C. R. Mecanique 342 (2014) 700–705

Contents lists available at ScienceDirect

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Eigenmode sensitivity of damped sandwich structures

Sensibilité des modes propres de structures sandwich amorties

Komlanvi Lampoh [a], Isabelle Charpentier [b,*], Daya El Mostafa [c]

[a] *Centre des matériaux, Mines ParisTech, 10, rue Henri-Desbruères, BP 87, 91003 Évry cedex, France*
[b] *ICube – Laboratoire des sciences de l'ingénieur, de l'informatique et de l'imagerie, Université de Strasbourg et CNRS, 300, bd Sébastien-Brant, CS 10413, 67412 Illkirch, France*
[c] *Laboratoire d'étude des microstructures et de mécanique des matériaux, UMR 7239, île du Saulcy, 57045 Metz cedex 01, France*

A R T I C L E   I N F O

Article history:
Received 25 June 2014
Accepted 1 August 2014
Available online 15 August 2014

Keywords:
Vibration
Sensitivity
Sandwich structures
Viscoelastic model
Complex nonlinear eigenvalue solver
Automatic differentiation

Mots-clés :
Vibration
Sensibilité
Structures sandwich
Modèle viscoélastique
Solveur de problèmes aux valeurs propres complexes
Différentiation automatique

A B S T R A C T

The modeling of the linear free vibration of a sandwich structure including viscoelastic layers yields a complex nonlinear eigenvalue problem. In this paper, the sensitivity of eigensolutions is computed using a homotopy-based asymptotic numerical method, then a first-order automatic differentiation. The generality of the proposed method enables us to consider any analytical frequency-dependent viscoelastic law in the modeling and the sensitivity computation. Its application potential is demonstrated by computing the sensitivity of eigenmodes, eigenfrequencies and modal loss factors of sandwich beams and plates to various perturbations.

R É S U M É

La modélisation des vibrations linéaires libres d'une structure sandwich comportant des couches visco-élastiques conduit à un problème aux valeurs propres non linéaires complexes. Dans cet article, la sensibilité des solutions propres est calculée en utilisant une méthode asymptotique numérique, puis une différentiation automatique d'ordre un. La généralité de la méthode proposée permet de considérer toute loi visco-élastique analytique avec dépendance en fréquence dans la modélisation et le calcul de sensibilité. Son potentiel applicatif est démontré en calculant la sensibilité des valeurs et vecteurs propres, et des facteurs de perte modaux de poutres et plaques sandwich à différentes perturbations.

1. Introduction

The vibrations of mechanical structures may induce undesirable phenomena such as fatigue and noise. These may be reduced incorporating passive damping in the structures through viscoelastic materials to augment energy dissipation and to avoid the resonance phenomenon. Viscoelastic laminated sandwich structures allow for passive damping [1,2]. Their

* Corresponding author.
E-mail address: icharpentier@unistra.fr (I. Charpentier).

http://dx.doi.org/10.1016/j.crme.2014.08.001

modeling should take into account nonlinear viscoelastic laws, geometric nonlinearities of thin structures and complex kinematics. In its automated version "Diamant" [3,4], the Asymptotic Numerical Method (ANM) [5] enables one to obtain an accurate solution to complex nonlinear eigenvalue problems [2], whatever the analytical nonlinear frequency-dependent viscoelastic law [6]. Sensitivity analysis of the dynamic behavior of sandwich structures is an important issue for optimal design. To the best of our knowledge, numerical methods [7–11] are concerned with linear eigenvalue problems only, and very few studies [10] are devoted to the sensitivity of vibration eigenmodes. None of these methods addresses the complex nonlinear eigenvalue problems arising in dynamic modeling of the viscoelastic sandwich structures involving a frequency-dependent law.

Based on Automatic Differentiation (AD) [12,13], our sensitivity method relies on the homotopy-based ANM algorithm [6] for the solution of the complex nonlinear eigenvalue problem, and on a first-order differentiation of their solutions. Sandwich structure modeling, AD and sensitivity implementation are introduced in Section 2. Some sensitivity results on eigenfrequencies, eigenmodes and loss factors are presented in Section 3 for sandwich beams and plates submitted to perturbations of the layer heights or the shear modulus, or to stiffness perturbations. Frequency-dependent viscoelastic models with small-to-moderate loss factors for the core are considered.

