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Theoretical modelling and experimental study of the fatigue of elastomers under cyclic loadings of variable amplitude

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ABSTRACT

Deviations from Miner's linear law of cumulative damage have been modelled and observed many times for the fatigue of metals, but almost no analogous studies have been performed for elastomers. Such a study is reported here.
A simple phenomenological model, applicable to any type of material and able to quantitatively reproduce such deviations, is presented first. This model is based on continuum damage mechanics. It relates the fatigue damage of the material to the number of cycles through some suitable evolution law, in which the derivative of damage is expressed as a non-factorizable function of the instantaneous load cycle and the damage itself.
Fatigue experiments performed on "diabolo" specimens made of two different elastomeric materials and subjected to two successive cyclic loads of different amplitudes are then reported. Significant deviations from Miner's rule are observed: Miner's "total cumulated damage" may be lower or larger than unity by a small or large amount, depending on the sequence of loadings and the type of material. As a rule, the deviation from Miner's rule systematically changes sign upon reversal of the sequence of loadings. The model is shown to allow an acceptable reproduction of the experimental results, and especially of this systematic change of sign.

1. Introduction

Rubber components often experience complex cyclic loadings involving 3D stress states varying arbitrarily in time. Prediction of fatigue under such general conditions is important for the prediction of the durability of these components.

Models for the fatigue of elastomers under general cyclic loadings have been reviewed by Mars and Fatemi [1], who distinguished between models based on consideration of crack initiation and propagation, respectively. We shall focus here on the first class of models, initiated by Cadwell et al. [2], since propagation-based models are applicable only when the location and size of the initial crack(s) are known, which is seldom the case in practice.

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In a previous paper (Brunac et al. [3]), we considered the problem of predicting the fatigue lifetime under general 3D but perfectly time-periodic cyclic loadings. We now envisage the question of description of fatigue under cyclic loadings varying in time.

The standard answer to this problem consists of using Miner's [4] heuristic, but appealingly simple and elegant linear rule of cumulative damage. The applicability and limits of this rule have been assessed in numerous experimental works in the case of metals; as a rule, in such materials, Miner's "total cumulated damage" is observed to be smaller than unity if the more severe load cycles are applied first, and larger than unity if the less severe cycles are applied first (Chaboche and Lesne [5]). In contrast, very few similar studies have been performed for elastomers. The purpose of this paper is therefore to describe a theoretical and experimental investigation of deviations from Miner's rule for the fatigue of elastomers.

The paper is organized as follows:

– Section 2 first describes a heuristic model retaining the basic simplicity of Miner's rule, but accounting for possible deviations from its predictions while allowing for an arbitrary dependence of the number of cycles at failure upon the load cycle. This model belongs to the category of continuum damage models, as described in the book of Lemaître and Chaboche [6], and relies on some evolution equation for the damage variable in which this variable and the instantaneous load cycle[1] appear in a "non-factorized" form. It involves a single adjustable parameter depending on the load cycle.
– Section 3 then explains how the load-cycle-dependent material parameter introduced in the model may be determined from experimental numbers of cycles at failure observed in successions of different cyclic loads.
– Section 4 presents the set of experiments performed. These experiments involve load histories consisting of two cyclic loads of different amplitudes applied in succession on "diabolo" specimens made of two distinct elastomeric materials. They evidence clear and significant deviations from Miner's rule in at least one of these materials.
– Finally Section 5 presents the application of the model to the experiments performed. Determination of the values of the model parameter for the two load cycles considered in these experiments is shown to allow a much better reproduction of the experimental results than Miner's standard rule.

2. Continuum-damage-based description of fatigue under varying cyclic loadings

A number of models accounting for possible deviations from Miner's rule of cumulative damage have been proposed. An extensive overview of the literature was provided by Fatemi and Yang [7]; historical references include the works of Marko and Starkey [8], Manson [9], Bui-Quoc et al. [10] and Subramanyan [11], among others. We shall focus here on models based on continuum damage mechanics.

