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Analytical expressions for odd-order anisotropic tensor dimension

Expressions analytiques donnant la dimension d'un tenseur anisotrope d'ordre impair

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According to the symmetries of the matter, the number of coefficients needed to define a tensorial relation varies. It is well known that in linear elasticity the number of generic coefficients varies from 21, for a complete anisotropic material, to 2, in case of isotropy. In a previous contribution, we provided analytical expressions that give the number of generic anisotropic coefficients in any anisotropic system for an even-order tensor. In the present note, we aim at extending the previous results to the case of odd-order tensors. As an illustration, the dimension of any anisotropic system for third-order piezoelectricity tensors and of the fifth-order coupling tensors of Mindlin's strain-gradient elasticity are determined.

R É S U M É

En fonction des symétries qu'un milieu possède, le nombre de coefficients génériques nécessaire à la définition d'une loi tensorielle varie. Dans le contexte de l'élasticité linéaire, si le milieu ne presente aucune symétrie, 21 coefficients élastiques sont génériquement nécessaires, tandis que, dans le cas de l'isotropie, ce nombre se réduit à 2. Dans une précédente note, nous avions dérivé des formules analytiques donnant, dans le cas d'un tenseur pair, le nombre de coefficients génériques nécessaire pour chaque type d'anisotropie. Le but de cette nouvelle contribution est de compléter ces formules, en les étendant au cas des tenseurs impairs. En guise d'illustration, nous calculerons, pour l'ensemble des systèmes d'anisotropie possibles, la dimension des tenseurs piezoélectriques (ordre 3) ainsi que des tenseurs de couplage de la théorie de l'élasticité à gradient (ordre 5).

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1. Introduction

In the multiphysical modeling of anisotropic behaviors [1], it is interesting to have prior information on the general characteristics of the sought model. The knowledge, for a given material anisotropy, of the number of independent coefficients in the tensorial law is interesting since, for example, it determines the number of elementary tests one needs to perform in a micro–macro homogenization procedure [2]. These results are also important to develop strategies for the experimental identification of constitutive laws [3,4]. If, in the case of classical elasticity, these results are well known [5,6], their extension to other situations is not straightforward. However, due to a growing interest in the modeling of size and non-local effects in materials and structures, the need for generalized continuum theories becomes evident [7–9]. This note, following a path already developed in some previous contributions [10–12], aims at providing some tools that can help the modeling of these non-conventional behaviors.

In the present paper, and following a previous one [12] in which only even-order tensors were considered, we provide analytical formulas that give the generic dimension of any anisotropic odd-order tensor.[1] To avoid any misunderstanding, it is worth noting that the present method does not solve the symmetry classification for a physical tensor, but only provides a way to compute, once the classification has been done, the number of generic coefficients for each anisotropic system. The solution of the classification problem can be found in the following references [23,25]. As an illustration, the obtained formula is applied to the space of third-order piezoelectric tensors [13,14], and to the fifth-order coupling tensor of Mindlin's strain-gradient elasticity [16].

This note is organized as follows. In Section 2, basic definitions about symmetries are summed up (see, e.g., [6] for an extended discussion). The decompositions of a tensor space into orthogonal elementary components and the notion of G-invariant space are developed in Section 3, and our main formula is provided there. Finally, physical illustrations are proposed in Section 4.

2. Physical and material symmetries

Hereafter E^3 will be the 3D Euclidean physical space. Let G be a closed subgroup of $\mathrm{O}(3)$, the orthogonal group in 3D, that is the group of isometries of E^3. Let us define a material $\mathcal{M}$ as an open subset of E^3. The set of operations $\mathrm{Q} \in \mathrm{O}(3)$ leaving $\mathcal{M}$ invariant is defined as

$$\mathrm{G}_{\mathcal{M}} = \left\{\mathrm{Q} \in \mathrm{O}(3),\ \mathrm{Q} \star \mathcal{M} = \mathcal{M}\right\}$$

where $\star$ stands for the Q action upon $\mathcal{M}$. This set, denoted $\mathrm{G}_{\mathcal{M}}$, is known as the material symmetry group. Now consider a physical property $\mathcal{P}$ defined on $\mathcal{M}$, the set of operations leaving $\mathcal{P}$ invariant is the physical symmetry group

$$\mathrm{G}_{\mathcal{P}} = \left\{\mathrm{Q} \in \mathrm{O}(3),\ \mathrm{Q} \star \mathcal{P} = \mathcal{P}\right\}$$