2. Sensitivity method

Assuming that the eigensolution satisfies $u(x,t)=u(x)\,\mathrm{e}^{\mathrm{i}\omega t}$, the modeling of the linear free vibration of a sandwich structure including viscoelastic layers results in the complex nonlinear eigenvalue problem [2]

$$\big(K(\omega)-\omega^2 M\big)u=0 \tag{1}$$

where ω^2, u, and M denote the eigenvalue, the eigenmode, and the mass matrix, respectively. When the sandwich structure is made of isotropic materials, the frequency-dependent stiffness matrix $K(\omega)$ may be written as

$$K(\omega)=K_0+E(\omega)K_\mathrm{v} \tag{2}$$

where $E(\omega)$ is the complex Young modulus of the core, K_0 (related to the delayed elasticity) and K_v are real constant stiffness matrices. The stiffness and mass matrices are issued from a finite-element method and depend on geometrical and material parameters such as layer heights or Young moduli. The sensitivity of the eigensolutions to parameter perturbations is thus a key issue.

ANM continuation procedure Let λ be equal to ω^2. The frequency-dependent eigenvalue problem (1)–(2) written under the residual form

$$\mathcal{R}(u,\lambda)=\big(K_0+E(\omega)K_\mathrm{v}-\lambda M\big)u=0 \tag{3}$$

is solved using the homotopy approach described in detail in [6]. In a nutshell, the homotopy (4)

$$\mathcal{R}(u,\lambda,a)=\mathcal{S}(u,\lambda)+a\mathcal{T}(u,\lambda)=0,\quad \text{for } a\in[0,1] \tag{4}$$

is driven from the solution $(u_\mathcal{S},\lambda_\mathcal{S})$ to the "undamped" real eigenvalue problem $\mathcal{S}(u_\mathcal{S},\lambda_\mathcal{S})=[K_0-\lambda_\mathcal{S}M]u_\mathcal{S}=0$, for $a=0$, to the solution (u,λ) to the residual problem (3), for $a=1$. The term $\mathcal{T}(u,\lambda)=E(\omega)K_\mathrm{v}u$ contains the nonlinearities. The branch of solutions $(u(a),\lambda(a))$ is calculated using truncated Taylor series, which requires a higher-order differentiation of $\mathcal{S}$ and $\mathcal{T}$.

Automatic differentiation AD [12] is obviously the more practicable approach for higher-order derivative computations, providing generality, efficiency, accuracy, and ease of use. Indeed, it views any computer code as a sequence of computational statements (assignments, elementary operations, and intrinsic functions), control and do-loop statements. Higher-order AD is performed through operator overloading as the vehicle to attach derivative computations to operators and intrinsic functions. The differentiation of the code is performed applying the chain rule to the computational statements, and the operations they contain. The interested reader is referred to the Diamanlab software [14] and the references therein for an AD implementation of the ANM.

Sensitivity of the complex eigensolutions The complex eigensolution (u,λ) being known from the ANM computation, its sensitivity $(u^\mathrm{d},\lambda^\mathrm{d})$ with respect to some parameter p is computed from a first-order differentiation of (3), and evaluated in the direction δ_p. This system is closed by differentiating the orthonormality condition $u^\mathrm{T}Mu=1$ with respect to p. More precisely, the sensitivity $(u^\mathrm{d},\lambda^\mathrm{d})$ is evaluated from (5), then (6)

$$\big(u^\mathrm{T}M\big)u^\mathrm{d}=-(1/2)u^\mathrm{T}M^\mathrm{d}u \tag{5}$$

$$\{\mathcal{R}_{1|\lambda^\mathrm{d}=1}\}\lambda^\mathrm{d}=-\{\mathcal{R}_{1|\lambda^\mathrm{d}=0}\} \tag{6}$$

where M^d is the tangent linear matrix computed with respect to p, $\{\mathcal{R}_{1|\lambda^\mathrm{d}=1}\}$ is the first-order derivative of $\mathcal{R}$ with respect to λ, and $\{\mathcal{R}_{1|\lambda^\mathrm{d}=0}\}$ contains the first-order derivatives with respect to p and u evaluated in the directions δ_p and u^d.