2.1. Generalities

The application of the phenomenological theory of continuum damage mechanics to the description of fatigue was explained in the book of Lemaître and Chaboche [6], and a more recent review was provided by Desmorat [12]. In this approach, the initiation of a macroscopic fatigue crack in an elementary volume of material is considered to be due to the progressive degradation of this volume, described through some damage variable D lying in the interval $(0, 1)$. This variable reduces the specific free energy by the factor $1 - D$ or some variant.[2] It obeys an evolution equation in which the number of cycles N plays the role of time. The simplest possible one reads

$$\frac{\mathrm{d}D}{\mathrm{d}N} = \frac{1}{N_\mathrm{f}}$$

where $N_\mathrm{f} \equiv N_\mathrm{f}(\mathcal{C})$ denotes the number of cycles at failure of the elementary volume, depending on the current load cycle $\mathcal{C}$.[3] This law reproduces Miner's rule since the damage after N non-identical cycles (with different values of N_f) is

$$D = \int_0^N \frac{\mathrm{d}N}{N_\mathrm{f}}$$

A more general form of the evolution law of D is

$$\frac{\mathrm{d}D}{\mathrm{d}N} = f(D, \mathcal{C}) \tag{1}$$

for some non-negative function f. When the load cycle $\mathcal{C}$ is invariable in time, integration of this evolution law yields

$$\int_0^D \frac{\mathrm{d}D'}{f(D', \mathcal{C})} = N$$

[1] This expression refers to the *sequence of loadings* undergone by the material during the cycle; the *number* of this cycle is immaterial.

[2] A factor of $1 - D^\alpha$, with $\alpha > 1$, was argued by Brunac [13] to be more appropriate in the case of elastomers, on the grounds that the elastic stiffness is experimentally found to remain almost constant during the major part of the degradation process and decrease significantly only at its very end.

[3] That is, again, on the sequence of loadings undergone by the material during the cycle, not on the number of this cycle.

It then follows from the definition of $N_f(\mathcal{C})$ as that number of cycles for which D reaches unity under a succession of identical load cycles $\mathcal{C}$, that

$$\int_0^1 \frac{\mathrm{d}D}{f(D,\mathcal{C})} = N_f(\mathcal{C}) \tag{2}$$

This is a necessary condition on the function f, which must hold for all possible load cycles $\mathcal{C}$.

A remarkable feature of the evolution equation (1) is the following equivalence:

$$(\text{Miner's rule holds}) \quad \Leftrightarrow \quad f(D,\mathcal{C}) \text{ is a "factorized" function of the form } \varphi(D)\psi(C). \tag{3}$$

Establishing the direct implication is more difficult than it may seem at first sight; a nice proof, briefly recalled in Appendix A for completeness, was provided by Stigh [14]. The proof of the converse implication is easier; see, e.g., Lemaître and Chaboche [6].

One drawback of an evolution equation of type (1), in this form, is that it inevitably places restrictions upon the way the number of cycles at failure, N_f, depends upon the load cycle, $\mathcal{C}$. Indeed it is necessary in practice to ascribe some more or less complex analytical form to the function f; a typical example, in the case of metals, is provided by Chaboche and Lesne's [5] model. This is annoying because at least in the case of elastomers, the dependence of N_f upon $\mathcal{C}$ is quite complex and not easily amenable to some analytic formula. (This is true even for 1D loadings, when the minimum stress reached during each cycle is nonzero.)

For this reason, we shall henceforward consider evolution equations of type (1), but written in the form

$$\frac{\mathrm{d}D}{\mathrm{d}N} = \frac{g(D,N_f)}{N_f}, \quad N_f \equiv N_f(\mathcal{C}) \tag{4}$$

for some non-negative function g. The advantage of thus incorporating the dependence of the rate of damage upon the load cycle in an implicit form, through the number of cycles at failure, is that any dependence of N_f upon $\mathcal{C}$ then becomes possible. It is not even necessary to assume any analytic form of dependence; the function $N_f(\mathcal{C})$ may be directly extracted from experiments. For 1D loadings for instance, it may be taken from Haigh's diagram, which provides the experimental number of cycles at failure as a function of the maximum and minimum stresses reached during each cycle. For 3D loadings, it may be deduced from any model providing N_f as a function of some characteristic elements of the load cycle; one may use for instance Brunac et al.'s [3] proposed heuristic extension of Haigh's diagram to completely arbitrary load cycles, inspired from Dang Van's [15] work on multiaxial fatigue of metals.

