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A direct approach for continuous topology optimization subject to admissible loading

Une approche directe pour l'optimisation de la topologie continue sous chargement admissible

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ABSTRACT

In the present paper, a method is proposed for topology optimization of continuum structures subject to static and plastic admissibility conditions relative to a prescribed load. A key feature of the method is that, using a finite-element discretization, the form of the resulting topology optimization problem is similar to that of the direct static approach of the limit analysis problem. The proposed method is formulated in plane strain using Tresca materials and is illustrated on example problems taken from the literature.

RÉSUMÉ

Dans le présent article, une méthode est proposée pour l'optimisation de la topologie des milieux continues soumis à des conditions d'admissibilité statique et plastique relativement à un chargement imposé. Une propriété essentielle de la méthode est qu'en utilisant une discrétisation par éléments finis, la forme du problème d'optimisation de la topologie résultant est similaire à celle du problème direct de l'analyse limite formulé selon l'approche statique. La méthode proposée est formulée en déformations planes en utilisant un matériau de Tresca. Elle est illustrée à travers des exemples de problèmes issus de la littérature.

1. Introduction

Research in topology optimization of continuum structures has witnessed a considerable development during the last decades [1–5]. This development led to near maturity, as demonstrated by the numerous successful applications in industry [6] and the emergence of powerful dedicated topology optimization software [7]. It is noted, however, that most of the work on continuum topology optimization has been restricted to linear elastic material behavior. Elastic design is historically the

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most common and most demanded type of design, and continuum topology design is not an exception in this regard. Elastoplastic analyses that seek to determine response quantities, especially evolution methods, are known for their high computational demand. On the contrary, direct methods of limit analysis are known to require lower computational effort to determine limit states in terms of either stress field or displacement/velocity field solutions. In an automated design context, where computational efficiency is a primary factor, direct methods of limit analysis become attractive for their considerable computational saving potential. Member sizing optimization of specific types of structures, e.g., trusses and frames, subject to plastic design constraints has occasionally been treated in the literature using direct methods of plastic collapse analysis [8]. Nevertheless, topology design of continuum media, involving this type of analyses, is nearly inexistent in the literature. Some research has dealt with continuum topology design of nonlinear elastic structures where the tools developed for the linear behavior were adapted and extended to the nonlinear case [2] and only a few tentatives were directed to design involving elastoplastic [9] or plastic analyses [10].

The present work is precisely concerned with the integration of direct methods of limit analysis into a methodology for plastic topology design of continuum structures. A judicious formulation for this class of problems is proposed such that the continuous design problem, which is expressed in terms of continuous material densities as design variables, takes on a mathematical form similar to that of a direct limit analysis problem. The computational demand for the topology design problem using the proposed approach is consequently expected to be in the order of that of the execution of a single limit analysis. In the present paper, the topology design problem of plastic continuum structures is formulated according to the microscopic, material approach and based on direct limit analysis. Some of the desirable properties of the design problem expressed in terms of continuous densities are highlighted. Formulation in these design variables leads to the so-called continuous or porous topologies. It has often been proposed as a preliminary to the ultimate goal of producing optimal black and white, i.e. 0–1, topologies. Owing to the recent breakthroughs in material technology, though, it is becoming possible to tailor the microstructure to suit a wide range of desired material properties and gradients. This development has regenerated interest in continuous topologies. Finally, a number of example design problems are treated to illustrate the capabilities of the proposed method and to compare the designs it generates with those produced using existing methods.

2. The static method of limit analysis

The following terminology defined in [11,12] will be adopted in the present paper. A stress field σ is said to be statically admissible (SA) if field equilibrium equations, stress vector continuity, and stress boundary conditions are satisfied. It is said to be plastically admissible if $f(\sigma) \leqslant 0$, where $f(\sigma)$ is the plasticity criterion of the material. A stress field σ that is both SA and plastically admissible will be said to be fully admissible or simply "admissible". A loading system $Q \in \mathbb{R}^{\mathrm{n}}$ in equilibrium with a statically admissible stress field σ, $Q = Q(\sigma)$, is said to be admissible. The n components of Q are called loading parameters. The relationship $Q = Q(\sigma)$, which usually describes either field equilibrium equations, when body forces are present, or boundary conditions on the stress vector, is linear in both cases. A solution of the limit analysis problem relative to the ith loading parameter is found by solving the following optimization problem for an admissible stress field σ such that:

$$\begin{aligned} &Q_{\mathrm{lim}} = \left(Q_1^{\mathrm{d}}, \ldots, \lambda_0 Q_i^{\mathrm{d}}, \ldots, Q_n^{\mathrm{d}}\right) \\ &\quad \lambda_0 = \max\left\{\lambda,\ Q(\sigma) = \left(Q_1^{\mathrm{d}}, \ldots, \lambda Q_i^{\mathrm{d}}, \ldots, Q_n^{\mathrm{d}}\right)\right\} \end{aligned} \tag{1}$$

where Q^{d} is a specified admissible loading. The resulting loading $Q(\sigma)$ is a limit loading of the mechanical domain. This formulation defines the static, lower bound problem of limit analysis, as it will be dealt with in the present work. Unlike the usual response-oriented analysis methods, the lower bound method of limit analysis determines the stress field at the limit state only. It provides neither information on the stress field at the intermediate stages of the loading process nor on the kinematic quantities at any loading step. For this reason, the method is classified as a direct method. This lack of information is compensated by lower computational demand which, in case the missing information is not necessary, becomes a paramount advantage. Another merit of the static approach is the status of rigorous lower bound of the limit load.