$\mathcal{P}$ is described, in the present paper, by an nth-order tensor $\mathrm{T}^{(n)} \in \mathbb{T}^{(n)}$. In that case the action $\star$ of $\mathrm{O}(3)$ on $\mathrm{T}^{(n)}$ is defined by the Rayleigh product:

$$\star : \mathrm{O}(3) \times \mathbb{T}^{(n)} \to \mathbb{T}^{(n)} : \left(\mathrm{Q}, \mathrm{T}^{(n)}\right) \quad \mapsto \quad \mathrm{Q} \star \mathrm{T}^{(n)} := Q_{i_1 j_1} \ldots Q_{i_n j_n} T^{(n)}_{j_1 \ldots j_n} \tag{1}$$

The material and the physical symmetry groups are related by the mean of Curie–Neumann's principle [17]:

$$\mathrm{G}_{\mathcal{M}} \subseteq \mathrm{G}_{\mathcal{P}}$$

meaning that each operation leaving the material invariant leaves the physical property invariant. Nevertheless, as shown for tensorial properties using Hermann's theorems [11], physical properties can be more symmetrical than the material.

In E^3 $\mathrm{G}_{\mathcal{P}}$ is conjugate to an $\mathrm{O}(3)$-closed subgroup [6,17]. Classification of $\mathrm{O}(3)$-closed subgroups is a classical result that can be found in many references, e.g. [18]:

Lemma 2.1. *Every closed subgroup of* $\mathrm{O}(3)$ *is conjugate to precisely one group of the following list, which has been divided into three classes:*

(i) *closed subgroups of* $\mathrm{SO}(3)$*:* $\mathbb{1}$, Z_n, D_n, $\mathcal{T}$, $\mathcal{O}$, $\mathcal{I}$, $\mathrm{SO}(2)$, $\mathrm{O}(2)$, $\mathrm{SO}(3)$*;*
(ii) $\tilde{K} := K \oplus \mathrm{Z}_2^c$, *where* K *is a closed subgroup of* $\mathrm{SO}(3)$ *and* $\mathrm{Z}_2^c = \{\mathbb{1}, -\mathbb{1}\}$*;*
(iii) *C closed subgroups not containing* $-\mathbb{1}$ *and not contained in* $\mathrm{SO}(3)$*:*

$$\mathrm{Z}_{2n}^- \ (n \geqslant 1), \quad \mathrm{D}_n^v \ (n \geqslant 2), \quad \mathrm{D}_{2n}^h \ (n \geqslant 2), \quad \mathcal{O}^- \text{ or } \mathrm{O}(2)^-$$

Let us now give a brief description of these different subgroups:

[1] For a given group $G \in \mathrm{O}(3)$, by generic G-invariant tensors we mean tensors that only satisfy G-invariance and no other constraint. This is the case of almost all G-invariant tensors.

Type-I subgroups Among SO(3)-closed subgroups, we can distinguish:

Planar groups: $\{\mathbb{1}, \mathrm{Z}_n, \mathrm{D}_n, \mathrm{SO}(2), \mathrm{O}(2)\}$, which are O(2)-closed subgroups;
Exceptional groups: $\{\mathcal{T}, \mathcal{O}, \mathcal{I}, \mathrm{SO}(3)\}$, which are the rotation groups of chiral Platonic polyhedrons completed by the rotation group of the sphere.

Let us detail first the set of planar subgroups. We fix a base $(\mathbf{i}; \mathbf{j}; \mathbf{k})$ of $\mathbb{R}^3$, and denote by $\mathbf{Q}(\mathbf{v}; \theta) \in \mathrm{SO}(3)$ the rotation about $\mathbf{v} \in \mathbb{R}^3$, with angle $\theta \in [0; 2\pi)$ we have:

- $\mathbb{1}$, the identity;
- Z_n $(n \geqslant 2)$, the cyclic group of order n, generated by the n-fold rotation $\mathbf{Q}(\mathbf{k}; \theta = \frac{2\pi}{n})$, which is the symmetry group of a chiral polygon;
- D_n $(n \geqslant 2)$, the dihedral group of order $2n$ generated by Z_n and $\mathbf{Q}(\mathbf{i}; \pi)$, which is the symmetry group of a regular polygon;
- SO(2), the subgroup of rotations $\mathbf{Q}(\mathbf{k}; \theta)$ with $\theta \in [0; 2\pi)$; it is the symmetry group of an oriented cone;
- O(2), the subgroup generated by SO(2) and $\mathbf{Q}(\mathbf{i}; \pi)$; it is the symmetry group of a twisted cylinder.