With the new form (4) of the evolution equation of D, the necessary condition (2) takes the form

$$\int_0^1 \frac{\mathrm{d}D}{g(D,N_f)} = 1 \tag{5}$$

for all possible load cycles $\mathcal{C}$. Also, property (3) takes the form:

$$(\text{Miner's rule holds}) \quad \Leftrightarrow \quad g(D,N_f) \text{ is a factorized function} \tag{6}$$

2.2. The proposed model

We wish to define the simplest possible evolution equation of type (4) predicting deviations from Miner's rule, that is involving some non-factorized function $g(D,N_f)$. In view of the necessary condition (5), the simplest option is to consider a function $g(D,N_f)$ possessing the property that for every value of N_f, its inverse varies linearly with D and takes the value 1 for $D = 1/2$. This function is of the form:

$$g(D,N_f) \equiv \frac{1}{1+2\beta(D-1/2)} = \frac{1}{1-\beta+2\beta D}, \quad \beta \equiv \beta(N_f) \tag{7}$$

Conditions (5) being then automatically satisfied, the only constraints on the possible values of the coefficient β are those arising from the necessary non-negativeness of the function g for $0 \le D \le 1$, which read

$$-1 \le \beta(N_f) \le 1 \tag{8}$$

for all values of N_f.

Note that the function g is non-factorized so that, by Eq. (6), deviations from Miner's rule are predicted as soon as the function $\beta(N_f)$ is not a constant.

Several evolution laws of type (1) involving non-factorized functions $f(D,C)$ have been proposed, notably by Chaboche and Lesne [5]. These laws generally included special functions f in analytical form, and therefore did not allow for an arbitrary dependence of the number of cycles at failure upon the load cycle. One interesting exception, however, is the law proposed in the book of Chaboche [16]:

$$\frac{\mathrm{d}D}{\mathrm{d}N}=\frac{D^{\alpha}}{(1-\alpha)N_{\mathrm{f}}},\quad \alpha\equiv\alpha(\mathcal{C}),\ N_{\mathrm{f}}\equiv N_{\mathrm{f}}(\mathcal{C}) \tag{9}$$

with $\alpha<1$ so as to ensure convergence of the integral (2) defining N_{f}. This evolution equation possesses the same basic features as that defined by Eqs. (4) and (7) since it allows for an arbitrary dependence of N_{f} upon $\mathcal{C}$, while predicting deviations from Miner's rule as soon as the function $\alpha(\mathcal{C})$ is not a constant. The difference with the model proposed here lies in the form of the function g, the inverse of which is a power function of D instead of a linear one. Some consequences of this difference are as follows:

- Chaboche's form (9) of the evolution equation distinguishes between the beginning and the end of the fatigue process (since the factor D^{α} particularizes the beginning), whereas formula (7) does not.
- In Chaboche's model, the initial damage rate can take only three possible values, $+\infty$, $1/N_{\mathrm{f}}$ and 0, depending on whether α is negative, zero or positive, whereas all values from $1/(2N_{\mathrm{f}})$ to $+\infty$ are possible in the model proposed here.

3. Determination of model parameters from experiments

Consider a fatigue experiment consisting of a succession of N_1 load cycles $\mathcal{C}_1$ followed by N_2 load cycles $\mathcal{C}_2$, leading to failure. Assume that the evolution of damage is governed by Eqs. (4) and (7). During the first phase, Eq. (4) reads $\mathrm{d}D/\mathrm{d}N=g(D,N_{\mathrm{f}1})/N_{\mathrm{f}1}$ where $N_{\mathrm{f}1}\equiv N_{\mathrm{f}}(\mathcal{C}_1)$. It follows through integration that the damage $\mathcal{D}$ at the end of this phase is given by the equation

$$G(\mathcal{D},N_{\mathrm{f}1})=\frac{N_1}{N_{\mathrm{f}1}},\qquad G(D,N_{\mathrm{f}})\equiv\int_0^D\frac{\mathrm{d}D'}{g(D',N_{\mathrm{f}})}\equiv\big[1-\beta(N_{\mathrm{f}})\big]D+\beta(N_{\mathrm{f}})D^2 \tag{10}$$

During the second phase, the evolution equation becomes $\mathrm{d}D/\mathrm{d}N=g(D,N_{\mathrm{f}2})/N_{\mathrm{f}2}$ where $N_{\mathrm{f}2}\equiv N_{\mathrm{f}}(\mathcal{C}_2)$. It again follows through integration that

$$\int_{\mathcal{D}}^1\frac{\mathrm{d}D}{g(D,N_{\mathrm{f}2})}=1-G(\mathcal{D},N_{\mathrm{f}2})=\frac{N_2}{N_{\mathrm{f}2}} \tag{11}$$

where Eq. (5) has been used.

