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High-frequency cell vibrations and spatial skin effect in thick cascade junction with heavy concentrated masses

Vibrations cellulaires de haute fréquence et effet spatial « peau » dans une jonction cascade épaisse avec des masses lourdes concentrées

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ABSTRACT

We study the asymptotic behavior of eigenvalues and eigenfunctions of the Laplacian in a 2D thick cascade junction with heavy concentrated masses. We present two-term asymptotic approximations, as $\varepsilon \to 0$, for the eigenelements in the case of "slightly heavy", "moderate heavy", and "super heavy" concentrated masses. Asymptotics of high-frequency cell-vibrations are found as well.

RÉSUMÉ

Nous étudions le comportement asymptotique des valeurs et fonctions propres du laplacien dans une jonction cascade épaisse bidimensionnelle, avec des masses lourdes concentrées. Si $\varepsilon \to 0$, nous présentons des approximations asymptotiques en deux termes pour les éléments propres dans les cas des masses concentrées « peu lourdes », « modérément lourdes » et « super-lourdes ». L'analyse asymptotique pour les vibrations à haute fréquence cellulaire est aussi trouvée.

1. Introduction

In this paper, we present the results obtained for a spectral problem with heavy concentrated masses in a *thick cascade junction*. Vibrating systems with a concentration of masses on a small set of diameter $\mathcal{O}(\varepsilon)$ have been studied for a long time. It was experimentally established that such concentration leads to the big reduction of the main frequencies and to the large localization of vibrations near concentrated masses. New impulse in this research was given by E. Sánchez-Palencia in

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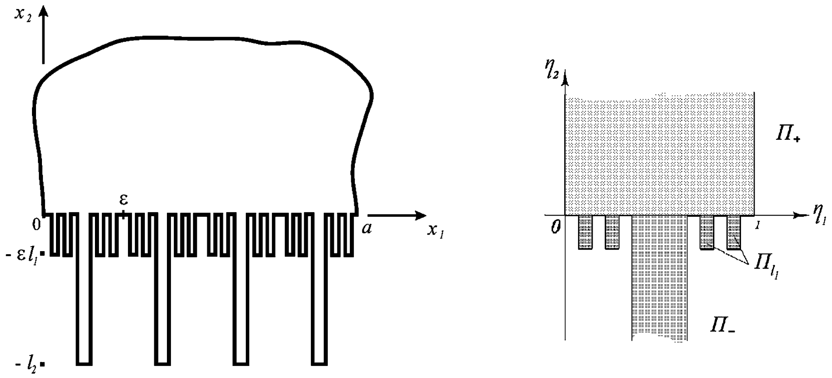

Fig. 1. Thick cascade junction Ω_ε (left) and cell of periodicity Π (right).

the paper [1], in which the effect of local vibrations was mathematically described. After this paper, many articles appeared. The reader can find a widely presented bibliography on spectral problems with concentrated masses in [2–11].

1.1. Statement of the problem

Let a, b_1, b_2, h_1, h_2 be positive numbers, $0 < b_1 < b_2 < \frac{1}{2}$, $0 < b_1 - \frac{h_1}{2}$, $b_1 + \frac{h_1}{2} < b_2 - \frac{h_1}{2}$, $b_2 + \frac{h_1}{2} < \frac{1}{2} - \frac{h_2}{2}$.

Let Ω_0 be a bounded domain in $\mathbb{R}^2$ with the Lipschitz boundary $\partial\Omega_0$ and $\Omega_0 \subset \{x := (x_1, x_2) \in \mathbb{R}^2\colon\ x_2 > 0\}$. Let $\partial\Omega_0$ contain the segment $I_0 = \{x\colon\ x_1 \in [0, a],\ x_2 = 0\}$. We also assume that there exists a positive number δ_0 such that $\Omega_0 \cap \{x\colon\ 0 < x_2 < \delta_0\} = \{x\colon\ x_1 \in (0, a),\ x_2 \in (0, \delta_0)\}$.

We divide $[0, a]$ into segments $[\varepsilon j, \varepsilon(j + 1)]$, $j = 0, \dots, N - 1$, $N \in \mathbb{N}$; $\varepsilon = a/N$ is a small discrete parameter.

A model *thick cascade junction* Ω_ε (see Fig. 1) consists of the junction's body Ω_0 and a large number of thin rods $G_j^{(1)}(d_k, \varepsilon) = \{x \in \mathbb{R}^2\colon\ |x_1 - \varepsilon(j + d_k)| < \frac{\varepsilon h_1}{2},\ x_2 \in (-\varepsilon l_1, 0]\}$, $k = 1, \dots, 4$, $G_j^{(2)}(\varepsilon) = \{x \in \mathbb{R}^2\colon\ |x_1 - \varepsilon(j + \frac{1}{2})| < \frac{\varepsilon h_2}{2},\ x_2 \in (-l_2, 0]\}$, $j = 0, \dots, N - 1$, where $d_1 = b_1$, $d_2 = b_2$, $d_3 = 1 - b_2$, $d_4 = 1 - b_1$, that is $\Omega_\varepsilon = \Omega_0 \cup G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}$, where $G_\varepsilon^{(1)} = \bigcup_{j=0}^{N-1} (\bigcup_{k=1}^{4} G_j^{(1)}(d_k, \varepsilon))$, $G_\varepsilon^{(2)} = \bigcup_{j=0}^{N-1} G_j^{(2)}(\varepsilon)$.

