C. R. Mecanique 342 (2014) 513–519

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Blow-up of solutions to quasilinear hyperbolic equations with $p(x,t)$-Laplacian and positive initial energy ☆

Explosion de solutions d'équations hyperboliques quasi linéaires avec $p(x,t)$-laplacien et énergie initiale positive

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ARTICLE INFO

Article history:
Received 18 April 2014
Accepted 2 June 2014
Available online 30 June 2014

Keywords:
Quasilinear hyperbolic
Blow-up in finite time
Positive initial energy

Mots-clés :
Quasilinéaire hyperbolique
Explosion en temps fini
Énergie initiale positive

ABSTRACT

The aim of this paper is to study an initial and homogeneous boundary value problem to a quasilinear hyperbolic equation with a $p(x,t)$-Laplacian and a positive initial energy. The authors prove that the solution blows up in a finite time under some conditions on the initial value, the exponents and the coefficients in the equation. The results generalize and improve that of S.N. Antonsev (2011) [6]. Besides, the conditions of positivity of the integral to the initial data and the boundedness of $p_t(x,t)$ are removed.

RÉSUMÉ

Le but de cet article est d'étudier un problème aux limites initial et homogène défini par une équation hyperbolique quasi linéaire avec un $p(x,t)$-Laplacien et une énergie initiale positive. Les auteurs montrent que la solution explose dans un temps fini sous certaines conditions sur la valeur initiale, les exposants et les coefficients de l'équation. Les résultats généralisent et améliorent celui de S.N. Antonsev (2011) [6]. En outre, les conditions de positivité de l'intégrale pour les données initiales et le caractère borné de $p_t(x,t)$ sont supprimées.

☆ The project is supported by NSFC (11271154, 11301211), by Fundamental Research Funds of Jilin University (450060501317) and by the 985 program of Jilin University.

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http://dx.doi.org/10.1016/j.crme.2014.06.001
1631-0721/

1. Introduction and main result

In this paper, we consider the following quasilinear hyperbolic problem:

$$\begin{cases} u_{tt} - \operatorname{div}\big(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u\big) - \Delta u_t = b(x,t)|u|^{q(x,t)-2}u, & (x,t) \in \Omega \times (0,T) := Q_T \\ u(x,t) = 0, & (x,t) = \partial\Omega \times (0,T) := \Gamma_T \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega \end{cases} \tag{1}$$

where $\Omega \subset R^N$ $(N \geqslant 1)$ is a bounded domain, $\partial\Omega$ is Lipschitz continuous. It will be assumed throughout the paper that the exponents $p(x,t)$, $q(x,t)$ and the coefficients $a(x,t)$, $b(x,t)$ satisfy the following conditions.

$$\begin{aligned} &1 < p^- \leqslant p(x,t) \leqslant p^+ < \infty, \qquad 1 < q^- \leqslant q(x,t) \leqslant q^+ < \infty \\ &0 < a^- \leqslant a(x,t) \leqslant a^+ < \infty, \qquad 0 < b^- \leqslant b(x,t) \leqslant b^+ < \infty \end{aligned} \tag{2}$$

The problem with variable exponent occurs in many mathematical models of applied science, for example, viscoelastic fluids, electro-rheological fluids, processes of filtration through a porous media, fluids with temperature-dependent viscosity etc. The interested readers may refer to [1–5] and the references therein. To the best of our knowledge, when p varies in space and time, only in [6], S.N. Antontsev discussed the blowing-up properties of solutions to the initial and homogeneous boundary value problem of quasilinear wave equations involving $p(x,t)$-Laplacian and a negative initial energy. However, it is natural to ask what happens when the initial energy is positive, whether the results of [6] do remain true. If so, is it a trivial generalization? In view of pure mathematics, we have to overcome the following difficulties in dealing with such problems. The main difficulties are the following: how we can establish the quantitative relationships among $\|\nabla u\|_{p(.)}$, $\|u\|_{q(.)}$ and the initial energy and what functional should be constructed to ensure that the initial energy may be controlled by the term $\int_\Omega |u|^{q(x,t)}\mathrm{d}x$. In this paper, we construct a new control function and apply suitable embedding theorems to establish the quantitative relationship between the term $\int_\Omega |u|^{q(x,t)}\mathrm{d}x$ and the initial energy. Furthermore, by modifying the functional constructed in [6] and utilizing the quantitative relationship between the term $\int_\Omega |u|^{q(x,t)}\mathrm{d}x$ and the initial energy, we prove that the solution blows up in finite time even in the case when the initial energy is positive. In addition, our results not only improve the conditions on the initial data and the exponents in the problem, but also remove the constrains of the boundedness of $p_t(x,t)$ and $q_t(x,t)$.