3. Finite-element formulation of the static problem

The numerical plane strain formulation of the static, lower bound problem is described in detail in [13]. Consider a triangular finite element discretization of the mechanical domain Ω in the global reference frame (x, y). The stress field is assumed to be linear in x and y within the element. Across interelement boundaries, it can be discontinuous, provided the stress vector acting on the element boundary remains continuous. In plane strain, the Tresca criterion is written as:

$$f(\sigma) = \sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2} - 2s \leqslant 0 \tag{2}$$

or equivalently as:

$$S(\sigma) = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \leqslant s \tag{3}$$

where s denotes the shear strength, or cohesion, of the material. In order to ensure static and plastic admissibility of the stress solution field, the following conditions are imposed:

- in each element, the equilibrium equations $\sigma_{ij,j} + \gamma_i = 0$ expressed in the Cartesian reference frame, where $\gamma = \rho g$ is the specific weight vector, ρ the mass density and g the acceleration of gravity;
- continuity conditions of the stress vector across discontinuity lines: for each discontinuity segment of normal n, the continuity of the stress vector $T_i = \sigma_{ij} n_j$ is imposed at the ends, defining this discontinuity segment;
- boundary conditions on the stress vector: $\sigma_{ij} n_j = T_i^{\mathrm{d}}$ at each end of the boundary element sides where the linearly varying stress vector T^{d} is imposed;
- stress field plastic admissibility at each triangle vertex. This ensures plastic admissibility over the total domain from the linear variation of the stress in a triangle and the convexity of the criterion (2).

Introducing a change of variables such that the stress vector σ is defined by the components $\frac{(\sigma_x+\sigma_y)}{2}$, denoted σ^+, $\frac{(\sigma_x-\sigma_y)}{2}$ and τ_{xy}, and writing the plasticity criterion directly in the conic form $X = s \geqslant \sqrt{Y^2 + Z^2}$, the numerical optimization problem, expressing the static limit analysis problem, can be written as a conic programming problem in the form:

$$\begin{aligned}
&\lambda_0 = \max \lambda \\
&Q(\sigma) = \left(Q_1^{\mathrm{d}}, \ldots, \lambda Q_i^{\mathrm{d}}, \ldots, Q_n^{\mathrm{d}}\right) \\
&S(\sigma) \leqslant s \\
&\sigma\ SA
\end{aligned} \tag{4}$$

and can, therefore, be solved using the conic programming code MOSEK [14] as in [15,16].

4. Topology optimization of continua

Research on topology optimization of continuum structures has been remarkably active since the publication of the papers by Bendsøe [17] and Bendsøe and Kikuchi [18]. Topology optimization of continuum structures aims at simultaneously optimizing the shape of external and internal boundaries, the number of holes and the connectivity within a specified 2D or 3D design domain Ω_0 with given boundary conditions. A design objective functional is predefined and a set of constraints are imposed. In the most popular formulations, the design objective consists of a global stiffness measure or a compliance functional. In this case, a bound is specified on the amount of structural material. Among the variety of solution procedures that have been developed, two major classes of approaches can be distinguished: the material, or microstructure approaches, and the geometrical or macrostructure approaches [2]. In the microstructure approaches, it is assumed that the material is distributed in some porous, microstructural form over the design domain Ω_0. It is customary to use a fixed, uniform finite-element mesh to model the structure within the entire, usually rectangular, design domain. The design variables are assumed to be constant within each finite element. For the analysis, finite elements are applied with properties that are related to material density based on physical modeling of the porous microstructures [2]. The discrete optimization consists in determining whether each element in the design domain should be solid or void, whereas the continuous optimization admits designs with intermediate densities.

Most of the work devoted to continuum topology optimization has been concerned with elastic material behavior. The mathematical approaches underlying the topology design methodologies for continuum structures often rely strongly on the linear elasticity assumption. The linearity assumption is essential to the equivalence of potential energy and compliance, which is the key to a large class of methods. It is also suitable for formulating interpolation schemes, such as SIMP (Solid Isotropic Material with Penalization) [17,19] and discards the difficulty associated with modeling the nonlinear behavior of materials at intermediate material densities. Regardless of the degree of complexity of the material behavior, topology optimization of continuum structures raised difficulties that took a considerable research effort to understand and overcome. These include the multiplicity of local minima, the issues of ill-posedness of the mathematical problem, the lack of convergence with respect to the size of the finite-element mesh [20], and the difficulty associated with local stress constraints [21].

The understanding reached on these issues and the strategies developed to resolve them through research on linear elastic topology design have been adapted and exploited in nonlinear elastic and plastic topology design involving evolution analysis methods. However, in dealing with topology design based on direct limit analysis, given the specificity of this type of analysis, these difficulties are expected to manifest themselves differently and will need to be addressed from a new perspective.

5. Topology optimization problem formulation

We consider a two-dimensional domain Ω_0 with a unit thickness, typically a bar with a rectangular section made out of a Tresca material characterized by a shear strength $\bar{s}$ and a density $\bar{\rho} = 1$ and subjected to a loading system

$Q = (Q_1, \ldots, Q_i, \ldots, Q_n)$. The aim of the design is to find the structural configuration included in the domain Ω_0 that achieves minimum weight while maintaining an admissible stress field associated with the specified loading Q. In a discrete formulation of the topology design problem, the solution is a structure with a black and white configuration. That is, the material density is unity wherever material is present and zero elsewhere. Considering the exact continuum (infinite dimensional) structural problem, the optimum design problem may be written as:

$$\begin{aligned} \min \quad & \int_{\Omega_0} \rho \, \mathrm{d}\Omega \\ \text{s. t.} \quad & Q(\sigma) = \left(Q_1^{\mathrm{d}}, \ldots, Q_i^{\mathrm{d}}, \ldots, Q_n^{\mathrm{d}}\right) \\ & S(\sigma) \leqslant \bar{s} \\ & \sigma \; SA \\ & \rho \in \{0, 1\} \end{aligned} \tag{5}$$

where $\rho(x, y)$ is a Boolean function and, in the hollow regions, i.e. where $\rho = 0$, it is understood that the stresses should be made to vanish. Furthermore, it is important to note that, in case the self-weight is taken into account, the condition $\sigma \; SA$ implicitly depends on the density given the design dependence of the specific weight $\gamma = \rho g$.