The classes of exceptional subgroups are: $\mathcal{T}$ the tetrahedral rotation group of order 12 that fixes a tetrahedron, $\mathcal{O}$ the octahedral rotation group of order 24 that fixes an octahedron (or a cube), and $\mathcal{I}$ the rotation group of order 60 that fixes an icosahedron (or a dodecahedron).

Type-II subgroups Type-II subgroups are of the form $\tilde{K} := K \oplus \mathrm{Z}_2^c$, where K is a closed subgroup of SO(3). Therefore, we directly know the collection of type-II subgroups.

Type-III subgroups The construction of type-III subgroups is more involved, and a description of their structure is provided in [18]. As for type-I subgroups, we can introduce subgroups of type III. Let $\sigma_{\mathbf{u}} \in \mathrm{O}(3)$ denote the reflection through the plane normal to the $\mathbf{u}$ axis.

- Z_2^- the order 2 reflection group generated by $\sigma_{\mathbf{i}}$;
- Z_{2n}^- $(n \geqslant 2)$ the group of order $2n$, generated by the $2n$-fold rotoreflection $\mathbf{Q}(\mathbf{k}; \theta = \frac{\pi}{n}) \cdot \sigma_{\mathbf{k}}$;
- D_{2n}^h $(n \geqslant 2)$ the prismatic group of order $4n$ generated by Z_{2n}^- and $\mathbf{Q}(\mathbf{i}, \pi)$. When n is odd it is the symmetry group of a regular prism, and when n is even it is the symmetry group of a regular antiprism;
- D_n^v $(n \geqslant 2)$ the pyramidal group of order $2n$ generated by Z_n and $\sigma_{\mathbf{i}}$, which is the symmetry group of a regular pyramid;
- $\mathrm{O}(2)^-$ is the limit group of D_n^v for continuous relation; it is therefore generated by $\mathbf{Q}(\mathbf{k}; \theta)$ and $\sigma_{\mathbf{i}}$. It is the symmetry group of a cone;
- $\mathcal{O}^-$, which is an achiral tetrahedral symmetry of order 24. This group has the same rotation axes as $\mathcal{T}$, but with six mirror planes, each through two 3-fold axes.

In order to have a better physical understanding of these subgroups, we reported in Appendix A the tables making correspondences between group notations and the classical crystallographic ones (Hermann–Mauguin, Schoenflies).

To study the symmetry classes of a tensor, we need to decompose it into O(3)-elementary parts.

3. Structure of tensor spaces

3.1. Harmonic decomposition

The O(3)-invariant decomposition of a tensor is known as harmonic decomposition; it is a higher-dimensional analogue of the Fourier decomposition. It allows us to decompose any finite-order tensor into a sum of irreducible ones [19,20]. Formally, this decomposition can be written as:

$$\mathrm{T}^n = \sum_{k,\tau} \mathrm{D}(n)^{k,\tau}$$

where tensors $\mathrm{D}(n)^{k,\tau}$ are components of the irreducible decomposition, k denotes the order of the harmonic tensor embedded in $\mathrm{D}(n)$ and τ separates same order terms. This decomposition establishes an isomorphism between $\mathbb{T}^n$ and a direct sum of harmonic tensor spaces $\mathbb{H}^k$ [6]:

$$\mathbb{T}^n \cong \bigoplus_{k,\tau} \mathbb{H}^{k,\tau}$$

but, as explained in [21], this decomposition may not be unique. Grouping together irreducible spaces of the same order, one obtains the O(3)-isotypic decomposition of a representation which is unique [22]:

$$\mathbb{T}^n \cong \bigoplus_{k=0}^{n} \alpha_k \mathbb{H}^k \tag{2}$$

where α_k is the multiplicity of the irreducible space $\mathbb{H}^k$ in the decomposition and n is the order of the highest-order irreducible space of the decomposition. $\mathbb{H}^k$ is the space of kth-order harmonic tensors, that is the space of totally symmetric, traceless tensors of order k. It is a vector space of dimension $2k+1$ in $\mathbb{R}^3$. The series $\{\alpha_k\}$ is a function of the order and of the intrinsic symmetries of the tensor space. Various methods exist to compute this family [10,19,20].

