outer walls:

$$\begin{aligned}
&-\mu\Delta_z\tilde{u}(z) + \nabla_z\tilde{p}(z) = \tilde{f}(z) && \text{in } \tilde{\Omega}\backslash\overline{\tilde{\Omega}_{\mathrm{S}}}\\
&\mathrm{div}_z\big(\tilde{u}(z)\big) = 0 && \text{in } \tilde{\Omega}\backslash\overline{\tilde{\Omega}_{\mathrm{S}}}\\
&\tilde{u}(z) = 0 && \text{on } \partial\tilde{\Omega}_{\mathrm{S}} \cup \partial\tilde{\Omega}\backslash\big(\{z_1 = 0\}\cup\{z_1 = L\}\big)\\
&\tilde{u}, \tilde{p} && \text{are } L\text{-periodic in } x_1
\end{aligned}$$

Here $\mu > 0$ denotes the dynamic viscosity. We are looking for a velocity field $\tilde{u} \in H^1(\tilde{\Omega})^2$ and a pressure $\tilde{p} \in L^2(\tilde{\Omega})/\mathbb{R}$. The Stokes equation is an approximation of the full Navier–Stokes equation which is valid for low Reynolds number flows. Using the transformation rules for the differential operators (see [8]), we obtain the following equation for the transformed quantities $u^\varepsilon(x) = \tilde{u}(\psi(x))$, $p^\varepsilon(x) = \tilde{p}(\psi(x))$ and $f(x) = \tilde{f}(\psi(x))$ in the rectangular domain Ω:

$$-\mu \cdot \operatorname{div}_x\big(F^{-1}(x)F^{-T}(x)\nabla_x u^\varepsilon(x)\big) + F^{-T}(x)\nabla_x p^\varepsilon(x) = f(x) \quad \text{in } \Omega\backslash\overline{\Omega_S} = \Omega^\varepsilon \tag{2a}$$

$$\operatorname{div}_x\big(F^{-1}(x)u^\varepsilon(x)\big) = 0 \quad \text{in } \Omega\backslash\overline{\Omega_S} = \Omega^\varepsilon \tag{2b}$$

$$u^\varepsilon(x) = 0 \quad \text{on } \partial\Omega_S \cup \partial\Omega\backslash\big(\{x_1 = 0\} \cup \{x_1 = L\}\big) \tag{2c}$$

$$u^\varepsilon, p^\varepsilon \quad \text{are } L\text{-periodic in } z_1 \tag{2d}$$

Here $\Omega_S := \psi^{-1}(\tilde{\Omega}_S)$ is the transformed solid inclusion, and F is defined as the Jacobian matrix of ψ, given by:

$$F(x) = \begin{bmatrix} 1 & 0 \\ g'(x_1) & 1 \end{bmatrix}$$

Since $\det(F) = 1$, ψ is a volume-preserving $\mathcal{C}^\infty$-coordinate transformation. Note that the transformed normal vector to Σ is given by $F^{-\mathrm{T}}(x)e_2$, and the transformed tangential vector by $F(x)e_1$, where e_i, $i \in \{1, 2\}$, denote the usual unit vectors in $\mathbb{R}^2$. For simplicity, we set $\mu = 1$ in the sequel. All subsequent derivations will be based on Eq. (2). We consider a sequence of scale parameters ε going to 0. In the limit, this leads to the formation of an effective porous medium in Ω_2 via homogenisation theory, where for $g \equiv 0$ the functions u^ε and p^ε converge to the Darcy velocity and pressure, respectively (see [10]).

For the general case presented here, the fluid behaviour can be approximated by the following effective equations: the velocity field in the free fluid domain Ω_1 is given by the following problem.