Eqs. (10) and (11) provide two relations connecting the unknown coefficients $\beta_1\equiv\beta(N_{\mathrm{f}1})$, $\beta_2\equiv\beta(N_{\mathrm{f}2})$, the unknown damage parameter $\mathcal{D}$ at the end of the first phase and the two experimentally known "damage parameters in the sense of Miner", or "Miner damages"

$$\begin{cases} D_1^{\mathrm{Miner}}\equiv N_1/N_{\mathrm{f}1}\\ D_2^{\mathrm{Miner}}\equiv N_2/N_{\mathrm{f}2}\end{cases} \tag{12}$$

To eliminate the undesired quantity $\mathcal{D}$ between these equations, one may first eliminate $\mathcal{D}^2$ to get

$$\mathcal{D}=\frac{\beta_1(1-D_2^{\mathrm{Miner}})-\beta_2D_1^{\mathrm{Miner}}}{\beta_1-\beta_2} \tag{13}$$

and then reinsert this value into the sum of Eqs. $(10)_1$ and (11) to get

$$\begin{aligned}F\big(D_1^{\mathrm{Miner}},D_2^{\mathrm{Miner}};\beta_1,\beta_2\big)\equiv{}&\frac{[\beta_1(1-D_2^{\mathrm{Miner}})-\beta_2D_1^{\mathrm{Miner}}]^2}{\beta_1-\beta_2}\\ &-\beta_1\big(1-D_2^{\mathrm{Miner}}\big)+\beta_2D_1^{\mathrm{Miner}}+1-D_1^{\mathrm{Miner}}-D_2^{\mathrm{Miner}}=0\end{aligned} \tag{14}$$

This is an equation connecting the sole unknown coefficients β_1, β_2 to the known Miner damages D_1^{Miner}, D_2^{Miner}.

Eq. (14) does not, of course, suffice to determine β_1 and β_2. However, assume that another fatigue experiment consisting of a succession of N_2' load cycles $\mathcal{C}_2$ followed by N_1' load cycles $\mathcal{C}_1$, again leading to failure, is available. The same reasoning as before leads to the new, symmetric relation

$$\begin{aligned}F\big(D_2^{\mathrm{Miner}\,\prime},D_1^{\mathrm{Miner}\,\prime};\beta_2,\beta_1\big)={}&\frac{[\beta_2(1-D_1^{\mathrm{Miner}\,\prime})-\beta_1D_2^{\mathrm{Miner}\,\prime}]^2}{\beta_2-\beta_1}\\ &-\beta_2\big(1-D_1^{\mathrm{Miner}\,\prime}\big)+\beta_1D_2^{\mathrm{Miner}\,\prime}+1-D_2^{\mathrm{Miner}\,\prime}-D_1^{\mathrm{Miner}\,\prime}=0\end{aligned} \tag{15}$$

Fig. 1. (Colour online.) "Diabolo" specimen.

where $D_2^{\text{Miner}\,\prime} \equiv N_2'/N_{f2}$, $D_1^{\text{Miner}\,\prime} \equiv N_1'/N_{f1}$ are the new Miner damages. Eqs. (14) and (15), put together, allow us to determine coefficients β_1 and β_2. If more experiments are available, the set of equations on these coefficients becomes over-determined, but may be solved in a least-squares sense. In all cases it is necessary to check that the necessary conditions (8) are satisfied, and if not to replace β_1 or β_2 by the maximum or minimum value allowed.