Define the energy functional as the following:

$$E(t) = \frac{1}{2}\int_\Omega |u_t|^2\mathrm{d}x + \int_\Omega \frac{a(x,t)}{p(x,t)}|\nabla u|^{p(x,t)}\mathrm{d}x - \int_\Omega \frac{b(x,t)}{q(x,t)}|u|^{q(x,t)}\mathrm{d}x$$

By [6], we find that the energy functional $E(t)$ satisfies the following identity.

Lemma 1.1. *Suppose that $u \in L^{q(x,t)}(Q_T) \cap L^\infty(0,T;W_0^{1,p(x,t)}(\Omega))$, $u_t \in L^2(0,T;H^1(\Omega))$ is a solution to Problem (1), then $E(t)$ satisfies the following identity:*

$$\begin{aligned} E(t) + \int_0^t\int_\Omega |\nabla u_s|^2\,\mathrm{d}x\,\mathrm{d}s &= E(0) + \int_0^t\int_\Omega \frac{a(x,s)p_s}{p^2}|\nabla u|^p\big(\ln|\nabla u|^p - 1\big)\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_0^t\int_\Omega \left(\frac{a_s(x,s)}{p}|\nabla u|^p - \frac{b_s(x,s)}{q}|u|^q\right)\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega \frac{b(x,s)q_s}{q^2}|u|^q\big(\ln|u|^q - 1\big)\mathrm{d}x\,\mathrm{d}s \end{aligned} \tag{3}$$

For simplicity, we give some notations and the embedding inequality to be used later. By Corollary 3.34 in [2], we know that $W_0^{1,p(x)}(\Omega) \hookrightarrow W_0^{1,p^-}(\Omega) \hookrightarrow L^r(\Omega)$ $(1 < r < \frac{Np^-}{N-p^-})$. Let B be the best constant of the embedding inequality:

$$\|u\|_r \leqslant B\|\nabla u\|_{p(.)}, \quad \forall u \in W_0^{1,p(x)}(\Omega)$$

Set $E_1 = \frac{(r-p^+)a^-}{rp^+}\alpha_1$, $\alpha_1 = (\frac{a^-}{b^+B_1^r})^{\frac{p^+}{r-p^+}}$, where $B_1 = \max\{B, 1, (\frac{a^-}{b^+})^{\frac{1}{r}}\}$.

First, we consider the simplest case when $p(x,t)$ is independent of t and $q(x,t) \equiv r$ is a fixed constant. Our main result is as following.

Theorem 1.1. *Assume that the exponent $p(x)$, the coefficients $a(x,t)$, $b(x,t)$ and $u_0(x)$, $u_1(x)$ satisfy (2) and the following conditions:*

(H_1) $\quad 0 \not\equiv u_0 \in L^2(\Omega) \cap W_0^{1,p(x)}(\Omega), u_1 \in L^2(\Omega), E(0) < E_1, \min\{\|\nabla u_0\|_{p(x)}^{p^-}, \|\nabla u_0\|_{p(x)}^{p^+}\} > \alpha_1$

(H_2) $\quad r > \max\{2, p^+\}, a_t(x,t) \leqslant 0, b_t(x,t) \geqslant 0, \forall x \in \Omega, t \geqslant 0$

then the solution of Problem (1) blows up in finite time.