Before closing this rather short introduction, it is important to note that in the decomposition of an even-order (respectively odd-order) tensor, odd-order (respectively even-order) components are pseudo-tensors,[2] i.e. they change sign if the space orientation is reversed.

3.2. Dimension of G-invariant harmonic spaces

Let G be any subgroup of O(3). The set of tensors $\mathrm{T} \in \mathbb{T}$ which are fixed by G:

$$\mathbb{F}\mathrm{ix}_{\mathbb{T}}(G) := \{\mathrm{T} \in \mathbb{T} \mid g.\mathrm{T} = \mathrm{T} \text{ for all } g \in G\},$$

is called the *fixed point set*.[3] It is the biggest *linear subspace* of $\mathbb{T}$ that contains G-invariant tensors. Elements of $\mathbb{F}\mathrm{ix}_{\mathbb{T}}(G)$ only defined in terms of G-invariance without any further constraint will be referred to as generic. Since non-generic elements constitute a null set, they will not be considered here.

It is worth noting that the dimension of type-II invariant subspaces is always 0. And for O(3)-subgroups of types I and III, we have the following results[4] concerning harmonic tensor spaces.

Type-I subgroups[5]

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(\mathrm{Z}_p) = 2\left\lfloor \frac{k}{p} \right\rfloor + 1; \qquad \dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(\mathrm{D}_p) = \begin{cases} \lfloor \frac{k}{p} \rfloor + 1 & \text{for } k \text{ even} \\ \lfloor \frac{k}{p} \rfloor & \text{for } k \text{ odd} \end{cases}$$

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(\mathcal{T}) = 2\left\lfloor \frac{k}{3} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor - k + 1; \qquad \dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(\mathcal{O}) = \left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{k}{3} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor - k + 1$$

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(\mathcal{I}) = \left\lfloor \frac{k}{5} \right\rfloor + \left\lfloor \frac{k}{3} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor - k + 1$$

Type-III subgroups

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(\mathrm{Z}_{2p}^-) = 2\left\lfloor \frac{k+p}{2p} \right\rfloor; \qquad \dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(\mathrm{D}_p^v) = \begin{cases} \lfloor \frac{k}{p} \rfloor & \text{if } k = 2n \\ \lfloor \frac{k}{p} \rfloor + 1 & \text{if } k = 2n+1 \end{cases}$$

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(\mathrm{D}_{2p}^h) = \left\lfloor \frac{k+p}{2p} \right\rfloor; \qquad \dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(\mathrm{SO}(2)^-) = \begin{cases} 0 & k = 2n \\ 1 & k = 2n+1 \end{cases}$$

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{H}^k}(\mathcal{O}^-) = \left\lfloor \frac{k}{3} \right\rfloor - \left\lfloor \frac{k}{4} \right\rfloor$$

These elementary results, combined with the knowledge of the isotypic decomposition of a tensor space (2), allow us to determine the dimension of any G-invariant tensor subspaces. Analytical expressions are constructed according to that procedure.

3.3. G-invariant tensor subspaces

Applying this process, the following formulas are obtained:

[2] The related tensors are multiplied by the Levi–Civita symbol ϵ_{ijk}.

[3] As G as an action on the space $\mathbb{T}$, there is a homeomorphism ψ from G to GL($\mathbb{T}$). Hence the notation $g.\mathrm{T}$ should be understood as a classical shortcut to the more rigorous one $\psi(g).\mathrm{T}$.

[4] More details can be found in [12,21].

[5] The formulas for type-I subgroup have already been provided in [12]. They are summed-up here for the sake of completeness.

Type-I subgroups

- Z_p**-invariance**

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(\mathrm{Z}_p) = 2\sum_{k=0}^{n}\alpha_k\left\lfloor\frac{k}{p}\right\rfloor + \sum_{k=0}^{n}\alpha_k$$

When $p > k$, we obtain $\lfloor\frac{k}{p}\rfloor = 0$ and so $\beta_{\mathrm{oth}} = \sum_{k=0}^{n}\alpha_k$ is the number of oriented transverse hemitropic coefficients. β_{oth} is the dimension of a SO(2)-invariant tensor.

- D_p**-invariance**

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(\mathrm{D}_p) = \sum_{k=0}^{n}\alpha_k\left\lfloor\frac{k}{p}\right\rfloor + \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\alpha_{2k}$$

When $p > k$, we obtain $\lfloor\frac{k}{p}\rfloor = 0$ and so $\beta_{\mathrm{tti}} = \sum_{k=0}^{n}\alpha_k$ is the number of twisted transverse isotropic coefficients. β_{tti} is the dimension of an O(2)-invariant tensor.