Remark. Consider the same two experiments as above, with the same notations plus $\mathcal{D}'$, which represents the damage at the end of the first phase of the second experiment. By Eqs. $(10)_1$ and (11),

$$\begin{cases} D_1^{\text{Miner}} + D_2^{\text{Miner}} = 1 + G(\mathcal{D}, N_{f1}) - G(\mathcal{D}, N_{f2}) \\ D_2^{\text{Miner}\,\prime} + D_1^{\text{Miner}\,\prime} = 1 + G(\mathcal{D}', N_{f2}) - G(\mathcal{D}', N_{f1}) \end{cases}$$

Now it is easy to check that for the specific function G defined by Eq. $(10)_2$, the expression $G(\mathcal{D}, N_{f1}) - G(\mathcal{D}, N_{f2})$, considered as a function of $\mathcal{D}$, has a constant sign; hence $G(\mathcal{D}, N_{f1}) - G(\mathcal{D}, N_{f2})$ and $G(\mathcal{D}', N_{f2}) - G(\mathcal{D}', N_{f1})$ are of opposite signs, whatever the values of $\mathcal{D}$ and $\mathcal{D}'$. It follows that if $D_1^{\text{Miner}} + D_2^{\text{Miner}} < 1$, then $d_2^{\text{Miner}\,\prime} + D_1^{\text{Miner}\,\prime} > 1$; and vice-versa. Hence the model predicts that *the deviation from Miner's rule, measured by the quantity* $D_1^{\text{Miner}} + D_2^{\text{Miner}} - 1$, *necessarily changes sign upon reversal of the sequence of loadings*.

4. Experimental procedure

What follows is just a brief presentation of the experiments performed. More details are provided in Jardin's thesis [17].

4.1. General presentation

Fatigue experiments are performed on "diabolo" specimens. Fig. 1 shows a photograph of one such specimen. The height and minimum diameter are 18 mm and 8 mm, respectively. The specimens are made of two elastomeric materials, denoted A and B in the sequel, which differ through their carbon contents.

The specimen are subjected to various 1D cyclic loadings at room temperature. The frequency of the cycles is 3 or 6 Hz, depending on the experiment; such low frequencies warrant negligible heating of the specimens. In each cyclic loading considered, the force applied varies between zero and some maximum value. Each experiment is performed at least three times on different specimens, in order to get an estimate of the experimental scatter.

The number of cycles at failure of the specimen is recorded for each experiment. This number is identified to that corresponding to initiation of a macroscopic crack in an elementary volume of material. The underlying hypothesis is that many more cycles are required for crack initiation than for crack propagation. Results shown below will be seen to support this approximation. Another argument is that it only leads to some *systematic* overestimation of the number of cycles necessary for initiation of a macroscopic crack, which should not affect the qualitative conclusions drawn.

4.2. Determination of numbers of cycles at failure

Prior to studying deviations from Miner's rule, it is necessary to determine the number of cycles at failure for a given cyclic load of constant amplitude. The loads considered are chosen so as to lead to complete failure within a "reasonable" number of cycles N_f. (Loads yielding very small values of N_f are simply not acceptable, whereas those yielding very large

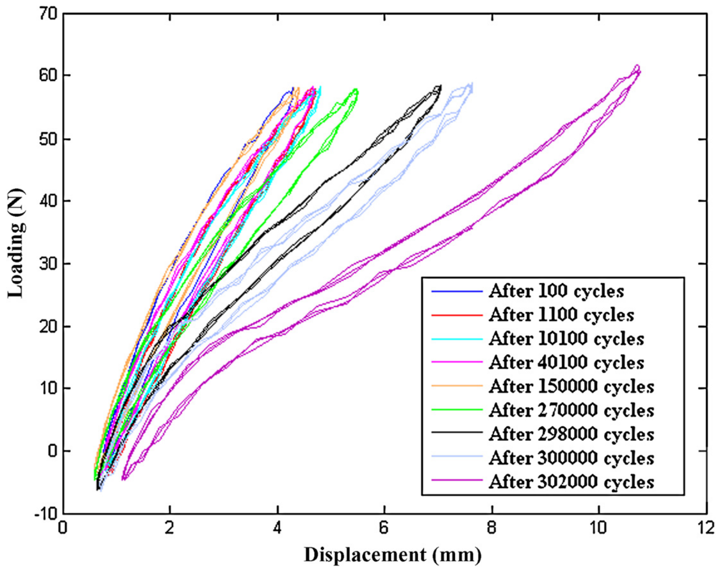

Fig. 2. (Colour online.) Load-displacement curve during a typical cyclic loading of constant amplitude.

values are uninteresting since they have no effect on the fatigue behaviour.) The maximum forces used are 60 and 90 N for material A, and 110 and 130 N for material B.

