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A Riemannian approach to strain measures in nonlinear elasticity

Une approche riemannienne vers des mesures des deformation en élasticité non linéaire

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ABSTRACT

The isotropic Hencky strain energy appears naturally as a distance measure of the deformation gradient to the set $SO(n)$ of rigid rotations in the canonical left-invariant Riemannian metric on the general linear group $GL(n)$. Objectivity requires the Riemannian metric to be left-$GL(n)$-invariant, isotropy requires the Riemannian metric to be right-$O(n)$-invariant. The latter two conditions are only satisfied for a three-parameter family of Riemannian metrics on the tangent space of $GL(n)$. Surprisingly, the final result is basically independent of the chosen parameters.
In deriving the result, geodesics on $GL(n)$ have to be parameterized and a novel minimization problem, involving the matrix logarithm for non-symmetric arguments, has to be solved.

RÉSUMÉ

L'énergie isotrope de Hencky est une mesure naturelle de la distance du gradient de déformation à l'ensemble des rotations rigides $SO(n)$ dans la métrique riemanienne canonique du groupe linéaire $GL(n)$. Le principe d'indifférence matérielle exige que la métrique soit $GL(n)$-invariante à gauche, et l'isotropie implique son invariance à droite par $O(n)$. Ces deux conditions sont uniquement satisfaites par une famille à trois paramètres de métriques riemaniennes sur l'espace tangent à $GL(n)$. On note cependant que le résultat final se révèle, en essence, indépendant des paramètres choisis. Pour obtenir ce résultat, on effectue une paramétrisation des géodésiques de $GL(n)$ et l'on résout un problème de minimisation qui fait intervenir le logarithme de matrices non symétriques.

1. Introduction

For the deformation gradient $F = \nabla\varphi \in GL^+(n)$ let $U = \sqrt{F^T F}$ be the symmetric right Biot-stretch tensor. We show that the isotropic Hencky strain energy, which was introduced by H. Hencky in 1928 [1] and is defined on the logarithmic strain tensor $\log U$ via

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$$W(F) = \mu \| \operatorname{dev} \log U \|^2 + \frac{\kappa}{2} \big[\operatorname{tr}(\log U) \big]^2 = \mu \| \operatorname{dev} \log U \|^2 + \frac{\kappa}{2} (\log \det F)^2 \tag{1}$$

measures the geodesic distance of F to the group of rotations $\mathrm{SO}(n)$ where $\mathrm{GL}(n)$ is viewed as a Riemannian manifold endowed with a left-invariant metric that is also right $\mathrm{O}(n)$-invariant (isotropic), and where the coefficients $\mu, \kappa > 0$ correspond to the shear modulus and the bulk modulus, respectively. Furthermore, the results provide yet another characterization of the polar decomposition $F = RU$, $R \in \mathrm{SO}(n)$, $U \in \mathrm{PSym}(n)$, since U also provides the minimal Euclidean distance to $\mathrm{SO}(n)$, i.e. [2],

$$\operatorname{dist}^2_{\text{euclid}}\big(F, \mathrm{SO}(n)\big) := \min_{Q \in \mathrm{SO}(n)} \operatorname{dist}^2_{\text{euclid}}(F, Q) = \min_{Q \in \mathrm{SO}(n)} \| F - Q \|^2 = \| F - R \|^2 = \| U - \mathbb{1} \|^2 \tag{2}$$

where the Euclidean distance $\operatorname{dist}^2_{\text{euclid}}(X, Y) := \| X - Y \|^2$ is the length of the line segment joining X and Y in $\mathbb{R}^{n^2}$, $\mathbb{1} \in \mathrm{GL}^+(n)$ is the identity and $\| X \| = \sqrt{\operatorname{tr}(X^T X)}$ denotes the Frobenius matrix norm here and henceforth. For both the Euclidean and the geodesic distance, the orthogonal factor $R = \operatorname{polar}(F)$ in the polar decomposition of F is the nearest rotation to F.