Find a velocity u^{eff} and a pressure p^{eff} such that:

$$-\operatorname{div}\big(F^{-1}F^{-\mathrm{T}}\nabla u^{\mathrm{eff}}\big) + F^{-\mathrm{T}}\nabla p^{\mathrm{eff}} = f \quad \text{in } \Omega_1 \tag{3a}$$

$$\operatorname{div}\big(F^{-1}u^{\mathrm{eff}}\big) = 0 \quad \text{in } \Omega_1 \tag{3b}$$

$$\int_{\Omega_1} p^{\mathrm{eff}}\,\mathrm{d}x = 0 \tag{3c}$$

$$u^{\mathrm{eff}} = 0 \quad \text{on } (0, L) \times \{h\} \tag{3d}$$

$$u^{\mathrm{eff}}, p^{\mathrm{eff}} \quad \text{are } L\text{-periodic in } x_1 \tag{3e}$$

$$u^{\mathrm{eff}} = -\varepsilon C^{\mathrm{bl}} \quad \text{on } \Sigma \tag{3f}$$

C^{bl} is the decay function of the boundary layer function β^{bl} defined in Section 4. It can be calculated by:

$$C^{\mathrm{bl}}(x) = -\int_0^1 \beta^{\mathrm{bl}}(x, y_1, +0)\,\mathrm{d}y_1$$

It holds $C^{\mathrm{bl}}(x) \cdot F^{-\mathrm{T}}(x)e_2 = 0$. The effective Darcy pressure $\tilde{p}^{\mathrm{eff}}$ in Ω_2 is given by:

$$\operatorname{div}\big(F^{-1}A\big(f - F^{-\mathrm{T}}\nabla\tilde{p}^{\mathrm{eff}}\big)\big) = 0 \quad \text{in } \Omega_2 \tag{4a}$$

$$A\big(f - F^{-\mathrm{T}}\nabla\tilde{p}^{\mathrm{eff}}\big) \cdot F^{-\mathrm{T}}e_2 = 0 \quad \text{on } (0, L) \times \{-K\} \tag{4b}$$

$$\tilde{p}^{\mathrm{eff}} = p^{\mathrm{eff}} + C_\omega^{\mathrm{bl}} \quad \text{on } \Sigma \tag{4c}$$

$$\tilde{p}^{\mathrm{eff}} \quad \text{is } L\text{-periodic in } x_1 \tag{4d}$$

C_ω^{bl} is the pressure stabilisation function of the boundary layer pressure ω^{bl} defined in (6). It can be calculated by $C_\omega^{\mathrm{bl}}(x) = \int_0^1 \omega^{\mathrm{bl}}(x, y_1, +0)\,\mathrm{d}y_1$. A is obtained with the help of the cell problem for the transformed Darcy law (see also [8]): fix $x \in \Omega$ and let $(w^j(x, \cdot), \pi^j(x, \cdot)) \in H^1_\#(Y^*)^2 \times L^2(Y^*)/\mathbb{R}$ be a solution of the parameter-dependent cell problem:

$$-\operatorname{div}_y\big(F^{-1}(x)F^{-T}(x)\nabla_y w^j(x, y)\big) + F^{-T}(x)\nabla_y \pi^j(x, y) = e_j \quad \text{in } Y^*$$

$$\operatorname{div}_y\big(F^{-1}(x)w^j(x, y)\big) = 0 \quad \text{in } Y^*$$

$$w^j(x, y) = 0 \quad \text{in } Y_S$$

$$w^j(x, y), \pi^j(x, y) \quad \text{are } Y\text{-periodic in } y$$

Define the matrix-valued function $A : \Omega \to \mathbb{R}^{2\times 2}$ by:

$$\big[A(x)\big]_{ji} = \int_{Y^*} w_i^j(x, y)\,\mathrm{d}y, \quad i, j = 1, 2$$

The cell problem is a partial differential equation in $y \in Y^*$, depending on $x \in \Omega$ as a parameter. Using the implicit function theorem for Banach spaces, one can derive differentiability properties of these functions in x-direction, see [9]. Furthermore, one can show that the matrix $A(x)$ is symmetric and (uniformly) positive definite.