Fig. 2 shows the load-displacement curve during a typical test. The minimum force applied is not strictly zero but slightly negative, but this is harmless since negative stresses have no effect on the fatigue behaviour. The maximum force applied can be seen to remain almost constant, as desired, even when the displacement increases considerably during propagation of the crack and final failure of the specimen. Also, this final phase can be observed to approximately extend from cycle 270,000 to cycle 300,000, which represents only 10% of the total number of cycles; this justifies the approximate identification of the numbers of cycles at initiation of a macroscopic crack and at failure of the specimen.

The numbers of cycles at failure found are as follows: for material A, 135,000 with a scatter of about 10% for a maximum force of 60 N and 20,000 with a scatter of about 35% for a maximum force of 90 N; for material B, 300,000 with a scatter of about 30% for a maximum force of 110 N and 37,000 with a scatter of about 5% for a maximum force of 130 N.

4.3. Experimental study of deviations from Miner's law

Once the values of N_f are known for the cyclic loads considered, deviations from Miner's law are studied by subjecting specimens to successions of two different cyclic loads. The effect of the sequence of loads is studied by systematically reverting it. Two possibilities are envisaged for each sequence: $0.33N_f$ cycles of the first cyclic load, then enough cycles of the second to break the specimen; and similarly with $0.66N_f$ cycles of the first cyclic load.

Fig. 3 shows the load-displacement curve during a typical succession of cyclic loads. Again, the maximum force applied can be observed to remain almost constant during each of the cyclic loads applied in succession, as desired.

The numbers of cycles at failure thus determined experimentally are presented and compared to model predictions in Section 5 below.

5. Results and comparison with the model proposed

Table 1 shows deviations from Miner's rule for material A. The quantities provided are the experimental and theoretical values of Miner's damages for the two phases of each experiment plus their sum, denoted as "Miner's total damage". There are 12 experiments, 6 for the loading sequence $(\mathcal{C}_1, \mathcal{C}_2)$ and 6 for the opposite one $(\mathcal{C}_2, \mathcal{C}_1)$; $\mathcal{C}_1$ and $\mathcal{C}_2$ here denote the cyclic loads with maximum forces of 60 N and 90 N respectively. The model parameters β_1 and β_2 are determined through a least-squares fitting of the experimental results and amount to 0.437 and −0.340 respectively. Once this is done, one uses Eq. (13) with these values of β_1 and β_2 to calculate the damage $\mathcal{D}$ at the end of the first phase of the loading, then Eqs. (10) and (11) to calculate the model values of the Miner damages of the two phases.

The model can be seen to reproduce the experimental results tolerably well. One interesting observation is the general experimental tendency toward a lower deviation from Miner's rule ($D_1^{\text{Miner}} + D_2^{\text{Miner}} < 1$) when the load with the smaller maximum force is applied first, and an upper deviation ($D_1^{\text{Miner}} + D_2^{\text{Miner}} > 1$) when the load with the larger maximum force is applied first. (There are some exceptions, but the deviation from Miner's rule is modest and probably not significant in these cases.) Note that there is a difference with the case of metals here since as mentioned in the Introduction, for the latter materials, lower deviations from Miner's rule are systematically obtained by applying the larger load first.

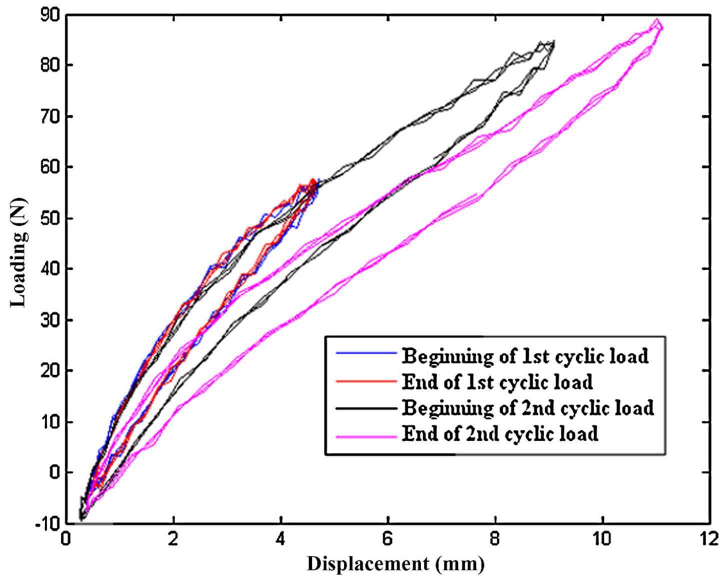

Fig. 3. (Colour online.) Load-displacement curve during a typical cyclic loading of variable amplitude.