An obvious necessary condition for the existence of a feasible solution to this design problem is that the specified applied load be no larger than the limit load corresponding to the fully dense bar, i.e. $\rho = \bar{\rho}$ everywhere. The existence of an optimum solution to the infinite-dimensional problem remains an open question, since it depends on the issue of closedness of the design space that needs to be addressed. The problem statement in Eqs. (5) may express as well the optimization problem of the discretized medium by interpreting the piecewise constant function ρ as a Boolean N-vector, where N is the number of finite elements spanning the domain Ω_0 and the piecewise linear field σ as a vector of nodal stress components.

The present work is concerned with the continuous formulation of the topology design problem. A continuation approach is utilized to model the continuous range of intermediate solutions between void and solid at a given point in the domain. The "amount" of material is expressed by the material density ρ of a fictitious material, e.g., a porous material, which continuously spans the range $0 \leqslant \rho \leqslant 1$. The conditions on the stress vector remain linear in the problem variables, precisely in terms of stresses. In the material approach for topology optimization achievement of problem, well-posedness is based on the ability to model porous materials with a well-defined microstructure. In the SIMP approach, the interpolated form of the mechanical properties of the material at intermediate density was originally chosen artificially for mathematical convenience in formulating the design problem, away from a genuine physical interpretation. Despite this limitation, the SIMP model was generally successful in terms of practical results and research advancement. Only later was the issue of physical feasibility addressed and conditions on the penalization power were identified [22], under which evidence of physical realization of the SIMP model was established. For the present plastic design problem, the continuation scheme requires the interpolation of the failure criterion. Assuming that the intermediate material obeys the same form of failure criterion, the only parameter that is affected by the change in density is the shear strength s. A reasonable choice for the shear strength s as a function of density is a proportional law of the form:

$$s(\rho) = \rho \bar{s} \tag{6}$$

This choice obviously satisfies the limit values $s(0) = 0$ and $s(1) = \bar{s}$. Another issue that arises from the continuation scheme is the need to control the vanishing of stresses in the void regions. Setting the density to zero is equivalent to forcing the shear strength and, consequently, the maximum shear to vanish. This leads to a spherical stress state, but not necessarily to a null stress tensor because the Tresca criterion is one that allows arbitrary spherical stress. Therefore a further condition should be imposed on the stress field to bring the stress tensor to zero when the density vanishes. A possible condition is to bound the spherical stress or equivalently, the trace of the stress tensor, by a multiple of the density as follows:

$$-K\rho \leqslant (\sigma_x + \sigma_y) \leqslant K\rho \tag{7}$$

where K is a constant sufficiently large so that the constraint tends to be activated near zero density only. This linear constraint pair fulfills its underlying purpose while maintaining the conic nature of the problem. However, it introduces a first-order approximation in ρ by truncating the failure criterion for lower densities. The following is an alternative condition that may be written in conic form and results in a higher-order error in the neighborhood of zero density:

$$(\sigma_x + \sigma_y)^2 \leqslant K\rho \tag{8}$$

Its implementation using MOSEK requires two additional simple constraints per design variable, one that is necessary to transform the quadratic formula to a "rotated" conic form and another to satisfy the MOSEK restriction that a single variable should not appear in more than one conic constraint. This leads to the equivalent constraint set:

$$\begin{aligned}&x_1^2+x_2^2 \leqslant 2x_3x_4\\&x_1=\sigma^+ \qquad x_2=\sigma^+ \qquad x_3=K\rho \qquad x_4=\frac{1}{4}\end{aligned} \tag{9}$$

The interpolation of the intermediate material having been defined, the continuous topology design problem takes the form:

$$\begin{aligned}\min\ &\int_{\Omega_0}\rho\,\mathrm{d}\Omega\\ \text{s. t.}\ \ &Q(\sigma)=\left(Q_1^{\mathrm{d}},\dots,Q_i^{\mathrm{d}},\dots,Q_n^{\mathrm{d}}\right)\\&S(\sigma)\leqslant\rho\bar{s}\\&\sigma\ SA\\&0\leqslant\rho\leqslant 1\\&\left|\left(2\sigma^+\right)^q\right|\leqslant K\rho\end{aligned} \tag{10}$$

where, with $q=1$ or 2, the last constraint expresses either of the conditions (7) and (8), which ensure null stress tensor in void regions, respectively, and the *SA* condition is understood to be design-dependent if gravity loads are considered. Premultiplying the objective by the real shear strength $\bar{s}$, redefining the constant K as $K\bar{s}$ and substituting s for $\rho\bar{s}$, the optimum design problem can be rewritten in the following alternative form, where the shear strengths replace the densities as design variables:

$$\begin{aligned}\min\ &\int_{\Omega_0}s\,\mathrm{d}\Omega\\ \text{s. t.}\ \ &Q(\sigma)=\left(Q_1^{\mathrm{d}},\dots,Q_i^{\mathrm{d}},\dots,Q_n^{\mathrm{d}}\right)\\&S(\sigma)\leqslant s\\&\sigma\ SA\\&0\leqslant s\leqslant\bar{s}\\&\left|\left(2\sigma^+\right)^q\right|\leqslant Ks\end{aligned} \tag{11}$$

where, again, the *SA* condition is design-dependent in the presence of gravity loads.