In Ω_ε we consider the following spectral problem

$$\begin{cases} -\Delta_x u(\varepsilon, x) = \lambda(\varepsilon)\rho_\varepsilon(x)u(\varepsilon, x), & x \in \Omega_\varepsilon; \qquad u(\varepsilon, x) = 0, \qquad x \in \Gamma_1 \\ -\partial_\nu u(\varepsilon, x) = 0, & x \in \partial\Omega_\varepsilon \setminus \Gamma_1; \qquad [u]|_{x_2=0} = [\partial_{x_2} u]|_{x_2=0} = 0, \quad x_1 \in Q_\varepsilon \end{cases} \tag{1}$$

Here $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative; the brackets denote the jump of the enclosed quantities; Γ_1 is a curve on $\partial\Omega_0$, located in $\{x\colon\ x_2 > \delta_0\}$; the density $\rho_\varepsilon(x) = 1$, $x \in \Omega_0 \cup G_\varepsilon^{(2)}$ and $\rho_\varepsilon(x) = \varepsilon^{-\alpha}$, $x \in G_\varepsilon^{(1)}$; the parameter $\alpha > 0$; $Q_\varepsilon = Q_\varepsilon^{(1)} \cup Q_\varepsilon^{(2)}$, $Q_\varepsilon^{(i)} = G_\varepsilon^{(i)} \cap \{x\colon\ x_2 = 0\}$, $i = 1, 2$.

Obviously, that for each fixed value of ε there is a sequence of eigenvalues:

$$0 < \lambda_1(\varepsilon) < \lambda_2(\varepsilon) \leqslant \dots \leqslant \lambda_n(\varepsilon) \leqslant \dots \to +\infty \quad \text{as } n \to \infty \tag{2}$$

of problem (1). The corresponding eigenfunctions $\{u_n(\varepsilon, \cdot)\}_{n\in\mathbb{N}}$, which belong to $\mathcal{H}_\varepsilon$, can be orthonormalized as follows $(u_n, u_k)_{L_2(\Omega_0 \cup G_\varepsilon^{(2)})} + \varepsilon^{-\alpha}(u_n, u_k)_{L_2(G_\varepsilon^{(1)})} = \delta_{n,k}$, $\{n, k\} \in \mathbb{N}$. Here and below $\delta_{n,k}$ is the Kronecker delta, $\mathcal{H}_\varepsilon$ is the Sobolev space $\{u \in H^1(\Omega_\varepsilon)\colon\ u|_{\Gamma_1} = 0$ in sense of the trace$\}$ with the scalar product $(u, v)_{\mathcal{H}_\varepsilon} := \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v \, dx$ $\forall u, v \in \mathcal{H}_\varepsilon$.

Our aim is to study the asymptotic behavior of the eigenvalues $\{\lambda_n(\varepsilon)\}_{n\in\mathbb{N}}$ and the eigenfunctions $\{u_n(\varepsilon, \cdot)\}_{n\in\mathbb{N}}$ as $\varepsilon \to 0$ if $\alpha > 1$, to find other limiting points of the spectrum of problem (1), and to describe the corresponding eigenoscillations.

We establish five qualitatively different cases in the asymptotic behavior of eigenvalues and eigenfunctions of problem (1) as $\varepsilon \to 0$, namely the case of "light" concentrated masses ($\alpha \in (0, 1)$), "intermediate" concentrated masses ($\alpha = 1$), and "heavy" concentrated masses ($\alpha \in (1, +\infty)$) that we divide into "slightly heavy" concentrated masses ($\alpha \in (1, 2)$), "moderately heavy" concentrated masses ($\alpha = 2$), and "super heavy" concentrated masses ($\alpha > 2$).

In the cases of "light" and "intermediate" concentrated masses, the perturbation of domain plays the leading role in the asymptotic behavior of the eigenelements. These cases were completely studied in [2], where we proved the low- and high-frequency convergences of the spectrum of problem (1) as $\varepsilon \to 0$, we constructed and justified the leading terms of the asymptotics both for the eigenfunctions and the eigenvalues; in addition, as in the paper [9], we found *pseudovibrations* in problem (1), having a rapidly oscillating character, and in which different rods of the junction vibrate individually, i.e., each rod has its own frequency.

2. Low-frequency vibrations for $\alpha > 1$

For $\alpha > 1$, we discover that all eigenvalues $\{\lambda_n(\varepsilon)\}$ converge to zero with the rate $\varepsilon^{\alpha-1}$, i.e., for any $n \in \mathbb{N}$, $\lambda_n(\varepsilon) \sim \varepsilon^{\alpha-1}\lambda_0^{(n)}$ as $\varepsilon \to 0$. This fact was proved in the following lemma.

Lemma 2.1. *(See [3].) If $\alpha > 1$, then for any fixed $n \in \mathbb{N}$ there exist constants C_0, C_1 and ε_0 such that for all values of ε from the interval $(0, \varepsilon_0)$ the following estimates hold*:

$$0 < \lambda_n(\varepsilon) \leqslant C_0\varepsilon^{\alpha-1}, \qquad \|u_n\|_{\mathcal{H}_\varepsilon} \leqslant C_1\varepsilon^{\frac{\alpha-1}{2}}$$

In addition, there is a positive constant c_0 (depending neither on ε nor on n) such that $0 < c_0\varepsilon^{\alpha-1} \leqslant \lambda_n(\varepsilon)$ for all $n \in \mathbb{N}$ and $\varepsilon \in (0, \varepsilon_0)$.