Table 1
Eigenfrequency and modal loss factor sensitivities to a 20% perturbation of the core height. Cantilever sandwich beam, constant modulus model.

ω	ω^d	$(\omega+\omega^d)$	ω_p	η	η^d	$(\eta+\eta^d)$	η_p
65.80	−1.32	64.48	64.58	2.45E−01	1.99E−02	2.65E−01	2.63E−01
300.62	−6.47	294.15	294.96	2.32E−01	−1.01E−02	2.21E−01	2.22E−01
750.79	−11.09	739.70	741.24	1.52E−01	−1.61E−02	1.36E−01	1.38E−01

Table 2
Sensitivity of eigenfrequencies and modal loss factors to a perturbation of 20% of the core height h_c, the face height h_f or the shear modulus G_c. Al/PVB/Al cantilever.

Param.	ω	ω^d	$(\omega+\omega^d)$	ω_p	η	η^d	$(\eta+\eta^d)$	η_p
h_c	82.07	7.98E−1	82.86	82.87	1.40E−03	2.80E−04	1.68E−03	1.66E−03
	506.29	3.39	509.68	509.66	5.19E−03	9.12E−04	6.10E−03	6.13E−03
	1387.92	3.86	1391.78	1391.68	8.83E−03	1.44E−03	1.03E−02	1.03E−02
h_f	82.07	15.50	97.57	97.56	1.40E−03	1.82E−04	1.59E−03	1.56E−03
	506.29	94.32	600.61	600.32	5.19E−03	5.86E−04	5.77E−03	5.80E−03
	1387.92	253.88	1641.80	1640.20	8.83E−03	8.88E−04	9.72E−03	9.80E−03
G_c	82.07	−3.38E−03	82.06	82.07	1.40E−03	2.42E−05	1.43E−03	1.40E−03
	506.29	1.48E−02	506.30	506.29	5.19E−03	−5.02E−05	5.14E−03	5.18E−03
	1387.92	1.34E−01	1388.05	1387.93	8.83E−03	−8.54E−05	8.75E−03	8.82E−03

Table 3
Sensitivity of eigenfrequencies and modal loss factors to a 20% perturbation of the core height h_c or the shear modulus G_c^*. Al/ISD112/Al plate.

Param.	ω	ω^d	$(\omega+\omega^d)$	ω_p	η	η^d	$(\eta+\eta^d)$	η_p
h_c	43.10	−0.95	42.15	41.74	4.36E−01	2.62E−02	4.62E−01	4.38E−01
	55.75	−0.72	55.03	54.49	3.51E−01	1.37E−02	3.65E−01	3.55E−01
	110.57	−1.41	109.16	108.46	2.86E−01	1.01E−02	2.96E−01	3.00E−01
G_c^*	43.10	2.88	45.98	45.32	4.36E−01	4.49E−02	4.81E−01	4.50E−01
	55.75	3.17	58.92	58.10	3.51E−01	2.32E−02	3.75E−01	3.61E−01
	110.57	5.58	116.15	114.86	2.86E−01	−9.76E−03	2.76E−01	2.81E−01

The second equation is established following arguments similar to those developed for the ANM [13]. Contrarily to the ANM sensitivity methods presented in [15,16], the derivatives in the modeling parameters are uncoupled from the ANM higher-order derivatives in the path parameter. This significantly simplifies the computer implementation since the same operator overloading library can be used for both differentiation stages.

Very little work is needed to take into account a particular viscoelastic law or to operate a sensitivity computation with respect to any input parameter of the code. Derivative computations as well as the whole process were systematically verified through Taylor tests [16]. These compare the derivative evaluated thanks to AD to first-order finite difference approximations, assuming sufficiently small perturbations.