This mathematical form of the design problem is similar to that of the direct limit analysis problem (4). It recalls the so-called strength reduction method formulation of the static problem of limit analysis, where the shear strength, treated as a variable, is minimized for a given loading parameter as in [23]. It can be noted that the two problems (4) and (10) differ only in that the roles of the shear strengths and the load parameter are interchanged. It is important to further note that the design problem is convex, which should discard convergence difficulties that are commonly encountered in the, usually nonconvex, elastic and elastoplastic topology design problems. Since the number of constraints, except for the simple bounding constraints imposed on the shear strengths, is the same in both problems, the computational effort is expected to be of the same order of magnitude. A major consequence of convexity is that the optimum value is unique and that any optimum solution obtained is a global one. Another merit of the proposed design problem formulation combining variable material densities with direct limit analysis is that no numerical difficulties arise when densities vanish. Consequently, there is no need for imposing a finite lower bound on the density, a routine practice in continuum elastic design to avoid singularities in the stiffness matrix.

6. Numerical examples

In this section, the proposed method for plastic topology optimization of continuum media subject to a specified limit load is tested through a selection of examples of topology design problems that are treated in the literature in the framework of elastic material behavior. These tests are performed for the following purposes: (i) demonstrate the capability of the method to generate sound optimum topologies, (ii) compare the optimum topologies produced using the proposed plastic design method with published topologies that have been generated using design methods for elastic media, (iii) compare the design computational time with that of the (limit) analysis. For convenience, the proposed design method, based on limit analysis, will be labeled LADM. For the comparison with published optimum topologies, to be meaningful, the reference examples are chosen as continuous solutions insofar as they are available. A series of problems dealing with cantilever beams having different aspect ratios, and loadings will be presented first. It will be followed by the classical Mitchell truss problem. The loading Q will be restricted to a single-load component denoted F. The design domain is discretized into a uniform grid of elementary rectangles, each divided into four triangular finite elements separated by the two diagonals. Each element has its own and unique density parameter. All examples are treated with version 5 of the commercial code MOSEK using an Intel Core Duo processor (1.66 GHz) based on the formulation in Eqs. (9).

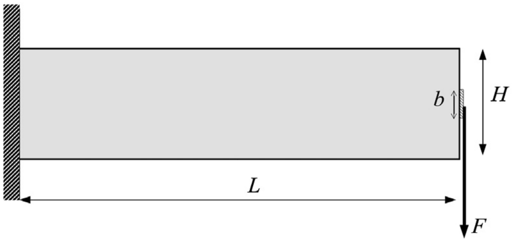

Fig. 1. Long cantilever beam.

Table 1
Results of limit analysis for the long cantilever beam.

Mesh	N	$F_{\lim}$ (kN)	CPU (s)	Error
20 × 10	800	7.759×10^{-2}	0.9	7.97×10^{-11}
20 × 20	1600	9.988×10^{-2}	2.1	4.97×10^{-11}
20 × 40	3200	1.000×10^{-1}	4.5	1.79×10^{-10}
40 × 80	12,800	1.000×10^{-1}	33.7	1.10×10^{-10}
80 × 160	51,200	9.999×10^{-2}	222.3	4.56×10^{-10}

Table 2
Effect of discretization on optimum – long cantilever beam.

Mesh	N	Objective	CPU (s)	Error
20 × 20	1600	1.2816	3.9	3.9×10^{-9}
20 × 40	3200	1.2629	9.1	5.0×10^{-10}
40 × 80	12,800	1.2666	52.6	1.3×10^{-8}
60 × 120	28,800	1.2613	177.4	2.2×10^{-8}
80 × 160	51,200	1.2412	479.2	4.8×10^{-8}
100 × 200	80,000	1.2556	925.4	3.2×10^{-8}

6.1. The long cantilever beam

This example is a first of a series of problems dealing with the design of cantilever beams having different aspect ratios. The problem of the cantilever beam is described in Fig. 1. The beam is loaded at its free end by a transverse line load that is distributed along a line element of length b. Its rectangular design domain dimensions are the length denoted by L and the height by H. The material obeys the Tresca criterion with a shear strength $\bar{s} = 1$ kPa. In this first example the beam is long, with height $H = 1$ m and length $L = 4$ m. It is loaded at its free end by a centered, practically concentrated load ($b = 10$ cm).

6.1.1. Preliminary

As a preliminary, the lower bound for the domain with full density is determined by solving the direct limit analysis problem. This load value is useful to know, since it represents an upper bound on the specified loads for which the topology optimization problem is feasible. In order to determine the degree of discretization required to achieve a reasonable accuracy for the lower bound, the domain is discretized into several meshes with different numbers of finite elements. The accuracy, or error, is defined by the largest constraint violation in the numerical limit analysis problem. The results for the lower bound and CPU times are reported in Table 1. They show that a 3200-finite-element mesh is a good choice for the limit analysis, since it achieves convergence of the lower bound within a CPU time orders of magnitude lower than required by finer meshes, without loss of accuracy. The lower bound for the full density domain is found to be $\bar{F} = 0.1$ kN. It is worthwhile noting that this is actually the exact solution, since it can be verified to be also the upper bound by considering a simple shearing block mechanism along the line of loading of length b.

Next, the degree of discretization that achieves a good accuracy for the optimal topology is determined by running a series of topology design problems with different mesh sizes. At this stage, the constraints preserving a zero-stress state in voids are considered in the form given in Eq. (7) with $K = 10$. Solving the topology design problem subject to a load $F = 0.099$ kN using different discretizations, close optimum values (difference less than 2.1%) are obtained as reported in Table 2, within CPU times nearing double those spent on analysis. The visual inspection of the final topologies, as illustrated in Fig. 2, indicates that the convergence of the topology design is achieved for a 28,800-element mesh. As a conservative choice, the mesh size of 51,200 elements is adopted for the remaining topology optimization runs relative to the long cantilever beam.