In the case $\alpha \in (1, 2)$ (slightly heavy concentrated masses), by means of the method of matching of asymptotic expansions (see, for instance, [12]), we have proved (see [3]), that the eigenvibrations $\{u_n(\varepsilon, \cdot)\}_{n\in\mathbb{N}}$ have a new type of skin effect which we call *spatial skin effect*. This means that vibrations of the thin rods from the second class repeat the shape of vibrations of the joint zone in the first term of the asymptotics. This first term is equal to

$$v_0(x) = \begin{cases} v_0^+(x), & x \in \Omega_0 \\ v_0^-(x_1) \equiv v_0^+(x_1, 0), & x \in D_2 = (0, a) \times (-l_2, 0) \end{cases} \tag{3}$$

and it and the corresponding number λ_0 are solutions of the following Steklov problem:

$$\begin{cases} \Delta_x v_0^+(x) = 0, \quad x \in \Omega_0; & \partial_\nu v_0^+(x) = 0, \quad x \in \Gamma_2 \\ v_0^+(x) = 0, \quad x \in \Gamma_1, & \partial_{x_2} v_0^+(x_1, 0) = -4h_1 l_1 \lambda_0 v_0^+(x_1, 0), \quad x_1 \in (0, a) \end{cases} \tag{4}$$

where $\Gamma_2 := \partial\Omega_0 \setminus (\Gamma_1 \cup I_0)$. The number λ_0 is the first term in the asymptotic expansion $\lambda(\varepsilon) \approx \varepsilon^{\alpha-1}(\lambda_0 + \varepsilon^{\alpha-1}\lambda_{\alpha-1} + \cdots)$ for eigenvalues of problem (1). The second term is

$$\lambda_{\alpha-1} = -\frac{\lambda_0}{4h_1 l_1}\left(h_2 l_2 + \int\limits_{\Omega_0} \left(v_0^+\right)^2 \mathrm{d}x\right) \tag{5}$$

The corresponding second term $v_{\alpha-1}^-$ in the asymptotics for eigenfunctions in D_2 depends also on the geometrical parameters h_2 and l_2 and in addition on the variable x_2, which does not take place for the first term v_0^- (see (3)).

The case $\alpha \in (m, m+1)$, $m \in \mathbb{N}$, $m \geqslant 2$. The following theorem holds.

Theorem 2.2. *The leading terms of the asymptotic expansion for eigenvalues $\{\lambda_n(\varepsilon)\}_{n\in\mathbb{N}}$ of problem (1) is as follows (index n is omitted)*:

$$\lambda(\varepsilon) \approx \varepsilon^{\alpha-1}\left(\lambda_0 + \varepsilon^{\alpha-m}\lambda_{\alpha-m} + o\left(\varepsilon^{\alpha-m}\right)\right) \tag{6}$$

and the leading terms of the asymptotic expansion for the corresponding eigenfunctions reads:

$$u(\varepsilon, x) \approx \varepsilon^{\frac{\alpha-1}{2}}\left(v_0(x) + \varepsilon^{\alpha-m}v_{\alpha-m}(x) + o\left(\varepsilon^{\alpha-m}\right)\right) \quad \text{in the Sobolev space } H^1(\Omega_\varepsilon) \tag{7}$$

where λ_0 and v_0 (see (3)) are the n-th eigenvalue and the respective eigenfunction of problem (4), the second term in (6):

$$\lambda_{\alpha-m} = -\frac{1}{4h_1 l_1}\int\limits_{I_0} v_0^+(x_1, 0)\partial_{x_2} v_0^+(x_1, 0)\,\mathrm{d}x \tag{8}$$

and the second term in (7):

$$v_{\alpha-m}(x) = \begin{cases} v_{\alpha-m}^+(x), & x \in \Omega_0 \\ v_{\alpha-m}^-(x_1) = v_{\alpha-m}^+(x_1, 0), & x \in D_2 \end{cases} \tag{9}$$

where $v_{\alpha-m}^+$ is a solution to the following boundary-value problem:

$$\begin{cases} \Delta_x v_{\alpha-m}^+(x) = 0, \quad x \in \Omega_0 \\ \partial_\nu v_{\alpha-m}^+(x) = 0, \quad x \in \Gamma_2; \qquad v_{\alpha-m}^+(x) = 0, \quad x \in \Gamma_1 \\ \partial_{x_2} v_{\alpha-m}^+(x_1, 0) = -4h_1 l_1 \lambda_0 v_{\alpha-m}^+(x_1, 0) - 4h_1 l_1 \lambda_{\alpha-m} v_0^+(x_1, 0) - \partial_{x_2} v_0^+(x_1, 0), \quad x_1 \in (0, a) \end{cases} \tag{10}$$

Remark 1. The second term in (6)—see (8)—depends only on the geometrical characteristics of the thin rods from the first class $G_\varepsilon^{(1)}$, where the concentrated masses are presented. This means the growing influence of concentrated masses on the asymptotics of the eigenvalues of problem (1). For the eigenfunctions of problem (1), we observe the *enhancement* of the spatial skin effect. This means that both the first and second terms of the asymptotics (7) are independent of x_2 in D_2; namely, the first term is the same as v_0 for $\alpha \in (1,2)$ (see (3)) and the second one has the similar form (see (9)).