By Lemma 1.1, it is easy to verify that the following inequality holds.

Lemma 1.2. *Suppose that u is a solution to Problem (1) and the coefficients $a(x,t)$, $b(x,t)$ satisfy the conditions of Theorem 1.1, then $E(t)$ satisfies:*

$$E(t) + \int_0^t \int_\Omega |\nabla u_t|^2 \mathrm{d}x\,\mathrm{d}s \leqslant E(0), \quad t \geqslant 0$$

Next, we establish quantitative relationship between the term $\int_\Omega |u|^r \mathrm{d}x$ and the initial energy.

Lemma 1.3. *Let u be a solution to Problem (1). If (H_1) and (H_2) are satisfied, then there exists a positive constant $\alpha_2 > \alpha_1$ such that for all $t \geqslant 0$:*

$$\int_\Omega |\nabla u|^{p(x)} \mathrm{d}x \geqslant \alpha_2 \tag{4}$$

$$\int_\Omega |u|^r \mathrm{d}x \geqslant B_1^r \max\left\{\alpha_2^{\frac{r}{p^-}}, \alpha_2^{\frac{r}{p^+}}\right\} \tag{5}$$

Proof. A simple analysis shows that:

$$E(t) \geqslant \frac{a^-}{p^+}\alpha(t) - \frac{b^+ B_1^r}{r} \max\left\{\alpha^{\frac{r}{p^-}}(t), \alpha^{\frac{r}{p^+}}(t)\right\} = h\big(\alpha(t)\big) \tag{6}$$

with $\alpha(t) = \int_\Omega |\nabla u|^{p(x)} \mathrm{d}x$.

Following the lines of the proof of Lemma 2.2 in [7], we have that $h(\alpha)$ is increasing for $0 < \alpha < \alpha_1$ while $h(\alpha)$ is decreasing for $\alpha \geqslant \alpha_1$, and $\lim_{\alpha\to\infty} h(\alpha) = -\infty$. Due to $E(0) < E_1$, then there exists a positive constant $\alpha_2 > \alpha_1$ such that $h(\alpha_2) = E(0)$. By $\min\{\|\nabla u_0\|_{p(x)}^{p^-}, \|\nabla u_0\|_{p(x)}^{p^+}\} > \alpha_1$, we get $h(\alpha_0) \leqslant E(0) = h(\alpha_2)$, where $\alpha_0 = \int_\Omega |\nabla u_0|^{p(x)} \mathrm{d}x$. Once again applying the monotonicity of $h(\alpha)$, we have $\alpha_0 \geqslant \alpha_2$.

Define $F(t) \triangleq \int_\Omega |\nabla u(.,t)|^{p(x)} \mathrm{d}x$. According to $u \in L^\infty(0,T; W_0^{1,p(x)}(\Omega))$, we know that $F(t) \in L^\infty(0,T)$. Next, we prove (4) by arguing by contradiction. Suppose that there exists a $t_0 > 0$ such that $F(t_0) < \alpha_2$. Hence, it is easy to verify that Steklov averages

$$F_h(t) = \begin{cases} \frac{1}{h}\int_t^{t+h} F(s)\mathrm{d}s, & t \in (0, T-h] \\ \frac{1}{h}\int_{t-h}^t F(s)\mathrm{d}s, & t > T-h \end{cases}$$

is continuous with respect to time t and $\lim_{h\to 0^+} F_h(t) = F(t)$, for $t \geqslant 0$. So it is not difficult to check that there exists a $\delta > 0$ such that

$$F_h(t_0) < \alpha_2, \quad |h| < \delta$$

Furthermore, by choosing $0 < \varepsilon_0 < \alpha_2 - \alpha_1$ and using the continuity of $F_h(t)$, we get that there exists a $t_1 > 0$ such that

$$\alpha_1 < F_h(t_1) < \alpha_2 - \varepsilon_0 \tag{7}$$

Letting $h \to 0$ in (7), we have

$$\alpha_1 \leqslant F(t_1) \leqslant \alpha_2 - \varepsilon_0 < \alpha_2.$$

By the definitions of $E(t)$ and the monotonicity of $h(\alpha)$, we have

$$E(t_1) \geqslant h\left(\int_\Omega \left|\nabla u(.,t_1)\right|^{p(x)} \mathrm{d}x\right) > h(\alpha_2) = E(0)$$

which contradicts $E(t) \leqslant E(0)$, $\forall t \geqslant 0$.