Define the effective mass flow rates in transformed tangential direction as:

$$M^{\mathrm{eff}} := \int_{\Omega_1} u^{\mathrm{eff}} \cdot F e_1 \, \mathrm{d}x \quad \text{and} \quad M^{\varepsilon} := \int_{\Omega_1} u^{\varepsilon} \cdot F e_1 \, \mathrm{d}x$$

We obtain the following estimates.

Theorem 3.1. *Let $f \in C^\infty(\bar{\Omega})$ be L-periodic in the first variable. For $(u^\varepsilon, p^\varepsilon)$ as defined in Eq. (2) and $(u^{\mathrm{eff}}, p^{\mathrm{eff}})$ defined in (3) the estimates:*

$$\begin{aligned}
&\left\| u^\varepsilon - u^{\mathrm{eff}} \right\|_{L^2(\Omega_1)^2} + \left| M^\varepsilon - M^{\mathrm{eff}} \right| \leqslant C \varepsilon^{\frac{3}{2}} \\
&\left\| u^\varepsilon - u^{\mathrm{eff}} \right\|_{H^{\frac{1}{2}}(\Omega_1)^2} + \left\| p^\varepsilon - p^{\mathrm{eff}} \right\|_{L^1(\Omega_1)} + \left\| \nabla \left(u^\varepsilon - u^{\mathrm{eff}} \right) \right\|_{L^1(\Omega_1)^4} \leqslant C \varepsilon \\
&\left\| |x_2|^{\frac{1}{2}} \nabla \left(u^\varepsilon - u^{\mathrm{eff}} \right) \right\|_{L^2(\Omega_1)^4} + \left\| |x_2|^{\frac{1}{2}} \left(p^\varepsilon - p^{\mathrm{eff}} \right) \right\|_{L^2(\Omega_1)} \leqslant C \varepsilon
\end{aligned}$$

hold with a constant $C > 0$, independent of ε. On the interface Σ, we have:

$$\frac{1}{\varepsilon} \left(u^\varepsilon - u^{\mathrm{eff}} \right) \rightharpoonup 0 \quad \text{in } L^2(\Sigma), \qquad \left\| u^\varepsilon - u^{\mathrm{eff}} \right\|_{H^{-\frac{1}{2}}(\Sigma)} \leqslant C \varepsilon^{\frac{4}{3}} \tag{5}$$

In the porous medium Ω_2, we arrive at the following results.

Theorem 3.2. *For the effective pressure in the porous medium defined by (4), we have for all $\delta > 0$:*

$$\begin{aligned}
&\frac{1}{\varepsilon^2} u^\varepsilon - A \left(f - F^{-\mathrm{T}} \nabla \tilde{p}^{\mathrm{eff}} \right) \rightharpoonup 0 \quad \text{in } L^2 \left((0, L) \times (-K, -\delta) \right) \\
&p^\varepsilon - \tilde{p}^{\mathrm{eff}} \rightharpoonup 0 \quad \text{in } L^2(\Omega_2), \qquad \left\| p^\varepsilon - \tilde{p}^{\mathrm{eff}} \right\|_{H^{-\frac{1}{2}}(\Sigma)} \leqslant C \varepsilon^{\frac{1}{2}}
\end{aligned}$$

with a constant $C > 0$ independent of ε.

4. Aspects of the derivation

In this section, we show how the initial steps of the derivation of the main results are performed. In principle, we correct the velocity u^ε and pressure p^ε with a sequence of solutions to auxiliary problems, until we are finally able to get reasonable estimates. There are basically three types of auxiliary problems: Boundary layer functions, which are used to correct jumps across the boundary Σ; counterflow functions, which correct for the decay of the boundary layer functions towards a constant, and functions for the correction of the divergence.