3. Numerical results

Our sensitivity approach is illustrated by the transverse free vibration of three-layer sandwich beams and plates with viscoelastic cores. The kinematic model is based on a classical zigzag model coupling the Kirchhoff–Love plate theory in the elastic faces and the Mindlin plate theory in the viscoelastic core [17]. Thus, a common transverse displacement of the layers and the continuity of the displacement at the interfaces are assumed. A four-node plate element, see [18], is chosen for the sandwich plate discretization using a finite-element method. Classical linear and cubic shape functions are used for the interpolation of the rotation of the core and the transverse displacement, respectively. The ANM is carried out with a truncature order of 30 for the series, and a threshold of 1×10^{-6} for the residual. Different kinds of nonlinear viscoelastic laws are considered to demonstrate the abilities of the method.

Tables 1, 2 and 3 present eigenfrequencies ω and modal loss factors η computed using the ANM, and their sensitivities ω^d and η^d to a perturbation of $\delta_p = 20\%$ of the parameter p under study evaluated using (5)–(6). For the sake of comparison, tables also report the eigenfrequencies ω_p and the modal loss factors η_p computed using the ANM and the parameter $p + \delta_p$.

Constant viscoelastic model for the core The sensitivity approach is validated on the classical cantilever viscoelastic beam [18,19], the parameters of which are reported in Table 4. Elastic faces are in aluminum (Al). Table 1 presents eigenfrequencies ω and modal loss factors η and their sensitivities ω^d and η^d evaluated with respect to a perturbation of 20% of the core height. The eigenfrequencies ω_p and the modal loss factors η_p are computed with a core height equal to $1.2h_c$. A good

Table 4
Geometrical and material characteristics of the cantilever sandwich beam, aluminum faces and constant core modulus model.

Elastic faces	Viscoelastic core	Dimensions
Young modulus 6.9×10^{10} Pa	Young modulus $E_0(1 + i\eta_c)$ $E_0 = 1794 \times 10^3$ Pa, $\eta_c = 0.6$	Length $L = 177.8$ mm Width $l = 12.7$ mm Mesh 100 elements
Poisson coeff. 0.3	Poisson coeff. $\nu_c = 0.3$	
Density 2766 $\mathrm{kg\,m^{-3}}$	Density 968.1 $\mathrm{kg\,m^{-3}}$	
Height $h_f = 1.524$ mm	Height $h_c = 0.127$ mm	

Table 5
Geometrical and material characteristics of the Al/ISD112/Al plate.

Elastic faces	Viscoelastic core	Dimensions
Young modulus 6.89×10^{10} Pa	Shear modulus Eq. (8)	Length $L = 348$ mm
Poisson coeff. 0.3	Poisson coeff. $\nu_c = 0.5$	Width $l = 304.8$ mm
Density 2740 $\mathrm{kg\,m^{-3}}$	Density 1600 $\mathrm{kg\,m^{-3}}$	Mesh 32×28 elements
Height $h_f = 0.762$ mm	Height $h_c = 0.254$ mm	

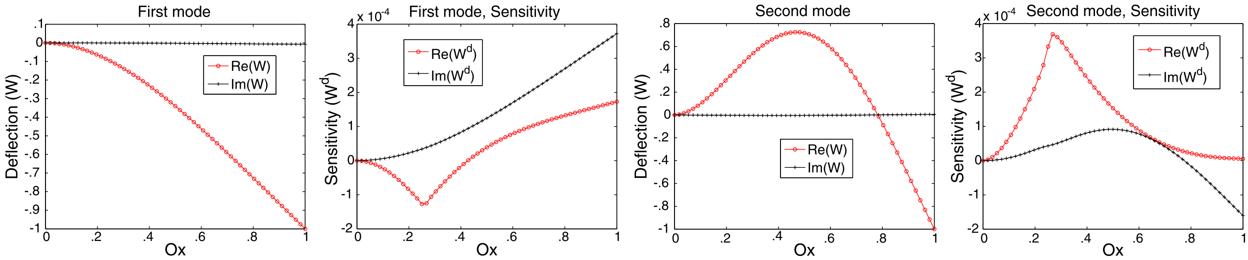

Fig. 1. (Color online.) Real and imaginary parts of the first two eigenmodes and their sensitivities to a perturbation of 20% of the stiffness element at $x = L/4$. Al/PVB/Al cantilever.

agreement exists between our first-order approximations $\omega + \omega^d$ and $\eta + \eta^d$, and perturbed solutions ω_p and η_p. Similar results were obtained for $\eta_c = 1.5$.