The rest is similar to that of Lemma 2.1 in [7], we omit the details. $\square$

Define

$$L(t)=\frac{1}{2}\int_{\Omega}|u|^2\mathrm{d}x+\frac{1}{2}\int_0^t\int_{\Omega}|\nabla u|^2\mathrm{d}x\,\mathrm{d}\tau-\frac{t}{2}\|\nabla u_0\|_2^2+\beta(t+t_0)^2$$

where $\beta>0$, $t_0>\max\{\frac{\|\nabla u_0\|_2^2}{\beta},\frac{\|u_1u_0\|_1}{2\beta}\}$ will be determined later. The idea of the following proof comes from [9]. A direct computation shows that:

$$L'(t)=\int_{\Omega}uu_t\mathrm{d}x+\int_0^t\int_{\Omega}\nabla u\nabla u_\tau\,\mathrm{d}x\,\mathrm{d}\tau+2\beta(t+t_0)$$

$$L''(t)=\int_{\Omega}u_tu_t\mathrm{d}x+\int_{\Omega}uu_{tt}\mathrm{d}x+\int_{\Omega}\nabla u\nabla u_t\mathrm{d}x+2\beta$$

According to the definition of $E(t)$, the expression of $L''(t)$ and Lemma 1.2, we have

$$\begin{aligned}L''(t)&\geqslant\frac{r+2}{2}\|u_t\|_2^2+\frac{r}{p^+}\int_{\Omega}|\nabla u|^{p(x)}\mathrm{d}x+r\int_0^t\int_{\Omega}|\nabla u_\tau|^2\mathrm{d}x\,\mathrm{d}\tau-rE_1+2\beta\\&\geqslant\frac{r+2}{2}\left[\|u_t\|_2^2+\int_0^t\int_{\Omega}|\nabla u_\tau|^2\mathrm{d}x\,\mathrm{d}\tau+4\beta\right]\end{aligned}$$

where $\beta=\frac{a^-\alpha_1}{2(r+1)p^+}$. Here we have used inequality (4) and the definition of E_1. Hence

$$\begin{aligned}\big(L'(t)\big)^2&\leqslant\Bigg[\|u_t\|_2^2\|u\|_2^2+\int_0^t\int_{\Omega}|\nabla u_\tau|^2\mathrm{d}x\,\mathrm{d}\tau\int_0^t\int_{\Omega}|\nabla u|^2\mathrm{d}x\,\mathrm{d}\tau\\&\quad+\|u_t\|_2^2\int_0^t\int_{\Omega}|\nabla u|^2\mathrm{d}x\,\mathrm{d}\tau+\|u\|_2^2\int_0^t\int_{\Omega}|\nabla u_\tau|^2\mathrm{d}x\,\mathrm{d}\tau\\&\quad+4\beta^2(t+t_0)^2+4\beta(t+t_0)\left(\left|\int_{\Omega}uu_t\mathrm{d}x\right|+\left|\int_0^t\int_{\Omega}\nabla u\nabla u_\tau\,\mathrm{d}x\,\mathrm{d}\tau\right|\right)\Bigg]\\&\leqslant2L(t)\left[\|u_t\|_2^2+\int_0^t\int_{\Omega}|\nabla u_t|^2\mathrm{d}x\,\mathrm{d}\tau+4\beta\right]\end{aligned}$$