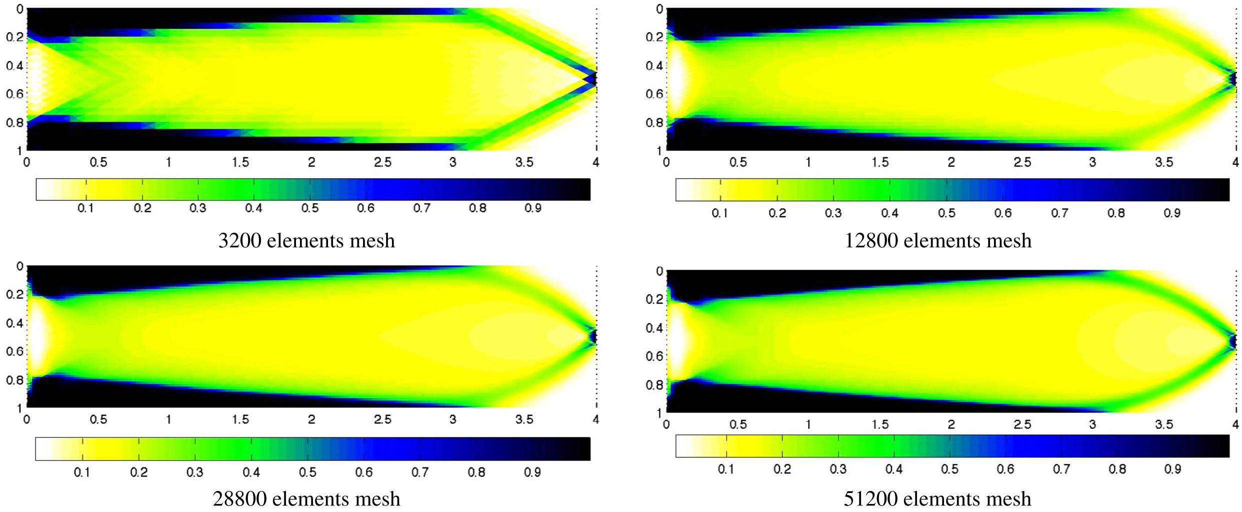

Fig. 2. (Color online.) Optimal configurations for the long cantilever beam (using Eq. (7), $K = 10$).

Table 3
Influence of factor K using linear equation (7).

K	Opt. obj.	Error	CPU	$\rho_{\min}$	$\rho_{\max}$
1	–	–	–	–	–
2	1.2723	6.17×10^{-06}	421	0.0054	1
3	1.2461	3.00×10^{-08}	436	0.0022	1
5	1.2429	5.63×10^{-07}	432	0.0015	1
10	1.2412	4.80×10^{-08}	479	0.0008	1
20	1.2402	2.33×10^{-06}	844	2.6×10^{-05}	1
40	1.2400	2.53×10^{-05}	755	0.0001	1
100	1.2400	7.15×10^{-04}	816	0.0001	1

Table 4
Influence of factor K using conic equation (8).

K	Opt. obj.	Error	CPU	$\rho_{\min}$	$\rho_{\max}$
2	–	–	–	–	–
3	1.3576	1.23×10^{-05}	515	0.0019	1
5	1.2586	1.03×10^{-09}	504	0.0028	1
10	1.2432	1.77×10^{-07}	530	0.0008	1
20	1.2405	1.08×10^{-05}	1084	4.7×10^{-06}	1
40	1.2469	8.11×10^{-08}	479	0.0043	1
100	1.2400	2.67×10^{-04}	726	0.0001	1

Table 5
Solution without constraints sets (7) or (8).

Objective	Error	CPU	$\rho_{\min}$	$\rho_{\max}$
1.2406	7.67×10^{-09}	281	0.0004	1

6.1.2. Selection of the factor K

Two series of runs were conducted with various values of K, one using the linear equation set (7) and another using the conic equation set (8) of constraints preserving a zero-stress state in voids. For comparison purposes, an additional run was performed ignoring both sets of constraints, which is equivalent to setting K to ∞ and is expected to yield lower objective values than with any finite choice for K. All these runs serve to select an appropriate constraint form and K value that will be adopted for the subsequent example problems.

The results obtained for the long cantilever beam subject to the imposed load $F = 0.099$ kN using a 51,200-element mesh are shown in Tables 3–5. It is noted that for values of K lower than a certain limit, the algorithm does not converge. This is associated with the reduction in the limit load to levels below the imposed one, as a result of the truncation of the criterion, which tends to render the problem infeasible. For very large values, difficulties of convergence are also observed. Considering the criteria of optimality in terms of objective value and constraint accuracy in addition to CPU time

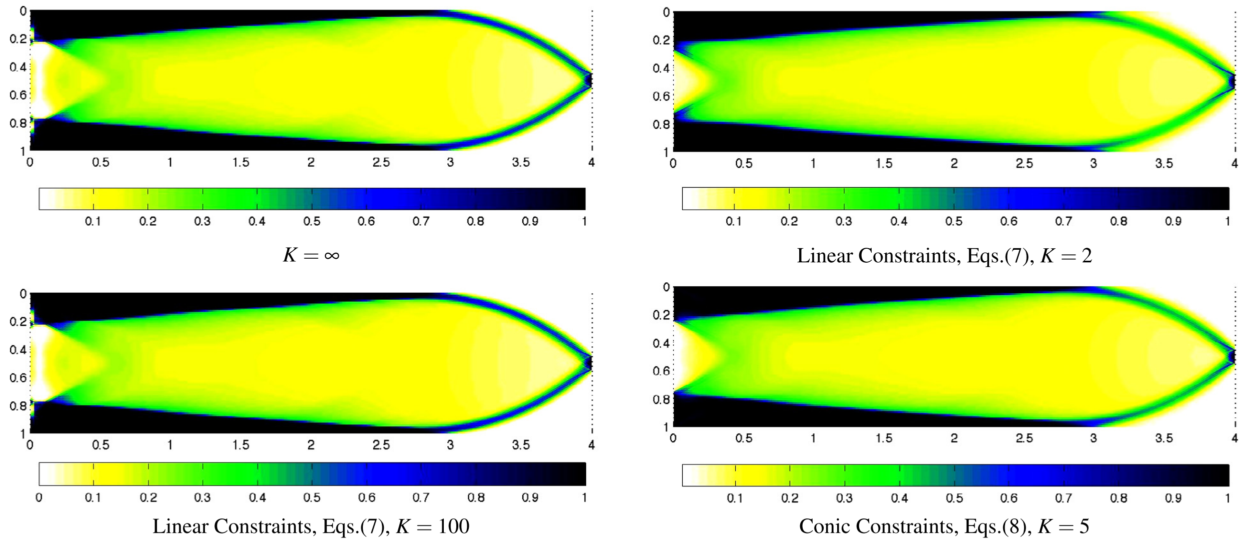

Fig. 3. (Color online.) The long cantilever beam: effect of factor K on design.