Fractional-derivative viscoelastic model The cantilever beam [20] is made of a polyvinyl butyral (PVB) core layered between aluminum elastic faces. Elastic faces are described in Table 4. The frequency-dependent shear modulus of the PVB core satisfies

$$G_c(\omega) = G_\infty + (G_0 - G_\infty)\left[1 + (\mathrm{i}\omega\tau)^{1-\alpha}\right]^{-\beta} \tag{7}$$

with $G_0 = 479 \times 10^3$ Pa, $G_\infty = 2.35 \times 10^8$ Pa, $\tau = 0.3979$, $\alpha = 0.46$ and $\beta = 0.1946$. The Poisson coefficient, the density and the height of the core are equal to 0.4, 999 $\mathrm{kg\,m^{-3}}$ and $h_c = 0.127$ mm, respectively. A perturbation of 20% of the parameter under study is applied. One notices that eigenfrequencies and loss factors are less sensitive to a perturbation of the shear modulus of the core. Moreover, eigenfrequencies are more sensitive than loss factors to a perturbation of the elastic face heights. This is an expected result, since the damping of sandwich structures with thin soft viscoelastic core is mainly due to the shear deformation taking place in the core.

Our sensitivity method applies to eigenmodes. Perturbations of the stiffness matrix K_v are introduced at the element level K_e of the assembly process by adding a null perturbation p_{K_e}. The stiffness matrix built during the assembly process is differentiated with respect to p_{K_e} and evaluated in the direction δ_{K_e}, the dimension of which is equal to the number of mesh elements. The real and imaginary parts of the first two eigenmodes and their sensitivities are plotted in Fig. 1 for a perturbation of 20% of the stiffness element at $x = L/4$ (*i.e.* 1% of the beam). One notices that the local effect of the perturbation is more visible on the real part $\mathrm{Re}(w^d)$ of the eigenmode sensitivity.

Generalized Maxwell model A sandwich plate with a 3M ISD112 core is studied under CFCF boundary conditions (C: clamped, F: free) [18]. Based on the generalized Maxwell model, the frequency-dependent shear modulus of the core at 20 °C verifies

$$G_c^*(\omega) = G_0\left(1 + \sum_{j=1}^{3} \frac{\Delta_j \omega}{\omega - \mathrm{i}\omega_j}\right) \tag{8}$$

where the shear modulus of the delayed elasticity $G_0 = 0.5 \times 10^6$, and the fitted parameters are $\Delta_1 = 2.8164$, $\Delta_2 = 13.1162$, $\Delta_3 = 45.46655$, and $\omega_1 = 31.1176$, $\omega_2 = 446.4542$ and $\omega_3 = 5502.5318$ [21]. Other geometrical and material characteristics are indicated in Table 5. Table 3 presents sensitivity analysis results performed with respect to the height and the shear