Table 1
Comparison of experimental and predicted Miner damages for material A.

Load order	D_1^{Miner}		D_2^{Miner}		D^{Miner} total exp.	D^{Miner} total pred.
	D_1^{Miner} exp.	D_1^{Miner} pred.	D_2^{Miner} exp.	D_2^{Miner} pred.		
$(\mathcal{C}_1, \mathcal{C}_2)$	0.33	0.17	0.81	0.70	1.14	0.87
	0.33	0.44	0.31	0.38	0.64	0.82
	0.33	0.55	0.13	0.28	0.46	0.83
	0.66	0.80	0.01	0.11	0.67	0.91
	0.66	0.50	0.44	0.32	1.10	0.82
	0.66	0.64	0.23	0.22	0.89	0.86
$(\mathcal{C}_2, \mathcal{C}_1)$	0.40	0.46	0.66	0.70	1.06	1.16
	0.63	0.56	0.66	0.62	1.29	1.18
	0.64	0.57	0.66	0.62	1.30	1.19
	0.57	0.73	0.33	0.44	0.90	1.17
	0.64	0.75	0.33	0.41	0.97	1.16
	1.10	0.90	0.33	0.19	1.43	1.09

Table 2 shows results for material B in a similar way. The symbols $\mathcal{C}_1$ and $\mathcal{C}_2$ now refer to the cyclic loads with maximum forces of 110 N and 130 N, respectively. The values of the model parameters β_1 and β_2 are -0.09 and 1 respectively; note that β_2 takes the maximum value allowed by inequality $(8)_2$. Experimental deviations from Miner's rule are notably larger than for material A. The reproduction of experimental results by the model is however still acceptable. It is interesting to note that a lower deviation is now obtained when the load with the larger maximum force is applied first, and an upper deviation when the load with the lower maximum force is applied first, in agreement with what is commonly observed in metals. The explanation of the different behaviours of materials A and B in this respect is unknown.

The following additional remarks are also in order:

– on average, the experiments on both materials confirm the model's prediction of a change of sign of the deviation from Miner's rule upon reversal of the sequence of loadings;
– in a few experiments (on material B), the Miner damage during the second phase of the loading exceeds unity. (In other words, the first cyclic loading seems to have paradoxically "reinforced" the material.) The model is unfortunately unable to reproduce such a feature. Indeed it is clear from Eq. (11), where $G(\mathcal{D}, N_{f2}) > 0$, that N_2/N_{f2} cannot exceed unity. This restriction is not tied to the specific form (7) of the function g, but would subsist for any evolution equation of type (4).

Table 2
Comparison of experimental and predicted Miner damages for material B.

Load order	D_1^{Miner}		D_2^{Miner}		D^{Miner} total exp.	D^{Miner} total pred.
	D_1^{Miner} exp.	D_1^{Miner} pred.	D_2^{Miner} exp.	D_2^{Miner} pred.		
$(\mathcal{C}_1, \mathcal{C}_2)$	0.33	0.27	1.66	0.94	1.99	1.21
	0.33	0.23	2.08	0.96	2.41	1.19
	0.33	0.27	1.56	0.94	1.89	1.21
	0.66	0.59	1.47	0.68	2.13	1.27
	0.66	0.68	0.40	0.57	1.06	1.25
	0.66	0.64	0.74	0.61	1.40	1.25
$(\mathcal{C}_2, \mathcal{C}_1)$	0.13	0.13	0.66	0.74	0.79	0.87
	0.06	0.07	0.66	0.86	0.72	0.93
	0.05	0.06	0.66	0.87	0.71	0.93
	0.15	0.18	0.33	0.65	0.48	0.83
	0.11	0.14	0.33	0.71	0.44	0.85
	0.08	0.12	0.33	0.76	0.41	0.88

6. Conclusion

A new, simple heuristic model for the fatigue of elastomers under variable cyclic loadings has been proposed. This model reproduces possible deviations from Miner's linear law of cumulative damage while allowing for an arbitrary dependence of the number of cycles at failure upon the instantaneous load cycle.