Table 6
Effect of load magnitude on the solution for the long cantilever case.

F	Objective	CPU	Error	$\rho_{\max}$
0.001	0.0112	379	1.5×10^{-09}	0.0161
0.005	0.0560	403	1.5×10^{-09}	0.8123
0.0062	0.0697	373	3.8×10^{-09}	0.9972
0.01	0.1124	396	2.0×10^{-08}	0.9997
0.05	0.5852	411	2.6×10^{-09}	1.0000
0.1	1.2804	450	2.8×10^{-07}	1.0000

and minimum density, the behavior of the solution appears to be more regular with the linear set of constraints, and the best results are obtained for the range of K values from 3 to 10. The value $K = 10$ stands out as the best for both sets of constraints. It can be noted that the CPU time for $K = 20$ is exceptionally large. This is due to the adoption of stricter stopping criteria in MOSEK for this case in order to obtain convergence. The final topologies, examples of which are shown in Fig. 3, are all visually identical. The additional constraints appear to nearly double the computational time and this time increase remains relatively insensitive to the factor K and the type of constraint. For the subsequent runs, the linear condition is chosen with $K = 3$, the lowest value in the good range.

6.1.3. Sensitivity to the load magnitude

To analyze the net effect of the load magnitude on the design, another series of topology optimization problems is run for the long cantilever beam. In each problem the beam is subject to a different value of the prescribed load. The optimal values obtained using a discretization with $N = 51{,}200$ finite elements are reported in Table 6 and optimal topologies are shown in Fig. 4.

For low loads, the density in the optimal design is less than 1 everywhere and the material distribution pattern is nearly independent of the load, with a large hole appearing in the interior. For higher loads, the pattern of material distribution tends to be more load-dependent. For loads that are larger than 0.006 kN, a zone with unit density develops at the top and bottom edges towards the support and grows thicker as the load increases. It is also noted from Table 6 that the computational effort increases with the magnitude of the applied load. Reported black and white optimal topologies for the long cantilever beam are abundant in the literature, whereas continuous solutions are rare. The solution given in Fig. 5, taken from [24], is intermediate in a process moving from a continuous solution to a black and white one. Some similarities are noted with the designs obtained using the LADM for the large loads.

6.2. Medium cantilever beam

The medium cantilever beam has the dimensions $L = 2$ m and $H = 1$ m. The load is centered and distributed along a distance $b = 10$ cm. Since the area of the design domain is now half that of the long beam, the medium beam is modeled with a 12,800-finite-element mesh. The lower bound for the fully dense domain is found to be $\bar{F} = 0.1$ kN for a unit shear strength. Based on the same discretization, the topology is optimized for various load values. The optimal weight is given in

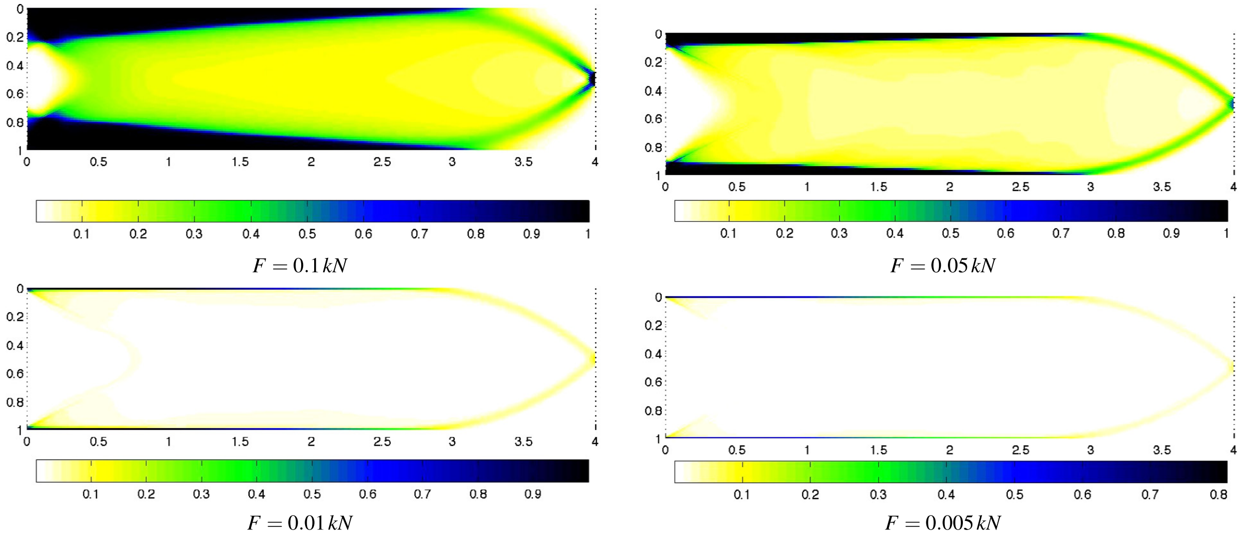

Fig. 4. (Color online.) The long cantilever beam: effect of load on design.