Proof. Combining the algorithm of constructing asymptotics in thin domains with the methods of homogenization theory, we seek the main terms of the asymptotics for the eigenvalue $\lambda_n(\varepsilon)$ and the eigenfunction $v_n(\varepsilon,\cdot)$ in the form (index n is omitted):

$$\lambda(\varepsilon) \approx \varepsilon^{\alpha-1}\big(\lambda_0 + \varepsilon^{\alpha-m}\lambda_{\alpha-m} + \cdots\big) \tag{11}$$

$$u(\varepsilon,x) \approx \varepsilon^{\frac{\alpha-1}{2}}\big(v_0^+(x) + \varepsilon^{\alpha-m}v_{\alpha-m}^+(x) + \cdots\big) \quad \text{in domain } \Omega_0 \tag{12}$$

$$v(\varepsilon,x) \approx \varepsilon^{\frac{\alpha-1}{2}}\left(v_0^-\left(x_1,x_2,\frac{x_1}{\varepsilon}-j\right) + \varepsilon^{\alpha-m}v_{\alpha-m}^-\left(x_1,x_2,\frac{x_1}{\varepsilon}-j\right) + \cdots\right) \tag{13}$$

in the thin rectangles $G_j^{(2)}, \varepsilon)$ $(j=0,\ldots,N-1$; and in the junction zone of the body and thin rectangles of both classes (which we call internal expansion) the series of the following type:

$$u(\varepsilon,x) \approx \varepsilon^{\frac{\alpha-1}{2}}\left(v_0^+(x_1,0) + \varepsilon^{\alpha-m}v_{\alpha-m}^+(x_1,0) + \cdots + \varepsilon\left(Z_1^{(0)}\left(\frac{x}{\varepsilon}\right)v_0^+(x_1,0) + \sum_{i=1}^{2} Z_1^{(i)}\left(\frac{x}{\varepsilon}\right)\partial_{x_i}v_0^+(x_1,0)\right) + \cdots\right) \tag{14}$$

Under the change of variables $\eta = \frac{x}{\varepsilon}$, the domain Ω_0 transforms to $\{\eta\colon\ \eta_i > 0,\ i=1,2\}$, the thin rectangle $G_0^{(2)}(\varepsilon)$ to the semistrip $\Pi^- = (\frac12 - \frac{h_2}{2}, \frac12 + \frac{h_2}{2}) \times (-\infty, 0]$ and rectangle $G_0^{(1)}(d_k,\varepsilon)$ to the fixed rectangle $\Pi_k = (d_k - \frac{h_1}{2}, d_k + \frac{h_1}{2}) \times (-l_1, 0]$ as $\varepsilon \to 0$.

Taking into account the periodic structure of Ω_ε in a neighborhood of I_0, we take the following cell of periodicity $\Pi = \Pi^- \cup \Pi^+ \cup \Pi_{l_1}$, in which we consider problems for coefficients Z from (14). Here $\Pi^+ = (0,1) \times (0,+\infty)$, $\Pi_{l_1} := \bigcup_{k=1}^{4} \overline{\Pi}_k$ (see Fig. 1).

Then substituting (11) and (12) in problem (1) and collecting terms with equal order of ε, we get equations and boundary conditions for functions $\{v_\gamma^+\}$ in Ω_0 and Γ_1, Γ_2 respectively; substituting the Taylor series for the functions $\{v_\gamma^-\}$ in (13) in a neighborhood of the point $x_1 = \varepsilon(j+\frac12)$ and (11) in (1) instead of $\lambda_n(\varepsilon)$ and $u_n(\varepsilon,\cdot)$ respectively, and collecting terms with equal powers of ε, we deduce problems whose solvability condition are equations for $\{v_\gamma^-\}$ in D_2; substituting the series (14) and (11) in (1) and collecting terms with equal powers of ε, we get problems for $Z_1^{(i)}$, $i=0,1,2$. Obviously, these solutions have to be 1-periodic in η_1. Therefore, we demand the following periodicity conditions:

$$\partial_{\eta_1}^s Z(0,\eta_2) = \partial_{\eta_1}^s Z(1,\eta_2), \quad \eta_2 > 0,\ s=0,1 \tag{15}$$

on the vertical sides of semistrip Π^+. In addition, all these solutions satisfy the Neumann conditions:

$$\partial_{\eta_2} Z(\eta_1,0) = 0, \quad (\eta_1,0) \in \partial\Pi, \qquad \partial_{\eta_2} Z(\eta_1,-l_1) = 0, \quad (\eta_1,-l_1) \in \partial\Pi \tag{16}$$

on the horizontal parts of the boundary of Π.

Denote by $\partial\Pi_\parallel$ the vertical part of $\partial\Pi$ laying in $\{\eta\colon\ \eta_2 < 0\}$.

Thus we get the following problems (to all those problems we must add the respective conditions (15) and (16)):

$$\begin{cases} -\Delta_\eta Z_1^{(0)}(\eta) = \begin{cases} 0, & \eta \in \Pi^+ \cup \Pi^-, \\ \lambda_0, & \eta \in \Pi_{l_1}, \end{cases} \\ \partial_{\eta_1} Z_1^{(0)}(\eta) = 0, \quad \eta \in \partial\Pi_\parallel; \end{cases} \quad \text{and} \quad \begin{cases} -\Delta_\eta Z_1^{(i)}(\eta) = 0, & \eta \in \Pi \\ \partial_{\eta_1} Z_1^{(i)}(\eta) = -\delta_{1i}, & \eta \in \partial\Pi_\parallel,\ i=1,2 \end{cases} \tag{17}$$

The existence and the main asymptotic relations for the solutions of those problems can be obtained from general results about the asymptotic behavior of solutions to elliptic problems in domains with different exits to infinity [13]. However, if a domain, where we consider a boundary-value problem, has some symmetry, then we can define more exactly the asymptotic relations and detect other properties of junction-layer solutions (see Lemma 4.1 and Corollary 4.1 from [14]). Using this approach, we prove the following lemma.