- $\mathcal{T}$, $\mathcal{O}$ **and** $\mathcal{I}$**-invariance**

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(\mathcal{T}) = \sum_{k=0}^{n}\alpha_k\left(2\left\lfloor\frac{k}{3}\right\rfloor + \left\lfloor\frac{k}{2}\right\rfloor - k + 1\right); \qquad \dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(\mathcal{O}) = \sum_{k=0}^{n}\alpha_k\left(\left\lfloor\frac{k}{4}\right\rfloor + \left\lfloor\frac{k}{3}\right\rfloor + \left\lfloor\frac{k}{2}\right\rfloor - k + 1\right)$$

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}(\mathcal{I}) = \sum_{k=0}^{n}\alpha_k\left(\left\lfloor\frac{k}{5}\right\rfloor + \left\lfloor\frac{k}{3}\right\rfloor + \left\lfloor\frac{k}{2}\right\rfloor - k + 1\right)$$

Type-III subgroups

- Z_{2p}^{-}**-invariance**

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}\left(\mathrm{Z}_{2p}^{-}\right) = 2\sum_{k=0}^{n}\alpha_k\left\lfloor\frac{k+p}{2p}\right\rfloor$$

When $p > k$, we obtain $\lfloor\frac{k+p}{2p}\rfloor = 0$ and therefore the tensor is null. In such a case, the tensor is obviously O(3)-invariant.

- D_{2p}^{h}**-invariance**

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}\left(\mathrm{D}_{2p}^{h}\right) = \sum_{k=0}^{n}\alpha_k\left\lfloor\frac{k+p}{2p}\right\rfloor = \frac{1}{2}\dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}\left(\mathrm{Z}_{2p}^{-}\right)$$

When $p > k$, we obtain $\lfloor\frac{k+p}{2p}\rfloor = 0$ and therefore the tensor is null. In such a case, the tensor is obviously O(3)-invariant.

- D_{p}^{v}**-invariance**

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}\left(\mathrm{D}_{p}^{v}\right) = \sum_{k=0}^{n}\alpha_k\left\lfloor\frac{k}{p}\right\rfloor + \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\alpha_{2k+1}$$

When $p > k$, we obtain $\lfloor\frac{k}{p}\rfloor = 0$ and so $\beta_{\mathrm{th}} = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\alpha_{2k+1}$ is the number of non-oriented transverse hemitropic coefficients. β_{th} is the dimension of an $\mathrm{O}(2)^{-}$-invariant tensor.

- $\mathcal{O}^{-}$**-invariance**

$$\dim \mathbb{F}\mathrm{ix}_{\mathbb{T}}\left(\mathcal{O}^{-}\right) = \sum_{k=0}^{n}\alpha_k\left(\left\lfloor\frac{k}{3}\right\rfloor - \left\lfloor\frac{k}{4}\right\rfloor\right)$$

4. Physical results

In order to illustrate the practical interest of these formulas, two examples will be considered: the third-order tensor of piezoelectricity and the fifth-order coupling tensor of Mindlin's strain gradient elasticity.

4.1. Piezoelectricity

Let us consider $\mathbb{P}\mathrm{iez}$ the vector space of piezoelectricity tensors, its elements are symmetric under the permutation of their two first indices: $\mathrm{P}_{(ij)k}$, where (..) stands for the minor symmetry. It has been shown [13,14] that this vector space is isomorphic to:

$$\mathbb{P}\mathrm{iez} \cong 2\mathbb{H}^1 \oplus \mathbb{H}^{\sharp 2} \oplus \mathbb{H}^3$$

where the $^\sharp$ notation indicates a pseudo-tensor. And so $\mathbb{P}\mathrm{iez}$ is defined by the following $\{\alpha_k\}$ family: $\{0, 2, 1, 1\}$. As determined in [13,14,23], the space of piezoelectric tensors can be divided into the following anisotropic systems:

$$[\mathbb{P}\mathrm{iez}] = \left\{[\mathbb{1}], [\mathrm{Z}_2], [\mathrm{Z}_3], \left[\mathrm{D}_2^v\right], \left[\mathrm{D}_3^v\right], \left[\mathrm{Z}_2^-\right], \left[\mathrm{Z}_4^-\right], [\mathrm{D}_2], [\mathrm{D}_3], \left[\mathrm{D}_4^h\right], \left[\mathrm{D}_6^h\right], \left[\mathrm{SO}(2)\right], \left[\mathrm{O}(2)\right], \left[\mathrm{O}(2)^-\right], \left[\mathcal{O}^-\right]\right\}$$