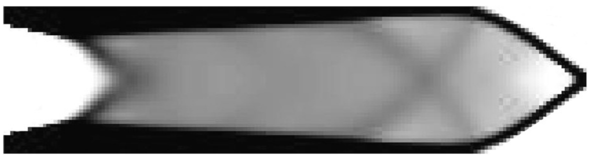

Fig. 5. A suboptimal solution for the long cantilever beam [24].

Table 7
Optimal objective and CPU for the medium cantilever beam problem.

F	Objective	CPU (s)	Error	$\rho_{\max}$
0.001	0.0071	27.6	7.77×10^{-12}	0.012
0.005	0.0202	36.9	8.95×10^{11}	0.164
0.01	0.0376	39.6	6.71×10^{11}	0.370
0.05	0.1838	43.6	3.88×10^{10}	0.998
0.1	0.3728	63.6	5.34×10^{08}	1.000

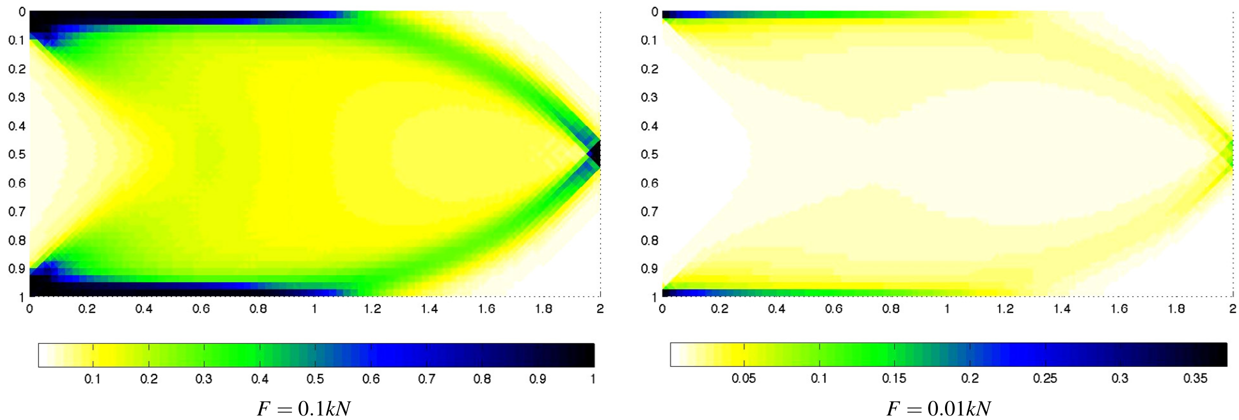

Fig. 6. (Color online.) Optimal topology for the medium cantilever beam using LADM.

Table 7 and the optimal topology is visualized in Fig. 6 for $F = 0.1$ kN and $F = 0.01$ kN. As observed for the long cantilever example, the optimal topology is load-dependent. For the larger load, the pattern of material distribution is visually similar to the distribution reported in [3] and shown in Fig. 7.

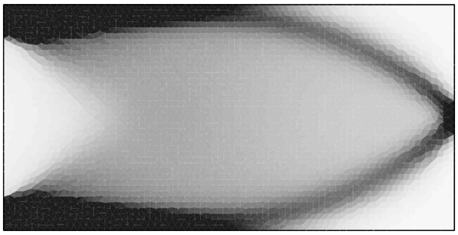

Fig. 7. Optimal topology for the medium cantilever beam.

From [3].

Table 8
Optimal objective and CPU for the short cantilever beam.

F	Objective	CPU	Error	ρ_{max}
0.001	0.0018	31.7	4.44×10^{11}	0.014
0.005	0.0069	39.4	5.53×10^{11}	0.064
0.01	0.0121	39.3	7.25×10^{11}	0.120
0.05	0.0550	44.3	1.95×10^{10}	0.570
0.1	0.1087	57.0	3.58×10^{09}	1.000

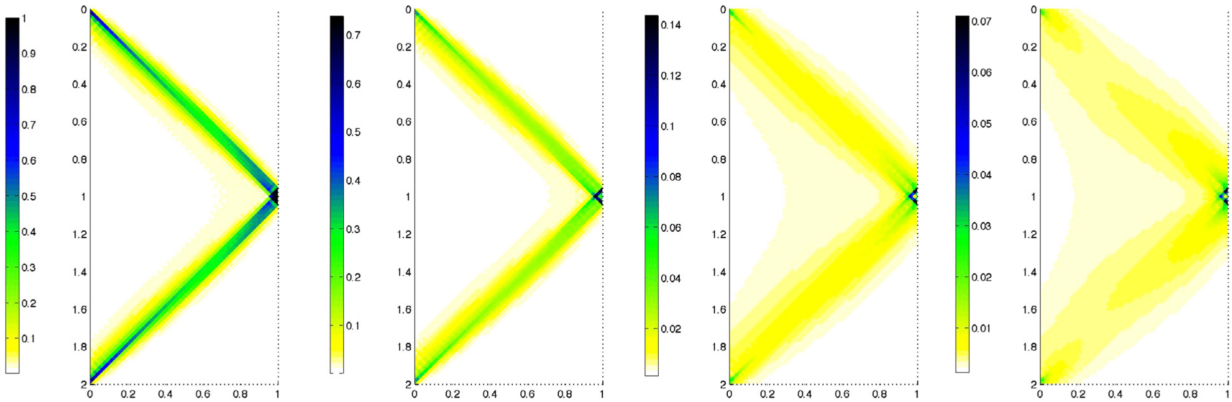

Fig. 8. (Color online.) Optimum topologies for the short cantilever beam.

6.3. The short cantilever beam

The short-cantilever domain is given the dimensions $L = 1$ m, $H = 2$ m with $b = 0.1$ m. Solving the LA problem for the domain with full density using a 12,800-element mesh, the lower bound is found to be $\bar{F} = 0.1$ kN within 22.9 s of CPU time. Employing the same discretization, the topology is optimized using the LADM for various values of the applied load. The results are summarized in Table 8. The CPU time increases with the load magnitude, but remains in the range of 1 to 2.5 times the CPU required by the analysis.