2. Strain tensors in linear and nonlinear elasticity

We consider an elastic body which in a reference configuration occupies the bounded domain $\Omega \subset \mathbb{R}^3$. Deformations of the body are prescribed by mappings $\varphi : \Omega \to \mathbb{R}^3$, where $\varphi(x)$ denotes the deformed position of the material point $x \in \Omega$. Central to elasticity theory is the notion of strain, which is a tensor depending on the deformation such that vanishing strain implies that the body Ω has been moved rigidly in space. Various such tensors exist, e.g. the *Green strain tensor* $\frac{1}{2}(U^2 - \mathbb{1})$, the *generalized Green strain tensor* $\frac{1}{m}(U^m - \mathbb{1})$, where m is a nonzero integer, and the *Hencky* (*or logarithmic*) *strain tensor* $\log U$.

In linearized elasticity, one considers $\varphi(x) = x + u(x)$, where $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ is the displacement. The classical linearized strain tensor is $\varepsilon = \operatorname{sym} \nabla u$. It appears through a matrix-nearness problem in the Euclidean distance

$$\operatorname{dist}^2_{\text{euclid}}\big(\nabla u, \mathfrak{so}(3)\big) := \min_{W \in \mathfrak{so}(3)} \| \nabla u - W \|^2 = \| \operatorname{sym} \nabla u \|^2 \tag{3}$$

where $\mathfrak{so}(3)$ denotes the set of all skew symmetric matrices in $\mathbb{R}^{3\times 3}$. Indeed, $\operatorname{sym} \nabla u$ qualifies as a linearized strain tensor: if $\operatorname{dist}^2_{\text{euclid}}(\nabla u, \mathfrak{so}(3)) = 0$ then $u(x) = \widehat{W}.x + \widehat{b}$ is a linearized rigid movement. This is the case since

$$\operatorname{dist}^2_{\text{euclid}}\big(\nabla u(x), \mathfrak{so}(3)\big) = 0 \quad \Rightarrow \quad \nabla u(x) = W(x) \in \mathfrak{so}(3) \tag{4}$$

and $0 = \operatorname{Curl} \nabla u(x) = \operatorname{Curl} W(x)$ implies that $W(x)$ is constant, see [3]. In nonlinear elasticity theory, one assumes that $\nabla \varphi \in \mathrm{GL}^+(3)$ (no self-interpenetration of matter) and may consider the matrix nearness problem

$$\operatorname{dist}^2_{\text{euclid}}\big(\nabla \varphi, \mathrm{SO}(3)\big) := \min_{Q \in \mathrm{SO}(3)} \| \nabla \varphi - Q \|^2 = \min_{Q \in \mathrm{SO}(3)} \big\| Q^T \nabla \varphi - \mathbb{1} \big\|^2 = \Big\| \sqrt{\nabla \varphi^T \nabla \varphi} - \mathbb{1} \Big\|^2, \tag{5}$$

where the last equality is due to (2). Indeed, the Biot strain tensor $\sqrt{\nabla \varphi^T \nabla \varphi} - \mathbb{1}$ qualifies as a nonlinear strain tensor: if $\operatorname{dist}^2_{\text{euclid}}(\nabla \varphi, \mathrm{SO}(3)) = 0$ then $\varphi(x) = \widehat{Q}.x + \widehat{b}$ is a rigid movement. This is the case since

$$\operatorname{dist}^2_{\text{euclid}}\big(\nabla \varphi, \mathrm{SO}(3)\big) = 0 \quad \Rightarrow \quad \nabla \varphi(x) = Q(x) \in \mathrm{SO}(3) \tag{6}$$

and $0 = \operatorname{Curl} \nabla \varphi(x) = \operatorname{Curl} Q(x)$ implies that $Q(x)$ is constant, see [3].

In geometrically nonlinear, physically linear isotropic elasticity, the formulation of a boundary value problem of place may now be based on minimizing the quadratic Biot strain energy

$$\mathcal{E}(\varphi) = \int_{\Omega} \mu \Big\| \operatorname{dev}\Big[\sqrt{\nabla \varphi^T \nabla \varphi} - \mathbb{1}\Big] \Big\|^2 + \frac{\kappa}{2} \Big(\operatorname{tr} \sqrt{\nabla \varphi^T \nabla \varphi} - \mathbb{1} \Big)^2 \, \mathrm{d}x, \quad \varphi|_{\Gamma_D} = \varphi_0, \tag{7}$$

where $\mu, \kappa > 0$ are the shear modulus and bulk modulus, respectively.