Straightforward applications of our formula give

- **Type-I subgroups**

$[\mathbb{P}\mathrm{iez}]$	$[\mathbb{1}]$	$[\mathrm{Z}_2]$	$[\mathrm{Z}_3]$	$[\mathrm{SO}(2)]$	$[\mathrm{D}_2]$	$[\mathrm{D}_3]$	$[\mathrm{O}(2)]$
dim	18	8	6	4	3	2	1

- **Type-III subgroups**

$[\mathbb{P}\mathrm{iez}]$	$[\mathrm{Z}_2^-]$	$[\mathrm{Z}_4^-]$	$[\mathrm{D}_2^v]$	$[\mathrm{D}_3^v]$	$[\mathrm{O}^-(2)]$	$[\mathrm{D}_4^h]$	$[\mathrm{D}_6^h]$	$[\mathcal{O}^-]$
dim	10	4	5	4	3	2	1	1

where the number of coefficients for each physical symmetry class has been determined.[6] These results are obviously in agreement with those in the literature [13–15].

4.2. Mindlin strain-gradient elasticity

Let us consider $\mathbb{E}\mathrm{la}_\mathrm{M}$ the vector space of the coupling tensors in Mindlin strain gradient elasticity. It is the vector space of fifth-order tensors endowed with the following index symmetries [16]: $\mathrm{M}_{(ij)(kl)m}$. It has been shown [11] that this vector space decomposed as follows:

$$\mathbb{E}\mathrm{la}_\mathrm{M} \cong \mathbb{H}^{\sharp 0} \oplus 6\mathbb{H}^1 \oplus 5\mathbb{H}^{\sharp 2} \oplus 5\mathbb{H}^3 \oplus 2\mathbb{H}^{\sharp 4} \oplus \mathbb{H}^5$$

and so $\mathbb{E}\mathrm{la}_\mathrm{M}$ is defined by the following $\{\alpha_k\}$ family: $\{1, 6, 5, 5, 2, 1\}$. As determined in [23], the space $\mathbb{E}\mathrm{la}_\mathrm{M}$ can be divided into the following anisotropic systems:

$$\begin{aligned}[\mathbb{E}\mathrm{la}_\mathrm{M}] = \Big\{&[\mathbb{1}], [\mathrm{Z}_2], \ldots, [\mathrm{Z}_5], \left[\mathrm{D}_2^v\right], \ldots, \left[\mathrm{D}_5^v\right], \left[\mathrm{Z}_2^-\right], \ldots, \left[\mathrm{Z}_8^-\right], [\mathrm{D}_2], \ldots, [\mathrm{D}_5],\\ &\left[\mathrm{D}_4^h\right], \ldots, \left[\mathrm{D}_{10}^h\right], \left[\mathrm{SO}(2)\right], \left[\mathrm{O}(2)\right], \left[\mathrm{O}(2)^-\right], [\mathcal{T}], \left[\mathcal{O}^-\right], [\mathcal{O}], \left[\mathrm{SO}(3)\right], \left[\mathrm{O}(3)\right]\Big\}\end{aligned}$$

Therefore $\mathbb{E}\mathrm{la}_\mathrm{M}$ is divided into 28 symmetry classes.[7] This large number of symmetry classes has to be compared with the eight symmetry classes of classical elasticity [6], and the 17 symmetry classes of second-order elasticity [24,25]. Straightforward applications of our formula now give:

- **type-I subgroups**

$[\mathbb{E}\mathrm{la}_\mathrm{M}]$	$[\mathbb{1}]$	$[\mathrm{Z}_2]$	$[\mathrm{Z}_3]$	$[\mathrm{Z}_4]$	$[\mathrm{Z}_5]$	$[\mathrm{SO}(2)]$	$[\mathrm{D}_2]$	$[\mathrm{D}_3]$	$[\mathrm{D}_4]$	$[\mathrm{D}_5]$	$[\mathrm{O}(2)]$	$[\mathcal{T}]$	$[\mathcal{O}]$	$[\mathrm{SO}(3)]$
dim	108	52	36	26	22	20	24	16	11	9	8	8	3	1