The optimal configurations generated are shown in Fig. 8. Topology optimization results for the short cantilever beam in elasticity are abundant in the literature [2,3,25]. Most interesting for the present comparison is the "continuous" configuration reported in [3] and shown here in Fig. 9a. It can be seen that its topology coincides with the topology relative to the larger loads given in Fig. 8.

6.4. Mitchell truss

In this example the classical Mitchell truss problem is considered as defined in [3]. The problem description is given in Fig. 10.

The rectangular design domain of length $L = 2$ m and height $H = 1$ m is pin supported at its lower ends and subjected to a concentrated vertical load F at the middle of the bottom edge. The domain is discretized into $N = 12{,}800$ finite elements. Each support is modeled as a substrate of width equal to the base of a triangular finite element. The limit analysis for the domain with full density yields a lower bound $\bar{F} = 0.1198$ kN within a CPU time of 21.3 s. The topology optimization problem is run using LADM for the Mitchell truss subject to a series of values of the prescribed load. The optimization results are summarized in Table 9. Figure 11 shows the reference optimum topology reported in [3] and the topology

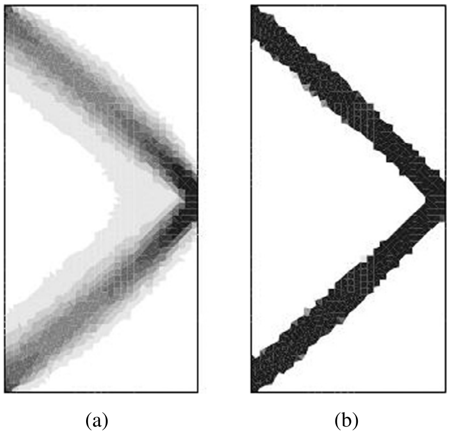

Fig. 9. Optimal topology for the short cantilever beam [3]. (a) Continuous. (b) Black and white.

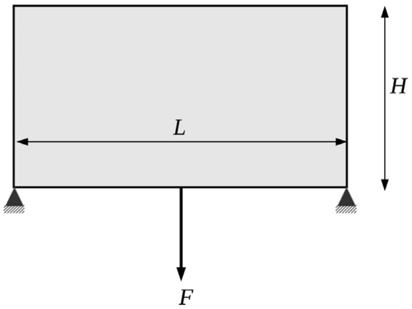

Fig. 10. Mitchell truss problem definition.

Table 9
Optimal objective and CPU for Mitchell truss problem.

F	Objective	CPU	Error	ρ_{max}
0.01	0.0128	53.74	2.06×10^{-10}	0.1911
0.015	0.0191	55.63	1.67×10^{-10}	0.2885
0.05	0.0636	60.07	2.87×10^{-10}	0.9717
0.1	0.1372	65.19	1.80×10^{-09}	1.0000
0.11	0.1558	63.45	7.48×10^{-09}	1.0000

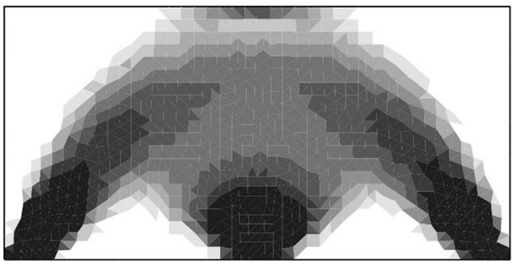

Fig. 11. Reference optimum topology for the Mitchell truss [3].

obtained using LADM is shown in Fig. 12 for $F = 0.01$ and 0.11 kN. Visual inspection reveals strong similarity between the configurations of the reference solution and the LADM solution corresponding to the highest load.

7. Conclusion

A topology optimization method is formulated and implemented for the minimum weight of continuum structures subject to a specified admissible loading. The method integrates the design task with the direct limit analysis in a single

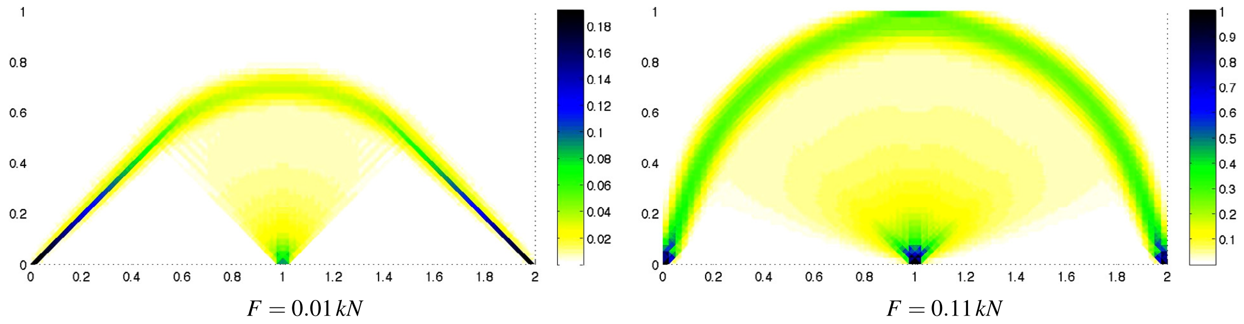

Fig. 12. (Color online.) Optimal topologies for the Mitchell truss.

optimization problem exhibiting the same algebraic structure and the same order of magnitude of computational demand as the static limit analysis problem. The proposed method is formulated in plane strain using Tresca materials and is illustrated on similar example problems taken from the literature. The comparison of the results of the proposed method with optimal topologies generated by elastic design methods shows a good agreement in the material distribution pattern, although the material behavior and the type of analysis are outstandingly different.

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