We start by eliminating the right-hand side of (2a) in Ω_1. Let (u^0, π^0) be a solution of:

$$\begin{aligned}
&-\mathrm{div}\left(F^{-1} F^{-\mathrm{T}} \nabla u^0 \right) + F^{-\mathrm{T}} \nabla \pi^0 = f && \text{in } \Omega_1 \\
&\mathrm{div}\left(F^{-1} u^0 \right) = 0 && \text{in } \Omega_1 \\
&u^0 = 0 && \text{on } \partial\Omega_1 \setminus \left(\{x_1 = 0\} \cup \{x_1 = L\} \right) \\
&u^0, \pi^0 && \text{are } L\text{-periodic in } x_1
\end{aligned}$$

There exists a unique solution in $u^0 \in H^1(\Omega_1)^2$, $\pi^0 \in L^2(\Omega_1)/\mathbb{R}$ by the results for the transformed Stokes equation. By regularity results (see, e.g., [11]), this solution is smooth for smooth f. We extend the velocity u^0 by 0 in Ω_2 and the pressure to a pressure $\tilde{\pi}^0$, defined as the Darcy pressure in Ω_2 given by:

$$\begin{aligned}
&\mathrm{div}\left(F^{-1} A \left(f - F^{-\mathrm{T}} \nabla \tilde{\pi}^0 \right) \right) = 0 && \text{in } \Omega_2 \\
&A \left(f - F^{-T} \nabla \tilde{\pi}^0 \right) \cdot F^{-\mathrm{T}} e_2 = 0 && \text{on } (0, L) \times \{-K\} \\
&\tilde{\pi}^0 = \pi^0 + C_\omega^{\mathrm{bl}} && \text{on } \Sigma \\
&\tilde{\pi}^0 && \text{is } L\text{-periodic in } x_1
\end{aligned}$$

The term $F^{-1} F^{-\mathrm{T}} \nabla u^0 e_2$ prevents better estimates in the weak formulation of the velocity difference $u^\varepsilon - u^0$ and the pressure difference $p^\varepsilon - \tilde{\pi}^0$. That is why we construct a correction for this term: consider the following parameter-dependent boundary layer functions $(\beta^{\mathrm{bl}}, \omega^{\mathrm{bl}})$ satisfying:

$$-\operatorname{div}_y\big(F^{-1}(x)F^{-\mathrm{T}}(x)\nabla_y\beta^{\mathrm{bl}}(x,y)\big)+F^{-\mathrm{T}}(x)\nabla_y\omega^{\mathrm{bl}}(x,y)=0 \quad \text{in } \Omega\times Z \tag{6a}$$

$$\operatorname{div}_y\big(F^{-1}(x)\beta^{\mathrm{bl}}(x,y)\big)=0 \quad \text{in } \Omega\times Z \tag{6b}$$

$$\big[\beta^{\mathrm{bl}}(x,y)\big]_S=0 \quad \text{on } \Omega\times S \tag{6c}$$

$$\big[\big(F^{-1}(x)F^{-\mathrm{T}}(x)\nabla_y\beta^{\mathrm{bl}}(x,y)-F^{-1}(x)\omega^{\mathrm{bl}}(x,y)\big)e_2\big]_S = F^{-1}(x)F^{-\mathrm{T}}(x)\nabla u^0(x)e_2 \quad \text{on } \Omega\times S \tag{6d}$$

$$\beta^{\mathrm{bl}}(x,y)=0 \quad \text{on } \Omega\times\bigcup_{k=1}^{\infty}\left\{\partial Y_S-\begin{pmatrix}0\\k\end{pmatrix}\right\} \tag{6e}$$

$$\beta^{\mathrm{bl}}(x,\cdot),\omega^{\mathrm{bl}}(x,\cdot) \quad \text{are 1-periodic in } y_1 \tag{6f}$$

and define $\beta^{\mathrm{bl},\varepsilon}(x)=\varepsilon\beta^{\mathrm{bl}}(x,\frac{x}{\varepsilon})$ as well as $\omega^{\mathrm{bl},\varepsilon}(x)=\omega^{\mathrm{bl}}(x,\frac{x}{\varepsilon})$. Here $[h]_S:=h|_{Z^+}-h|_{Z^-}$ denotes the jump of the function h across the boundary S. By the theory of the boundary layer functions, there exist constants $C^{\mathrm{bl}}(x)\in\mathbb{R}^2$, $C^{\mathrm{bl}}_{\omega}(x)\in\mathbb{R}$ such that, for $q\geqslant 1$:

$$\frac{1}{\varepsilon}\big\|\beta^{\mathrm{bl},\varepsilon}-\varepsilon C^{\mathrm{bl}}(x)H(x_2)\big\|_{L^q(\Omega)^2}+\big\|\omega^{\mathrm{bl},\varepsilon}-C^{\mathrm{bl}}_{\omega}(x)H(x_2)\big\|_{L^q(\Omega)}+\big\|\nabla_y\beta^{\mathrm{bl},\varepsilon}\big\|_{L^q(\Omega)^4} + \left\|\nabla_x\left(\beta^{\mathrm{bl}}\left(x,\frac{x}{\varepsilon}\right)-H(x_2)C^{\mathrm{bl}}(x)\right)\right\|_{L^q(\Omega)^4}+\left\|\operatorname{div}_x\left(F^{-1}F^{-T}\nabla_y\beta^{\mathrm{bl}}\left(x,\frac{x}{\varepsilon}\right)\right)\right\|_{L^q(\Omega)^2}\leqslant C\varepsilon^{\frac{1}{q}}$$

Here H denotes the Heaviside function. This correction introduces problems due to the stabilisation towards C^{bl}. Therefore we define the following counterflow:

$$\begin{aligned}
&-\operatorname{div}\big(F^{-1}(x)F^{-\mathrm{T}}(x)\nabla u^{\sigma}(x)\big)+F^{-\mathrm{T}}(x)\nabla\pi^{\sigma}(x)=0 && \text{in } \Omega_1\\
&\operatorname{div}\big(F^{-1}(x)u^{\sigma}(x)\big)=0 && \text{in } \Omega_1\\
&u^{\sigma}(x)=0 && \text{on } (0,L)\times\{h\}\\
&u^{\sigma}(x)=C^{\mathrm{bl}}(x) && \text{on } \Sigma\\
&u^{\sigma},\pi^{\sigma} && \text{are } L\text{-periodic in } x_1
\end{aligned}$$

Since $\int_{\Sigma}C^{\mathrm{bl}}(x)\cdot F^{-T}(x)e_2\,\mathrm{d}x=0$, there exist a unique velocity u^{σ} and a pressure π^{σ}, unique up to constants. Define as a first approximation:

$$\begin{aligned}
\mathcal{U}^{\varepsilon}&:=u^{\varepsilon}-u^0+\big(\beta^{\mathrm{bl},\varepsilon}-\varepsilon C^{\mathrm{bl}}H(x_2)\big)+\varepsilon u^{\sigma}H(x_2)\\
\mathcal{P}^{\varepsilon}&:=p^{\varepsilon}-\pi^0H(x_2)-\tilde{\pi}^0H(-x_2)+\big(\omega^{\mathrm{bl},\varepsilon}-C^{\mathrm{bl}}_{\omega}H(x_2)\big)+\varepsilon\pi^{\sigma}H(x_2)
\end{aligned}$$

Upon considering the weak formulation of $(\mathcal{U}^{\varepsilon},\mathcal{P}^{\varepsilon})$, one finds that these corrections above are not enough to derive reasonable estimates; therefore, higher order correctors (having a prefactor of ε^2 or more) are required:

- first, another boundary layer function is constructed that corrects the term $\operatorname{div}_x(F^{-1}(x)F^{-T}(x)\nabla_y\beta^{\mathrm{bl}}(x,y))$; this is accompanied by a corresponding counterflow;
- then, the term $F^{-\mathrm{T}}(x)\nabla_x(\omega^{\mathrm{bl}}(x,y)-H(x_2)C^{\mathrm{bl}}_{\omega}(x))$ stemming from the pressure is corrected in a similar fashion;
- finally, the divergence is corrected such that the final velocity $\mathcal{U}^{\varepsilon}_0$ satisfies $\operatorname{div}(F^{-1}\mathcal{U}^{\varepsilon}_0)=0$.