- **type-III subgroups**

$[\mathbb{E}\mathrm{la}_\mathrm{M}]$	$[\mathrm{Z}_2^-]$	$[\mathrm{Z}_4^-]$	$[\mathrm{Z}_6^-]$	$[\mathrm{Z}_8^-]$	$[\mathrm{D}_2^v]$	$[\mathrm{D}_3^v]$	$[\mathrm{D}_4^v]$	$[\mathrm{D}_5^v]$	$[\mathrm{O}^-(2)]$	$[\mathrm{D}_4^h]$	$[\mathrm{D}_6^h]$	$[\mathrm{D}_8^h]$	$[\mathrm{D}_{10}^h]$	$[\mathcal{O}^-]$
dim	56	26	16	6	28	20	15	13	12	13	8	3	1	5

6 A precise definition of symmetry classes can be found in the following references [6,23,25].

7 Or 29, if the class [O(3)] of null tensors is taken into account. But this point is only a question of convention.

5. Conclusion

In this note, analytical formulas giving the dimension of a subspace left fixed under O(3)-subgroup action have been provided and applied to the symmetry classes of some physical tensor spaces. To be applied, the only things that have to be known are the O(3)-isotypic decomposition of the studied tensor space, and the symmetry classes of the tensors spaces. Using the method proposed in [11,20], this decomposition is easily obtained, and the determination of the symmetry classes is completely solved in [23,25]. We believe that these simple formulas can be of great help to develop, for example, higher-order constitutive laws [8,26] and to design micro–macro homogenization procedure for anisotropic materials [2].

Appendix A. Dictionary between group notations and crystallographic systems

A.1. O(3) *type-I closed subgroups*

Hermann–Mauguin	Schonflies	Group
1	C_1	$\mathbb{1}$
2	C_2	Z_2
222	D_2	D_2
3	C_3	Z_3
32	D_3	D_3
4	C_4	Z_4
422	D_4	D_4
6	C_6	Z_6
622	D_6	D_6
∞	C_∞	SO(2)
$\infty 2$	D_∞	O(2)
23	T	$\mathcal{T}$
432	O	$\mathcal{O}$
532	I	$\mathcal{I}$
$\infty\infty$		SO(3)

A.2. O(3) *type-II closed subgroups*

Hermann–Mauguin	Schonflies	Group
$\bar{1}$	C_i	Z_2^c
$2/m$	C_{2h}	$Z_2 \oplus Z_2^c$
mmm	D_{2h}	$D_2 \oplus Z_2^c$
$\bar{3}$	S_6, C_{3i}	$Z_3 \oplus Z_2^c$
$\bar{3}m$	D_{3d}	$D_3 \oplus Z_2^c$
$4/m$	C_{4h}	$Z_4 \oplus Z_2^c$
$4/mmm$	D_{4h}	$D_4 \oplus Z_2^c$
$6/m$	C_{6h}	$Z_6 \oplus Z_2^c$
$6/mmm$	D_{6h}	$D_6 \oplus Z_2^c$
$m\bar{3}$	T_h	$\mathcal{T} \oplus Z_2^c$
$m\bar{3}m$	O_h	$\mathcal{O} \oplus Z_2^c$
$\bar{5}\bar{3}m$	I_h	$\mathcal{I} \oplus Z_2^c$
∞/m	$C_{\infty h}$	$SO(2) \oplus Z_2^c$
∞/mm	$D_{\infty h}$	$O(2) \oplus Z_2^c$
$\infty/m\infty/m$		O(3)

A.3. O(3) *type-III closed subgroups*

Hermann–Mauguin	Schonflies	Group
m	C_s	Z_2^-
$2mm$	C_{2v}	D_2^v
$3m$	C_{3v}	D_3^v
$\bar{4}$	S_4	Z_4^-
$4mm$	C_{4v}	D_4^v
$\bar{4}2m$	D_{2d}	D_4^h
$\bar{6}$	C_{3h}	Z_6^-
$6mm$	C_{6v}	D_6^v
$\bar{6}2m$	D_{3h}	D_6^h
$\bar{4}3m$	T_d	$\mathcal{O}^-$
∞m	$C_{\infty v}$	$O(2)^-$