Lemma 2.3. *There exist solutions $Z_1^{(i)} \in H^1_{loc,\eta_2}(\Pi)$, $i = 0, 1, 2$, of the problems (17), which have the following differentiable asymptotics*

$$Z_1^{(0)}(\eta) = \begin{cases} \mathcal{O}\big(\exp(-2\pi\eta_2)\big), & \eta_2 \to +\infty \\ \dfrac{4h_1 l_1 \lambda_0}{h_2}\eta_2 + C_1^{(0)} + \mathcal{O}\big(\exp(\pi h_2^{-1}\eta_2)\big), & \eta_2 \to -\infty \end{cases} \tag{18}$$

$$Z_1^{(1)}(\eta) = \begin{cases} \mathcal{O}\big(\exp(-2\pi\eta_2)\big), & \eta_2 \to +\infty \\ \big(-\eta_1 + \frac{1}{2}\big) + \mathcal{O}\big(\exp(\pi h_2^{-1}\eta_2)\big), & \eta_2 \to -\infty \end{cases} \tag{19}$$

$$Z_1^{(2)}(\eta) = \begin{cases} \eta_2 + \mathcal{O}\big(\exp(-2\pi\eta_2)\big), & \eta_2 \to +\infty \\ \dfrac{\eta_2}{h_2} + C_1^{(2)} + \mathcal{O}\big(\exp(\pi h_2^{-1}\eta_2)\big), & \eta_2 \to -\infty \end{cases} \tag{20}$$

Moreover $Z_1^{(1)}$ is odd in η_1 with respect to $\frac{1}{2}$; $Z_1^{(0)}$ and $Z_1^{(2)}$ are even in η_1 with respect to $\frac{1}{2}$.

Then, applying the method of matching of asymptotic expansions for the series (12), (14) and (13), (14), we derive the conditions on I_0 in problems for functions $\{v_\gamma\}$ (see (4) and (10)).

Then, estimating the discrepancy in the equation and boundary conditions in (1) and applying the scheme proposed in [15], we rigorously verify the constructed asymptotics. □

The asymptotic expansion for eigenvalues $\{\lambda_n(\varepsilon)\}_{n\in\mathbb{N}}$ of problem (1) is as follows (index n is omitted):

$$\lambda(\varepsilon) \approx \varepsilon^{\alpha-1}\big(\lambda_0 + \varepsilon^{\alpha-m}\lambda_{\alpha-m} + \cdots + \varepsilon\lambda_1 + \varepsilon^{\alpha-m+1}\lambda_{\alpha-m+1} + \cdots\big) \tag{21}$$

Moreover, α is the nearest to m, the more terms are between $\varepsilon^{\alpha-m}\lambda_{\alpha-m}$ and $\varepsilon\lambda_1$ in (21). Therefore, we write down "$\cdots$" between $\varepsilon^{\alpha-m}\lambda_{\alpha-m}$ and $\varepsilon\lambda_1$ in (21) and (14). This means that for integer α we cannot use (6) at $\alpha = m$ and it is necessary to reapply a formal procedure for this case.

Thus, for $\alpha = m$, $m \in \mathbb{N}$, $m \geqslant 2$, the following theorem holds.

Theorem 2.4. *The leading terms of the asymptotic expansion for eigenvalues $\{\lambda_n(\varepsilon)\}_{n\in\mathbb{N}}$ of problem (1) are as follows (index n is omitted):*

$$\lambda(\varepsilon) \approx \varepsilon^{\alpha-1}\big(\lambda_0 + \varepsilon\lambda_1 + o(\varepsilon)\big) \tag{22}$$

and the leading terms of the asymptotic expansion for the corresponding eigenfunctions read:

$$u(\varepsilon, x) \approx \varepsilon^{\frac{\alpha-1}{2}}\big(v_0(x) + \varepsilon v_1(x) + o(\varepsilon)\big) \quad \text{in the Sobolev space } H^1(\Omega_\varepsilon) \tag{23}$$

where λ_0 and v_0 (see (3)) are the n-th eigenvalue and the respective eigenfunction of problem (4), the second term if $m = 2$ in (22) reads:

$$\begin{aligned} \lambda_1 = &-\frac{\lambda_0}{4h_1 l_1}\bigg(h_2 l_2 + \int\limits_{\Omega_0} \big(v_0^+\big)^2 \,\mathrm{d}x\bigg) - \frac{\varsigma_{(0,0)}}{4h_1 l_1} - \varsigma_{(2,0)} \int\limits_{I_0} \partial^2_{x_1 x_1} v_0^+(x_1, 0) v_0^+(x_1, 0)\,\mathrm{d}x_1 \\ &- \lambda_0 \int\limits_{\Pi_{l_1}} Z_1^{(2)}(\eta)\,\mathrm{d}\eta \int\limits_{I_0} v_0^+(x_1, 0)\partial_{x_2} v_0^+(x_1, 0)\,\mathrm{d}x_1 \end{aligned} \tag{24}$$

and if $m \geqslant 3$, then:

$$\lambda_1 = -\frac{\varsigma_{(0,0)}}{4h_1 l_1} - \varsigma_{(2,0)} \int\limits_{I_0} \partial^2_{x_1 x_1} v_0^+(x_1, 0) v_0^+(x_1, 0)\,\mathrm{d}x_1 - \lambda_0 \int\limits_{\Pi_{l_1}} Z_1^{(2)}(\eta)\,\mathrm{d}\eta \int\limits_{I_0} v_0^+(x_1, 0)\partial_{x_2} v_0^+(x_1, 0)\,\mathrm{d}x_1 \tag{25}$$

where

$$\varsigma_{(0,0)} = \lambda_0 \int\limits_{\Pi_{l_1}} Z_1^{(0)}(\eta)\,\mathrm{d}\eta, \qquad \varsigma_{(2,0)} = \int\limits_{\Pi_{l_1}\cup\Pi^-} \big(1 + \partial_{\eta_1} Z_1^{(1)}(\eta)\big)\,\mathrm{d}\eta \tag{26}$$

and $Z_1^{(k)}$, $k = 0, 1, 2$, are solutions of (17) with asymptotics (18), (19), (20) respectively.