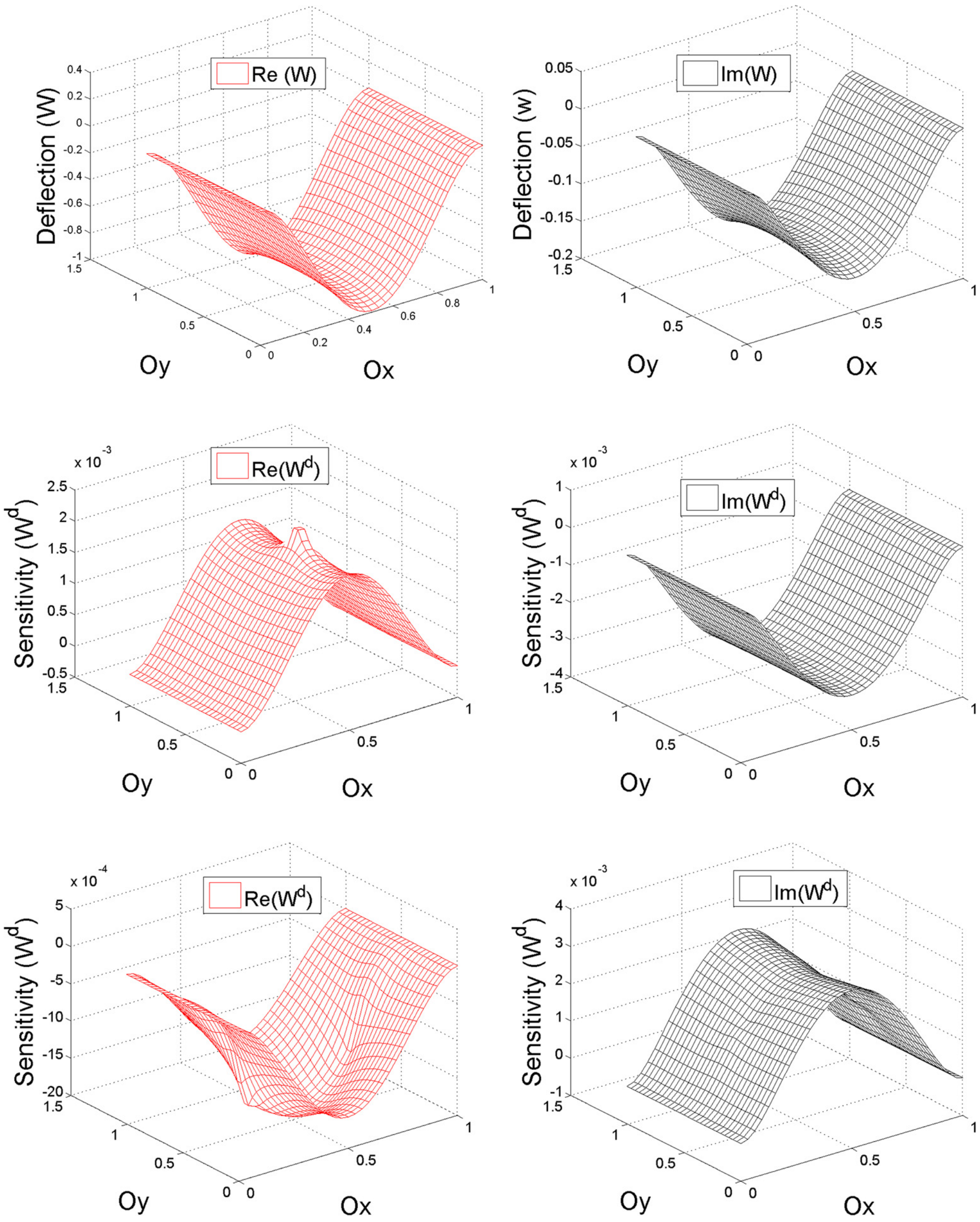

Fig. 2. (Color online.) First eigenmode (upper row), sensitivity to a 40% perturbation applied at point $(L/2, l/2)$ (middle row), sensitivity to a 40% perturbation applied at points $(L/4, l/2)$ and $(3L/4, l/2)$ (lower row). Left: Real parts. Right: Imaginary parts. Al/ISD112/Al plate.

modulus of the core. The interpretation is similar to what has been discussed for the sandwich beam. Some eigenmode sensitivities to local stiffness perturbations are plotted in Fig. 2. A very small perturbation is applied to the stiffness matrix when a unique element stiffness matrix is considered, because each element represents about 0.11% of the plate area. As in the sandwich beam case, local effects of the perturbation are more visible on the real part of w^{d} of the eigenmode. Similar results were obtained for CCCC, SSSS, and CSCS boundary conditions (S: Simply supported).

4. Conclusion

This article presents a method for the sensitivity analysis of complex nonlinear eigensolutions, with applications to viscoelastic sandwich beams and plates, and different viscoelastic laws. The sensitivity of modal parameters (frequency and loss factor) as well as the sensitivity of eigenmodes are computed with respect to a perturbation of some material characteristics or local perturbations of the stiffness matrix. Numerical results demonstrate that our first-order approximations are in good agreement with direct computations. Following [22,23], such an accurate method provides useful qualitative and quantitative information about eigenmode sensitivity for further active damping and patch location studies, for instance.

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