The last inequality above follows from

$$4\beta(t+t_0)\left|\int_{\Omega}uu_t\mathrm{d}x\right|\leqslant2\|u_t\|_2^2\left(\beta(t+t_0)^2-\frac{t}{2}\|\nabla u_0\|_2^2\right)+2\beta^2(t+t_0)^2\|u\|_2^2\left(\beta(t+t_0)^2-\frac{t}{2}\|\nabla u_0\|_2^2\right)^{-1}$$

So we have

$$L(t)L''(t)-\frac{r+2}{4}\big(L'(t)\big)^2\geqslant0$$

where $r>2$, which implies

$$\left(L^{1-\frac{r+2}{4}}(t)\right)''\leqslant0,\quad\text{for }t>0$$

Noting that $L^{1-\frac{r+2}{4}}(0)>0$, $(L^{1-\frac{r+2}{4}})'(0)<0$, then

$$L^{1-\frac{r+2}{4}}\big(T^*\big)=0,\quad\text{for some }T^*\in\left(0,\frac{-L^{1-\frac{r+2}{4}}(0)}{(L^{1-\frac{r+2}{4}})'(0)}\right)$$

In the case when $q(x,t)=r(x)$, we have the following theorem.

Theorem 1.2. *Assume that $p(x)$, $r(x)$ and $u_0(x)$ satisfy the following conditions:*

$$(H_3)\quad 0\not\equiv u_0\in L^2(\Omega)\cap W_0^{1,p(x)}(\Omega),\, u_1\in L^2(\Omega),\, E(0)<E_1,\, \min\left\{\|\nabla u_0\|_{p(x)}^{p^-},\|\nabla u_0\|_{p(x)}^{p^+}\right\}>\alpha_1$$

$$(H_4)\quad \max\{2,p^+\}<q^-\leqslant r(x)\leqslant q^+,\, a_t(x,t)\leqslant 0,\, b_t(x,t)\geqslant 0,\, \forall x\in\Omega,\, t\geqslant 0$$

then the solution of Problem (1) blows up in finite time.

We only need to replace $\int_\Omega |u|^{r(x)}\mathrm{d}x$ by $\max\{\|u\|_{r^+}^{r^+},\|u\|_{r^-}^{r^-}\}$, the rest of the proof of Theorem 1.2 is the same as that to the previous theorem. Next, we give an example to illustrate that there exists a u_0 satisfying the condition of Theorem 1.2. For simplicity, we assume that $a(x,t)=b(x,t)\equiv 1$. Denote

$$\widetilde{E}=\left\{(u_0,u_1)\in W_0^{1,p(x)}(\Omega)\times L^2(\Omega):\begin{cases}\frac{1}{2}\int_\Omega |u_1|^2\mathrm{d}x+\int_\Omega \frac{|\nabla u_0|^{p(x)}}{p(x)}\mathrm{d}x-\int_\Omega\frac{|u_0|^{r(x)}}{r(x)}\mathrm{d}x>0,\\ \frac{1}{2}\int_\Omega |u_1|^2\mathrm{d}x+\int_\Omega \frac{|\nabla u_0|^{p(x)}}{p(x)}\mathrm{d}x-\int_\Omega\frac{|u_0|^{r(x)}}{r(x)}\mathrm{d}x<E_1,\\ \min\left\{\|\nabla u_0\|_{p(x)}^{p^-},\|\nabla u_0\|_{p(x)}^{p^+}\right\}>\alpha_1.\end{cases}\right\}$$

Then we get Proposition 1.1.

Proposition 1.1. *The set $\widetilde{E}$ is not empty.*

Proof. First, we consider the simplest cases when $r(x)=r$ is a fixed constant and $r(p^+-p^-)<p^-(r-p^+)$. And then we may choose a suitable positive constant B_1 satisfying

$$\max\{B,1\}<B_1<\left(\frac{p^-r-p^-p^+}{rp^+-rp^-}\right)^{\frac{r-p^+}{r(p^+-p^-)}}$$

In fact, the inequalities above hold by choosing a suitable domain Ω, since the embedding constant B is dependent on $|\Omega|$.