Proof. In this case we seek the main terms of the asymptotics for the eigenvalue $\lambda_n(\varepsilon)$ and the eigenfunction $u_n(\varepsilon,\cdot)$ in the form (index n is omitted):

$$\Lambda(\varepsilon) \approx \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots \tag{27}$$

$$v(\varepsilon, x) \approx v_0^+(x) + \varepsilon v_1^+(x) + \varepsilon^2 v_2^+(x) + \cdots \quad \text{in domain } \Omega_0 \tag{28}$$

$$v(\varepsilon, x) \approx v_0^-\left(x_1, x_2, \frac{x_1}{\varepsilon} - j\right) + \varepsilon v_1^-\left(x_1, x_2, \frac{x_1}{\varepsilon} - j\right) + \varepsilon^2 v_2^-\left(x_1, x_2, \frac{x_1}{\varepsilon} - j\right) + \cdots \tag{29}$$

in the thin rectangle $G_j^{(2)}(\varepsilon)$ $(j = 0, \ldots, N-1)$; and in the junction zone of the body and thin rectangles of both classes (which we call internal expansion) the series of the following type:

$$v(\varepsilon, x) \approx v_0^+(x_1, 0) + \varepsilon\left(v_1^+(x_1, 0) + Z_1^{(0)}\left(\frac{x}{\varepsilon}\right)v_0^+(x_1, 0) + \sum_{i=1}^{2} Z_1^{(i)}\left(\frac{x}{\varepsilon}\right)\partial_{x_i} v_0^+(x_1, 0)\right) + \cdots \tag{30}$$

Then acting in the same way as in the proof of Theorem 2.2, we complete the proof. □

Remark 2. Comparing formulas for the second terms in the asymptotics for eigenvalues of problem (1) (see (5) for $\alpha \in (1, 2)$, (24) for $\alpha = 2$, and (8) and (25) for $\alpha > 2$), we see the reduction of the influence of the geometry of domain Ω_0 and of the thin rods from the second class $G_\varepsilon^{(2)}$ on the asymptotic behavior of the eigenvalues.

This and the facts mentioned above justify the separation of the "heavy" concentrated masses into "slightly heavy" ($\alpha \in (1, 2)$), "moderate heavy" ($\alpha = 2$), and "super heavy" concentrated masses ($\alpha > 2$).

3. High-frequency cell vibrations

It is known (see for instance [1,4,5]) that for spectral problems with concentrated masses, there exist converging sequences of eigenvalues $\lambda_{n(\varepsilon)}(\varepsilon)$ $(n(\varepsilon) \to +\infty$ as $\varepsilon \to 0)$; the corresponding vibrations are usually called *high-frequency vibrations*.

As for vibrations of fastened membranes with concentrated masses on a small set of diameter $\mathcal{O}(\varepsilon)$ (see for instance [6–8] and reference therein), there exist three qualitatively different cases for such spectral problems: $\alpha < 2$, $\alpha = 2$, $\alpha > 2$. It was proved in these papers that there are two kinds of eigenvibrations: the local vibrations, for which the corresponding eigenfunctions are of order $O(1)$ only in a region near the concentrated masses, and the global vibrations, for which the corresponding eigenfunctions are located on the whole membrane. The local and global vibrations can exist only for $\alpha \geqslant 2$, and the local vibrations are always low-frequency vibrations. Local vibrations are not found in the case $\alpha < 2$. The associated eigenvalues for the local vibrations have the asymptotics

$$\lambda_n(\varepsilon) = \varepsilon^{\alpha-2}\lambda_n + o\left(\varepsilon^{\alpha-2}\right) \tag{31}$$

where λ_n is an eigenvalue of the corresponding spectral local problem. The formula (31) shows the structure of the low-frequency convergence of the spectrum.

In contrast to the results of papers [6,7], we show that there are free-vibrations in problem (1), so-called *high-frequency cell-vibrations*, which correspond to local vibrations of the concentrated masses; they present at each value of the parameter $\alpha \in (0, +\infty)$; and they are always high-frequency vibrations. This kind of high-frequency vibrations is connected with the following *spectral cell-problem*:

$$\begin{cases} -\Delta_\eta \mathcal{Z}(\eta) = \begin{cases} 0, & \eta \in \Pi^+ \cup \Pi^- \\ \Lambda\mathcal{Z}, & \eta \in \Pi_{l_1} \end{cases} \\ \partial_{\eta_1}^s \mathcal{Z}(0, \eta_2) = \partial_{\eta_1}^s \mathcal{Z}(1, \eta_2), \quad \eta_2 > 0,\ s = 0, 1 \\ \partial_{\nu_\eta} \mathcal{Z}(\eta) = 0, \qquad \eta \in \partial\Pi \cap \{\eta\colon\ \eta_2 \leqslant 0\} \end{cases} \tag{32}$$

Let $\widehat{C}_0^\infty(\overline{\Pi})$ be a space of infinitely differentiable functions in $\overline{\Pi}$ that satisfy the periodicity conditions (15) and are finite in η_2, i.e., $\forall v \in \widehat{C}_0^\infty(\overline{\Pi})$ $\exists R > 0$ $\forall \eta \in \overline{\Pi} |\eta_2| \geqslant R\colon\ v(\eta) = 0$. Let $\mathcal{H}$ be the completion of the space $\widehat{C}_0^\infty(\overline{\Pi})$ with respect to the norm $\|u\|_{\mathcal{H}} = (\|\nabla_\eta u\|^2_{L_2(\Pi)} + \|\rho u\|^2_{L_2(\Pi)})^{1/2}$, where $\rho(\eta_2) = (1 + |\eta_2|)^{-1}$, $\eta_2 \in \mathbb{R}$.