References

[1] A. Thionnet, C. Martin, A new constructive method using the theory of invariants to obtain material behavior laws, Int. J. Solids Struct. 43 (2006) 325–345.
[2] D.-K. Trinh, R. Jänicke, N. Auffray, S. Diebels, S. Forest, Evaluation of generalized continuum substitution models for heterogeneous materials, Int. J. Multiscale Comput. Eng. 10 (2012) 527–549.
[3] M. Francois, G. Geymonat, Y. Berthaud, Determination of the symmetries of an experimentally determined stiffness tensor: application to acoustic measurements, Int. J. Solids Struct. 35 (1998) 4091–4106.
[4] M. Francois, Y. Berthaud, G. Geymonat, Une nouvelle analyse des symétries d'un matériau élastique anisotrope. Exemple d'utilisation à partir de mesures ultrasonores, C. R. Acad. Sci. Paris, Ser. IIb 322 (1996) 87–94.
[5] S.C. Cowin, M.M. Mehrabadi, Eigentensors of linear anisotropic elastic materials, Q. J. Mech. Appl. Math. 43 (1990) 15–41.
[6] S. Forte, M. Vianello, Symmetry classes for elasticity tensors, J. Elast. 43 (1996) 81–108.
[7] N.A. Fleck, J.W. Hutchinson, Strain gradient plasticity, in: J.W. Hutchinson, T.Y. Wu (Eds.), in: Adv. Appl. Mech., vol. 33, Academic Press, New York, 1997.
[8] J. Alibert, P. Seppecher, F. dell'Isola, Truss modular beams with deformation energy depending on higher displacement gradients, Math. Mech. Solids 8 (2003) 51–73.
[9] G. Sciarra, F. dell'Isola, O. Coussy, Second gradient poromechanics, Int. J. Solids Struct. 44 (2007) 6607–6629.
[10] N. Auffray, Démonstration du théorème d'Hermann à partir de la méthode Forte–Vianello, C. R. Mecanique 336 (2008) 458–463.
[11] N. Auffray, Décomposition harmonique des tenseurs – Méthode spectrale, C. R. Mecanique 336 (2008) 370–375.
[12] N. Auffray, Analytical expressions for anisotropic tensor dimension, C. R. Mecanique 338 (2010) 260–265.
[13] T. Weller, Etude des symétries et modèles de plaques en piézoélectricité linéarisée, PhD thesis, Université Montpellier-2, 2004.
[14] T. Weller, G. Geymonat, Piezomagnetic tensors symmetries: an unifying tentative approach, in: Configurational Mechanics: Proceedings of the Configurational Mechanics, Taylor & Francis, 2004.
[15] W.N. Zou, C.X. Tang, E. Pan, Symmetry types of the piezoelectric tensor and their identification, Proc. R. Soc. A 469 (2013).
[16] R.D. Mindlin, N.N. Eshel, On first strain-gradient theories in linear elasticity, Int. J. Solids Struct. 4 (1968) 109–124.
[17] Q.-S. Zheng, J.-P. Boehler, The description, classification, and reality of material and physical symmetries, Acta Mech. 102 (1994) 73–89.
[18] S. Sternberg, Group Theory and Physics, Cambridge University Press, Cambridge, UK, 1994.
[19] W. Zou, Q.-S. Zheng, D. Du, J. Rychlewski, Orthogonal irreducible decompositions of tensors of high orders, Math. Mech. Solids 6 (2001) 249–267.
[20] J. Jerphagnon, D. Chemla, R. Bonneville, The description of the physical properties of condensed matter using irreducible tensors, Adv. Phys. 27 (1978) 609–650.
[21] M. Golubitsky, I. Stewart, D. Schaeffer, Singularities and Groups in Bifurcation Theory, vol. II, Springer-Verlag, 1989.
[22] Y. Kosmann-Schwarzbach, Groupes et symétries. Groupes finis, groupes et algébres de Lie, représentations, Éditions de l'École polytechnique, Paris, 2005.
[23] M. Olive, N. Auffray, Symmetry classes for odd-order tensors, Z. Angew. Math. Mech. (2013), http://dx.doi.org/10.1002/zamm.201200225.
[24] N. Auffray, H. Le Quang, Q.-C. He, Matrix representations for 3D strain-gradient elasticity, J. Mech. Phys. Solids 61 (2013) 1202–1223.
[25] M. Olive, N. Auffray, Symmetry classes for even-order tensors, Math. Mech. Complex Syst. 1 (2) (2013) 177–210.
[26] F. dell'Isola, P. Sciarra, S. Vidoli, Generalized Hooke's law for isotropic second gradient materials, Proc. R. Soc. A 465 (2009) 2177–2196.