Experiments have been performed on "diabolo" specimens made of two different elastomeric materials and subjected to two different successive cyclic loads. These experiments evidence significant deviations from Miner's rule in at least one of the materials considered. The agreement between the experimental results and the model predictions, for suitable values of the adjustable material parameter involved, is acceptable. The experiments confirm in particular the model's prediction that the deviation from Miner's rule should change sign upon reversal of the sequence of loadings.

Appendix A. Stigh's proof that Miner's rule implies factorization of the function $f(D, \mathcal{C})$

Assume that fatigue is governed by the evolution law (1) of the damage variable D, and consider a loading history consisting of a succession of N_1 load cycles $\mathcal{C}_1$ followed by N_2 load cycles $\mathcal{C}_2$, leading to failure. During the first phase of this loading history, D increases from 0 to some value $\mathcal{D}$, following the evolution equation $\mathrm{d}D/\mathrm{d}N = f(D, \mathcal{C}_1)$. Therefore

$$N_1 = \int_0^{\mathcal{D}} \frac{\mathrm{d}D}{f(D, \mathcal{C}_1)} \tag{16}$$

Also, the number of cycles at failure for a succession of load cycles $\mathcal{C}_1$ is given by

$$N_{f1} \equiv N_f(\mathcal{C}_1) = \int_0^1 \frac{\mathrm{d}D}{f(D, \mathcal{C}_1)} \tag{17}$$

During the second phase of the loading history, D further increases from $\mathcal{D}$ to 1, following the evolution equation $\mathrm{d}D/\mathrm{d}N = f(D, \mathcal{C}_2)$. Therefore

$$N_2 = \int_{\mathcal{D}}^1 \frac{\mathrm{d}D}{f(D, \mathcal{C}_2)} = N_{f2} - \int_0^{\mathcal{D}} \frac{\mathrm{d}D}{f(D, \mathcal{C}_2)} \tag{18}$$

where

$$N_{f2} \equiv N_f(\mathcal{C}_2) = \int_0^1 \frac{\mathrm{d}D}{f(D, \mathcal{C}_2)} \tag{19}$$

It follows from Eqs. (16)–(19) that

$$\frac{N_1}{N_{f1}} + \frac{N_2}{N_{f2}} = 1 + \frac{\int_0^{\mathcal{D}} \frac{\mathrm{d}D}{f(D,\mathcal{C}_1)}}{\int_0^1 \frac{\mathrm{d}D}{f(D,\mathcal{C}_1)}} - \frac{\int_0^{\mathcal{D}} \frac{\mathrm{d}D}{f(D,\mathcal{C}_2)}}{\int_0^1 \frac{\mathrm{d}D}{f(D,\mathcal{C}_2)}}$$

and therefore that

$$(\text{Miner's rule holds}) \quad \Rightarrow \quad \frac{N_1}{N_{f1}} + \frac{N_2}{N_{f2}} = 1 \quad \Rightarrow \quad \frac{\int_0^{\mathcal{D}} \frac{dD}{f(D,\mathcal{C}_1)}}{\int_0^1 \frac{dD}{f(D,\mathcal{C}_1)}} = \frac{\int_0^{\mathcal{D}} \frac{dD}{f(D,\mathcal{C}_2)}}{\int_0^1 \frac{dD}{f(D,\mathcal{C}_2)}} \tag{20}$$

for all load cycles $\mathcal{C}_1$ and $\mathcal{C}_2$. By Eq. (20), if Miner's rule applies, the function $\int_0^D \frac{dD'}{f(D',\mathcal{C})} / \int_0^1 \frac{dD'}{f(D',\mathcal{C})}$ is independent of the load cycle $\mathcal{C}$. Denote this function $\Phi(D)$. Then

$$\int_0^D \frac{dD'}{f(D',\mathcal{C})} \equiv \Phi(D)\Psi(\mathcal{C}), \quad \Psi(\mathcal{C}) \equiv \int_0^1 \frac{dD'}{f(D',\mathcal{C})}$$

Differentiating this equation with respect to D, one gets

$$\frac{1}{f(D,\mathcal{C})} = \Phi'(D)\Psi(\mathcal{C}) \quad \Rightarrow \quad f(D,\mathcal{C}) = \frac{1}{\Phi'(D)} \frac{1}{\Psi(\mathcal{C})} \tag{21}$$

which shows that the function $f(D,\mathcal{C})$ is factorized and concludes the proof.

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