A number Λ is an eigenvalue of problem (32) if there exists a function $\mathcal{Z} \in \mathcal{H} \setminus \{0\}$ such that the following integral identity holds:

$$\int_{\Pi} \nabla_\eta \mathcal{Z} \cdot \nabla_\eta v \,\mathrm{d}\eta = \Lambda \int_{\Pi_{l_1}} \mathcal{Z} v \,\mathrm{d}\eta, \quad \forall v \in \mathcal{H} \tag{33}$$

With the help of Hardy's inequality:

$$\int_0^{+\infty} (1+\eta_2)^{-2}\phi^2(\eta_2)\,d\eta_2 \leqslant 4\int_0^{+\infty} |\partial_{\eta_2}\phi|^2\,d\eta_2, \quad \forall \phi \in C^1\big([0,+\infty)\big),\ \phi(0)=0$$

we can show (see for instance Lemma 3.1 in [16]) that problem (32) is equivalent to a spectral problem for some positive, self-adjoint, compact operator. Thus, the eigenvalues of problem (32) form the sequence:

$$0 = \Lambda_0 < \Lambda_1 < \Lambda_2 \leqslant \cdots \leqslant \Lambda_n \leqslant \cdots \to +\infty \quad \text{as } n \to \infty$$

with the classical convention of repeated eigenvalues. The respective sequence of the corresponding eigenfunctions $\{\mathcal{Z}_n\}_{n\in\mathbb{Z}_+} \subset \mathcal{H}$ can be orthonormalized as follows: $\int_{\Pi_{l_1}} \mathcal{Z}_n \mathcal{Z}_k\,d\eta = \delta_{n,k}$, $\{n,k\} \in \mathbb{Z}_+$. Also, it follows from the results of [16, §3.1] that the eigenfunctions have the asymptotics $\mathcal{Z}_n(\eta) = \mathcal{O}(\exp(-2\pi\eta_2))$ as $\eta_2 \to +\infty$ in Π^+, and $\mathcal{Z}_n(\eta) = C_n(h_2) + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2))$ as $\eta_2 \to -\infty$ in $\eta \in \Pi^-$, for $n \in \mathbb{N}$. But now taking into account the harmonicity of $\mathcal{Z}_n(\eta)$ in $\Pi^+ \cup \Pi^-$, we can state that the constant $C_n(h_2) = 0$ and in addition $\mathcal{Z}_n(\eta) = 0$ for $\eta \in \Pi^-$ and there exists $\varrho_0 > 0$ such that for all $\eta \in \Pi^+$, $\eta_2 \geqslant \varrho_0$ we have $\mathcal{Z}_n(\eta) = 0$.

Now let us take any positive eigenvalue Λ of problem (32) and the corresponding eigenfunction $\mathcal{Z}$ that is even in η_1 with respect to $\frac{1}{2}$ (due to the symmetry of the domain Π with respect to the line $\{\eta:\ \eta_1 = \frac{1}{2}\}$, such an eigenfunction always does exist). Then we extend it periodically in the direction $O\eta_1$. Since the eigenfunctions are orthonormalized, we have

$$\left\| \mathcal{Z}\left(\frac{\cdot}{\varepsilon}\right)\right\|_\varepsilon = \sqrt{(\mathcal{Z},\mathcal{Z})_{\mathcal{H}_\varepsilon}} \sim c\Lambda^{\frac{1}{2}} \quad \text{as } \varepsilon \to 0 \tag{34}$$

Substituting $\mathcal{Z}(\frac{\cdot}{\varepsilon})$ and $\varepsilon^{\alpha-2}\Lambda$ in the differential equation of problem (1) instead of $u(\varepsilon,\cdot)$ and $\lambda(\varepsilon)$ respectively and taking into account properties of $\mathcal{Z}$ mentioned above, we get

$$\begin{aligned}
&\Delta_x\left(\mathcal{Z}\left(\frac{x}{\varepsilon}\right)\right) + \varepsilon^{\alpha-2}\Lambda\mathcal{Z}\left(\frac{x}{\varepsilon}\right) = \varepsilon^{\alpha-2}\Lambda\mathcal{Z}\left(\frac{x}{\varepsilon}\right) \quad \text{in } \Omega_0\\
&\Delta_x\left(\mathcal{Z}\left(\frac{x}{\varepsilon}\right)\right) + \varepsilon^{\alpha-2}\Lambda\mathcal{Z}\left(\frac{x}{\varepsilon}\right) = 0 \quad \text{in } G_\varepsilon^{(2)}\\
&\Delta_x\left(\mathcal{Z}\left(\frac{x}{\varepsilon}\right)\right) + \varepsilon^{-\alpha}\varepsilon^{\alpha-2}\Lambda\mathcal{Z}\left(\frac{x}{\varepsilon}\right) = 0 \quad \text{in } G_\varepsilon^{(1)}
\end{aligned}$$