Details of these corrections together with existence and uniqueness results as well as decay estimates will be presented in forthcoming publications. Finally, one arrives at the following estimates.

Proposition 4.1. *For the corrected functions, it holds*:

$$\varepsilon\big\|\nabla\mathcal{P}^{\varepsilon}\big\|_{H^{-1}(\Omega^{\varepsilon})^2}+\varepsilon\big\|\nabla\mathcal{U}^{\varepsilon}\big\|_{L^2(\Omega^{\varepsilon})^4}+\big\|\mathcal{U}^{\varepsilon}\big\|_{L^2(\Omega_2^{\varepsilon})^2}+\varepsilon^{\frac{1}{2}}\big\|\mathcal{U}^{\varepsilon}\big\|_{L^2(\Sigma)^2}\leqslant C\varepsilon^2$$

as well as

$$\varepsilon\big\|\mathcal{P}^{\varepsilon}\big\|_{L^2(\Omega_1)}+\big\|\mathcal{U}^{\varepsilon}\big\|_{H^{\frac{1}{2}}(\Omega_1)^2}\leqslant C\varepsilon^{\frac{3}{2}}$$

Now the estimates of Theorem 3.1 follow by using the fact that $u^{\varepsilon}-u^{\mathrm{eff}}=\mathcal{U}^{\varepsilon}-(\beta^{\mathrm{bl},\varepsilon}-\varepsilon H(x_2)C^{\mathrm{bl}})$, the estimates above and the theory of very weak solutions for the transformed Stokes equation. Theorem 3.2 follows by showing that one can approximate the solution of Eq. (4) by the corresponding problem, replacing p^{eff} in (4c) by π^0 and using the two scale convergence towards the transformed Darcy law.

5. Final comments

Our results show that the following behaviour of a free fluid in contact with a flow in a curved porous medium can be expected for low Reynolds number flows: in the free fluid domain, the velocity and pressure are given by the Stokes equation. In the porous medium, the flow is pressure driven and given by Darcy's law. At the interface, the velocity is given with the help of the decay function $C^{\mathrm{bl}}(x)$ of an auxiliary boundary layer problem. This depends on parameters from the geometry of the interface. In tangential direction, a jump between the velocities and pressures occurs. In transformed normal direction $C^{\mathrm{bl}}(x) \cdot F^{-\mathrm{T}}(x)e_2 = 0$ holds, which is an approximation of continuity of the velocity in that direction.

Therefore, we showed that for curved interfaces, a generalisation of the boundary condition of Beavers and Joseph is valid. It has to incorporate effects stemming from the geometry of the fluid–porous interface. The estimate on the right-hand side of (5) corresponds to a generalised rigorous version of (1). Note, however, that contrary to [4] and [5], we are not able to write the function $C^{\mathrm{bl}}(x)$ as a product containing a factor of ∇u^0. The latter velocity appears instead in the equations for β^{bl}, see (6d). For the choice $g \equiv 0$, the matrix F equals the identity matrix, and the constants and auxiliary problems defined in this note match those given in [5]. Thus in the case of a planar boundary, we recover the boundary conditions derived by Marciniak-Czochra and Mikelić.

We can compare our results with those obtained by Beavers, Joseph and Saffman in [1] and [2]. They proposed the continuity of the normal velocity together with a condition of the form (1). The jump condition in Section 3 does not have this specific form; however, by looking at Eq. (6d), one sees that the gradient of the velocity in the free fluid still influences the jump of the velocity, albeit in a more complex way. In papers [12] and [13], Levi, Ene and Sanchez-Palencia also considered boundary effects. They distinguished the case when the pressure gradient on the side of the porous medium is normal to the porous surface or not. In the latter case, using a specific boundary layer approach, it was found that the pressure is continuous across the interface. In the former case, they deduced that the velocity is continuous in normal direction and the pressure is constant on the interface. However, in these papers, no body force was considered, and some additional assumptions on the pressure apply.

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