Subsequently, by Theorem 2 of [8], we know that for any $\lambda_0>0$, there exists a nontrivial solution $\varphi(x)\in W_0^{1,p(x)}(\Omega)$ to the following problem:

$$(\Delta)\quad\begin{cases}-\operatorname{div}\left(|\nabla\varphi|^{p(x)-2}\nabla\varphi\right)=\lambda_0|\varphi|^{r-2}\varphi, & x\in\Omega\\ \varphi=0, & x\in\partial\Omega\end{cases}$$

where $\max\{2,p^+\}<r<\frac{Np^-}{N-p^-}$. In particular, we choose λ_0 satisfying $B_1^{\frac{rp^-}{p^+-r}}<\lambda_0<\min\{\frac{p^+p^-}{p^+-p^-}B_1^{\frac{rp^+}{p^+-r}},\frac{p^-}{r}\}$ such that $\int_\Omega|\varphi(x)|^r\mathrm{d}x=1$. And then, multiplying the first identity in (Δ) by $\varphi(x)$ and integrating over Ω, we have $\int_\Omega|\nabla\varphi|^{p(x)}\mathrm{d}x=\lambda_0\int_\Omega|\varphi(x)|^r\mathrm{d}x$. Without loss of generality, we also assume that $\int_\Omega|\varphi|^r\mathrm{d}x=1$ (in fact, set $S=\{\varphi\in W_0^{1,p(x)},\|u\|_r=1\}$, $J(\varphi)=\int_\Omega\frac{1}{p(x)}|\nabla\varphi|^{p(x)}\mathrm{d}x-\lambda_0$. Then S is a weakly closed set of $W_0^{1,p(x)}(\Omega)$ and $J(\varphi)$ is coercive, weakly lower continuous and bounded from below, so $J(\varphi)$ attains its infimum value in S).

Next, we construct a function ψ satisfying $\frac{1}{r}-\frac{\lambda_0}{p^+}<\int_\Omega\frac{|\psi|^2}{2}\mathrm{d}x<E_1+\frac{1}{r}-\frac{\lambda_0}{p^-}$, for example $\psi=|\Omega|^{-\frac{1}{2}}(E_1-\frac{\lambda_0(p^++p^-)}{p^+p^-}+\frac{2}{r})^{\frac{1}{2}}$.

Finally, we prove $(\varphi,\psi)\in\widetilde{E}$. It is apparent from the proof above that the following conclusions hold:

$$\begin{cases}0<\frac{1}{2}\int\limits_\Omega|\psi|^2\mathrm{d}x+\int\limits_\Omega\frac{|\nabla\varphi|^{p(x)}}{p(x)}\mathrm{d}x-\int\limits_\Omega\frac{|\varphi|^r}{r}\mathrm{d}x<E_1\\ \min\left\{\|\nabla\varphi\|_{p(x)}^{p^-},\|\nabla\varphi\|_{p(x)}^{p^+}\right\}\geqslant\lambda_0^{\frac{p^+}{p^-}}>\alpha_1\end{cases}$$

which shows that the set $\widetilde{E}$ is not empty. Similarly, we may generalize the results to the case when r is a function with respect to a spatial variable. The proof is left to the reader. $\square$

At the end of this paper, we consider the case when p, q are dependent on t.