and $\mathcal{Z}(\frac{\cdot}{\varepsilon})$ satisfies all boundary conditions of problem (1). As a result, we have

$$\left(\mathcal{Z}\left(\frac{\cdot}{\varepsilon}\right), v\right)_{\mathcal{H}_\varepsilon} - \varepsilon^{\alpha-2}\Lambda\left(\mathcal{A}_\varepsilon\mathcal{Z}\left(\frac{\cdot}{\varepsilon}\right), v\right)_{\mathcal{H}_\varepsilon} = \varepsilon^{\alpha-2}\Lambda\int_{\Omega_0}\mathcal{Z}\left(\frac{\cdot}{\varepsilon}\right)v\,dx \quad \forall v \in \mathcal{H}_\varepsilon \tag{35}$$

where $\mathcal{A}_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_\varepsilon$ is the corresponding operator to problem (1) and it is defined by the following equality:

$$(\mathcal{A}_\varepsilon u, v)_{\mathcal{H}_\varepsilon} = \int_{\Omega_0\cup G_\varepsilon^{(2)}} uv\,dx + \varepsilon^{-\alpha}\int_{G_\varepsilon^{(1)}} uv\,dx \quad \forall u,v \in \mathcal{H}_\varepsilon$$

Since

$$\begin{aligned}
\left|\int_{\Omega_0}\mathcal{Z}\left(\frac{\cdot}{\varepsilon}\right)v\,dx\right| &= \left|\int_{\Omega_0^\varepsilon}\mathcal{Z}\left(\frac{\cdot}{\varepsilon}\right)v\,dx\right| \leqslant \sqrt{\int_{\Omega_0^\varepsilon}\left|\mathcal{Z}\left(\frac{\cdot}{\varepsilon}\right)\right|^2 dx}\sqrt{\int_{\Omega_0^\varepsilon}|v|^2\,dx}\\
&\leqslant \sqrt{\varepsilon}\sqrt{\int_{\Pi^+}\left|\mathcal{Z}(\eta)\right|^2 d\eta}\,\varepsilon^{\frac{1}{2}-\delta}\|v\|_{H^1(\Omega_0)} \leqslant C_0\varepsilon^{1-\delta}\|v\|_\varepsilon
\end{aligned} \tag{36}$$

(in the last line, we used Lemma 1.5 from [10]; here $\Omega_0^\varepsilon := \Omega_0 \cap \{x:\ x_2 \in (0,\varepsilon\varrho_0)\}$), with the help of the first statement of the Vishik–Lyusternik Lemma 12 [17] and (34), we deduce

$$\min_{n\in\mathbb{N}}\left|\frac{1}{\varepsilon^{\alpha-2}\Lambda} - \frac{1}{\lambda_n(\varepsilon)}\right| \leqslant \left\|\mathcal{Z}\left(\frac{\cdot}{\varepsilon}\right)\right\|_\varepsilon^{-1}\left\|\mathcal{A}_\varepsilon\mathcal{Z}\left(\frac{\cdot}{\varepsilon}\right) - \frac{1}{\varepsilon^{\alpha-2}\Lambda}\mathcal{Z}\left(\frac{\cdot}{\varepsilon}\right)\right\|_\varepsilon \leqslant C_1\varepsilon^{1-\delta} \tag{37}$$

where δ is arbitrary number from the interval $(0,1)$.

Taking into account the second statement of the Vishik–Lyusternik Lemma 12 [17], we prove the following theorem.

Theorem 3.1. *For any positive eigenvalue Λ of problem (32) and for any $\delta \in (0,1)$, there exists an eigenvalue $\lambda_{n(\varepsilon)}(\varepsilon)$ of problem (1) ($n(\varepsilon) \to +\infty$ as $\varepsilon \to 0$), such that*

$$\left| \frac{1}{\varepsilon^{\alpha-2}\Lambda} - \frac{1}{\lambda_{n(\varepsilon)}(\varepsilon)} \right| \leqslant C_1 \varepsilon^{1-\delta} \tag{38}$$

In addition, for any $\delta \in (0, \frac{1}{2})$ there exists a finite linear combination

$$\widetilde{U}_\varepsilon(x) = \sum_{i=0}^{p(\varepsilon)} d_i(\varepsilon) u_{k(\varepsilon)+i}(\varepsilon, x), \quad x \in \Omega_\varepsilon \quad \left(\|\widetilde{u}_\varepsilon\|_\varepsilon^2 = 1 = \sum_{i=0}^{p(\varepsilon)} d_i^2(\varepsilon) \lambda_{k(\varepsilon)+i}(\varepsilon) \right)$$

of eigenfunctions corresponding respectively to all eigenvalues $\lambda^{-1}_{k(\varepsilon)}(\varepsilon), \lambda^{-1}_{k(\varepsilon)+1}(\varepsilon), \ldots, \lambda^{-1}_{k(\varepsilon)+p(\varepsilon)}(\varepsilon)$ of the operator $\mathcal{A}_\varepsilon$ from the segment

$$\left[\frac{1}{\varepsilon^{\alpha-2}\Lambda} - C_1 \varepsilon^\delta, \frac{1}{\varepsilon^{\alpha-2}\Lambda} + C_1 \varepsilon^\delta \right]$$

such that

$$\left\| \frac{\mathcal{Z}(\frac{\cdot}{\varepsilon})}{\|\mathcal{Z}(\frac{\cdot}{\varepsilon})\|_\varepsilon} - \widetilde{U}_\varepsilon \right\|_\varepsilon \leqslant 2\varepsilon^{1-2\delta} \tag{39}$$

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