However, since the Euclidean distance in (5) is an arbitrary choice, novel approaches in nonlinear elasticity theory aim at putting more geometry (i.e. respecting the group structure of the deformation mappings) into the description of the strain a material endures. In our context, it is now natural to consider a strain measure induced by the geodesic distances stemming from choices for the Riemannian structure respecting also the algebraic group structure of $\mathrm{GL}^+(n)$, which we introduce next.

3. Left invariant Riemannian metrics on GL(n)

Viewing $\mathrm{GL}(n)$ as a Riemannian manifold endowed with a left invariant metric:

$$g_H : T_H \,\mathrm{GL}(n) \times T_H \,\mathrm{GL}(n) \to \mathbb{R} : \quad g_H(X, Y) = \big\langle H^{-1} X, H^{-1} Y \big\rangle, \quad H \in \mathrm{GL}(n), \tag{8}$$

for a suitable inner product $\langle\cdot,\cdot\rangle$ on the tangent space $T_{\mathbb{1}}\,\mathrm{GL}(n)=\mathfrak{gl}(n)=\mathbb{R}^{n\times n}$ at the identity $\mathbb{1}$, the distance between $F,P\in\mathrm{GL}^+(n)$ can be measured along sufficiently smooth curves. We denote by

$$\mathcal{A}=\left\{\gamma\in C^0\big([0,1];\mathrm{GL}^+(n)\big)\;\middle|\;\gamma\text{ piecewise differentiable},\;\gamma(0)=F,\;\gamma(1)=P\right\}\tag{9}$$

the admissible set of curves connecting F and P, and by $L(\gamma)=\int_0^1\sqrt{g_{\gamma(s)}(\dot\gamma(s),\dot\gamma(s))}\,\mathrm{ds}$ the length of $\gamma\in\mathcal{A}$. Then the geodesic distance

$$\mathrm{dist}_{\mathrm{geod}}(F,P)=\inf_{\gamma\in\mathcal{A}}L(\gamma)\tag{10}$$

defines a metric on $\mathrm{GL}^+(n)$. While it is generally difficult to explicitly compute this distance or to find length minimizing curves, it can be shown [4,5] that if the Riemannian metric is defined by an inner product of the form

$$\begin{aligned}
&\langle X,Y\rangle=\langle X,Y\rangle_{\mu,\mu_c,\kappa}:=\mu\langle\operatorname{dev}\operatorname{sym}X,\operatorname{dev}\operatorname{sym}Y\rangle_{n\times n}+\mu_c\langle\operatorname{skew}X,\operatorname{skew}Y\rangle_{n\times n}+\frac{\kappa}{2}\operatorname{tr}X\operatorname{tr}Y,\\
&\|X\|^2_{\mu,\mu_c,\kappa}:=\langle X,X\rangle_{\mu,\mu_c,\kappa}=\mu\|\operatorname{dev}\operatorname{sym}X\|^2+\mu_c\|\operatorname{skew}X\|^2+\frac{\kappa}{2}[\operatorname{tr}X]^2,\quad\mu,\mu_c,\kappa>0,\\
&\operatorname{dev}X:=X-\frac{1}{n}\operatorname{tr}X\cdot\mathbb{1},\quad\mu_c\text{ denoting the }\textit{spin modulus},
\end{aligned}\tag{11}$$

which is the case if and only if the metric g is right invariant under $\mathrm{O}(n)$ [4], then every geodesic γ connecting F and P is of the form

$$\gamma(t)=F\exp\left(t\left(\operatorname{sym}\xi-\frac{\mu_c}{\mu}\operatorname{skew}\xi\right)\right)\exp\left(t\left(1+\frac{\mu_c}{\mu}\right)\operatorname{skew}\xi\right)\tag{12}$$

for some $\xi\in\mathfrak{gl}(n)$, where $\exp:\mathfrak{gl}(n)\to\mathrm{GL}^+(n)$ denotes the matrix exponential, $\operatorname{sym}\xi=\frac{1}{2}(\xi+\xi^{\mathrm{T}})$ the symmetric part and $\operatorname{skew}\xi=\frac{1}{2}(\xi-\xi^{\mathrm{T}})$ the skew symmetric part of ξ.