Theorem 1.3. *Assume that* $p(x,t)$, $q(x,t)$, *the coefficients* $a(x,t)$, $b(x,t)$ *and* $u_0(x)$, $u_1(x)$ *satisfy* (2) *and the following conditions:*

$$(H_5)\quad 0 \not\equiv u_0 \in W_0^{1,p(x,0)}(\Omega), u_1 \in L^2(\Omega), E(0) + \left(\frac{a^+}{p^-} + \frac{b^+}{q^-}\right)|\Omega| < E_1, \min\left\{\|\nabla u_0\|_{p(x)}^{p^-}, \|\nabla u_0\|_{p(x)}^{p^+}\right\} > \alpha_1$$

$$(H_6)\quad \max\left\{2, p^+\right\} < q^- \leqslant q(x,t) \leqslant q^+ < \frac{Np^-}{N-p^-}, a_t(x,t) \leqslant 0, b_t(x,t) \geqslant 0, \forall x \in \Omega, t \geqslant 0$$

$$(H_7)\quad p_t \leqslant 0, q_t \geqslant 0, \left|\frac{p_t}{p^2}\right| + \left|\frac{q_t}{q^2}\right| \in L^1_{\text{loc}}\big((0,\infty); L^1(\Omega)\big)$$

then the solution of Problem (1) *blows up in finite time.*

In order to prove this theorem, we need to prove a new energy estimate.

Lemma 1.4. *Suppose that* (H_7) *holds, then the energy functional* $E(t)$ *satisfies:*

$$E(t) + \int_0^t \int_\Omega |\nabla u_t|^2 \mathrm{d}x\,\mathrm{d}s \leqslant E(0) + \left(\frac{a^+}{p^-} + \frac{b^+}{q^-}\right)|\Omega|, \quad t \geqslant 0$$

Proof. From Lemma 1.1, we have:

$$\begin{aligned} E'(t) + \int_\Omega |\nabla u_t|^2 \mathrm{d}x \leqslant & \int_\Omega \frac{a(x,t)|\nabla u|^{p(x,t)}}{p^2(x,t)}\big(p(x,t)\ln|\nabla u| - 1\big)p_t(x,t)\mathrm{d}x \\ & - \int_\Omega \frac{b(x,t)|u|^{q(x,t)}}{q^2(x,t)}\big(q(x,t)\ln|u| - 1\big)q_t(x,t)\mathrm{d}x := J_1 + J_2 \end{aligned} \tag{8}$$

Next, we estimate the value of J_1 and J_2, respectively:

$$\begin{aligned} J_1 & \leqslant \int_{\{|\nabla u|^p \leqslant \mathrm{e}\}} \frac{a(x,t)|\nabla u|^{p(x,t)}}{p^2(x,t)}\big(\ln|\nabla u|^{p(x,t)} - 1\big)p_t(x,t)\mathrm{d}x \\ & \leqslant \int_{\{|\nabla u|^p \leqslant \mathrm{e}\}} \frac{-p_t(x,t)a(x,t)}{p^2(x,t)}\mathrm{d}x \leqslant a^+ \int_\Omega \frac{-p_t(x,t)}{p^2(x,t)}\mathrm{d}x \end{aligned} \tag{9}$$

The second inequality above follows from:

$$-\frac{1}{\mathrm{e}} \leqslant s\ln s \leqslant 0, \quad 0 \leqslant s \leqslant 1$$

Similarly, we have:

$$J_2 \leqslant b^+ \int_\Omega \frac{q_t(x,t)}{q^2(x,t)}\mathrm{d}x \tag{10}$$

Eqs. (8)–(10) imply that

$$E'(t) + \int_\Omega |\nabla u_t|^2 \mathrm{d}x \leqslant \int_\Omega \frac{-p_t(x,t)}{p^2(x,t)}\mathrm{d}x + \int_\Omega \frac{q_t(x,t)}{q^2(x,t)}\mathrm{d}x$$

The conclusion of Lemma 1.4 follows easily. □

The rest of the proof of Theorem 1.3 is similar to that of Theorem 1.1, we omit the details here.

Remark 1.1. If $E(0)$ is bigger than zero, it seems that inequalities (19) and (21) in [6] do not hold, while (19) and (21) play an essential role in proving their main results. Particularly, when p is dependent on t, the argument of [6] is not applicable. However, it can be checked that the results in [6] may be obtained by the method in this paper.

Acknowledgements

The authors express their thanks to the referees for their helpful comments, which improved our original manuscript and for pointing reference [9].

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