Now, according to the classical Hopf–Rinow theorem of differential geometry, there exists a length minimizing geodesic in $\mathcal{A}$ for all $F,P\in\mathrm{GL}^+(n)$. To obtain such a minimizer γ (and thus the distance $\mathrm{dist}_{\mathrm{geod}}(F,P)=L(\gamma)$), it therefore remains to find $\xi\in\mathfrak{gl}(n)$ with

$$P=\gamma(1)=F\exp\left(\operatorname{sym}\xi-\frac{\mu_c}{\mu}\operatorname{skew}\xi\right)\exp\left(\left(1+\frac{\mu_c}{\mu}\right)\operatorname{skew}\xi\right).\tag{13}$$

The existence of such a ξ is clear from the above.

4. The geodesic distance to SO(n)

Although no closed form solution to (13) is known, the equation can be used to obtain a lower bound[1]

$$\mathrm{dist}^2_{\mathrm{geod}}\big(F,\mathrm{SO}(n)\big)=\min_{Q\in\mathrm{SO}(n)}\mathrm{dist}^2_{\mathrm{geod}}(F,Q)\geqslant\min_{Q\in\mathrm{SO}(n)}\big\|\mathrm{Log}\big(Q^TF\big)\big\|^2_{\mu,\mu_c,\kappa}\tag{14}$$

for the distance of $F\in\mathrm{GL}^+(n)$ to $\mathrm{SO}(n)$, as well as a simple upper bound

$$\begin{aligned}
\mathrm{dist}^2_{\mathrm{geod}}\big(F,\mathrm{SO}(n)\big)&\leqslant\mathrm{dist}^2_{\mathrm{geod}}\big(F,\operatorname{polar}(F)\big)\\
&\leqslant\big\|\log\big(\operatorname{polar}(F)^TF\big)\big\|^2_{\mu,\mu_c,\kappa}=\mu\big\|\operatorname{dev}\log(U)\big\|^2+\frac{\kappa}{2}\big[\operatorname{tr}(\log U)\big]^2
\end{aligned}\tag{15}$$

where $F=RU$, $R=\operatorname{polar}(F)\in\mathrm{SO}(n)$, $U=\sqrt{F^TF}\in\mathrm{PSym}(n)$ denotes the polar decomposition of F. Finally, we can use a surprising optimality result recently proved by Neff et al. [6]:

Theorem 4.1. *Let $\|\,.\,\|$ be the Frobenius matrix norm on $\mathfrak{gl}(n)$, $F\in\mathrm{GL}^+(n)$. Then the minimum*

$$\min_{Q\in\mathrm{SO}(n)}\big\|\mathrm{Log}\big(Q^TF\big)\big\|^2=\big\|\log\big(\operatorname{polar}(F)^TF\big)\big\|^2=\big\|\log\big(\sqrt{F^TF}\big)\big\|^2=\big\|\log(U)\big\|^2\tag{16}$$

is uniquely attained at $Q=\operatorname{polar}(F)$.

An extension of Theorem 4.1, combined with (11), (14) and (15), yields our main result [7]:

[1] We denote by log the principal matrix logarithm, while the expression Log is used to indicate that the infimum is taken over the whole inverse image under exp, i.e. $\min_{Q\in\mathrm{SO}(n)}\|\mathrm{Log}(Q^TF)\|^2_{\mu,\mu_c,\kappa}=\min\{\|\xi\|^2_{\mu,\mu_c,\kappa}:\ \xi\in\mathfrak{gl}(n),\exp(\xi)=QF\}$.

Theorem 4.2. *Let g be a left invariant Riemannian metric on* $\mathrm{GL}(n)$ *that is also right invariant under* $\mathrm{O}(n)$*, and let* $F \in \mathrm{GL}^+(n)$*. Then*:

$$\operatorname{dist}^2_{\mathrm{geod}}\big(F, \mathrm{SO}(n)\big) = \operatorname{dist}^2_{\mathrm{geod}}\big(F, \operatorname{polar}(F)\big) = \mu \left\| \operatorname{dev} \log(U) \right\|^2 + \frac{\kappa}{2} \left[\operatorname{tr}(\log U)\right]^2. \tag{17}$$

Thus the geodesic distance of the deformation gradient F to $\mathrm{SO}(n)$ *is the isotropic Hencky strain energy of F. In particular, the result is independent of the spin modulus $\mu_c > 0$.*

Furthermore, for $\mu_c = 0$ (in which case $\operatorname{dist}_{\mathrm{geod}}$ defines only a pseudometric on $\mathrm{GL}^+(n)$), Theorem 4.2 still holds.

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