C. R. Mecanique 343 (2015) 57–73

Contents lists available at ScienceDirect

Comptes Rendus Mecanique

www.sciencedirect.com

Elasticity of fractal materials using the continuum model with non-integer dimensional space

Vasily E. Tarasov

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow 119991, Russia

ARTICLE INFO

Article history:
Received 22 April 2014
Accepted 15 September 2014
Available online 18 October 2014

Keywords:
Fractal material
Non-integer dimensional space
Elasticity
Gradient elasticity
Thermoelasticity
Fractional continuum model

ABSTRACT

Using a generalization of vector calculus for space with non-integer dimension, we consider elastic properties of fractal materials. Fractal materials are described by continuum models with non-integer dimensional space. A generalization of elasticity equations for non-integer dimensional space, and its solutions for the equilibrium case of fractal materials are suggested. Elasticity problems for fractal hollow ball and cylindrical fractal elastic pipe with inside and outside pressures, for rotating cylindrical fractal pipe, for gradient elasticity and thermoelasticity of fractal materials are solved.

1. Introduction

Fractals are measurable metric sets with non-integer Hausdorff dimensions [1,2]. The main characteristic of a fractal set is the non-integer Hausdorff dimension that should be observed on all scales. The Hausdorff dimension is a local property, i.e. this dimension characterize (measure) property of a set of distributed points in the limit of a vanishing diameter, which is used to cover subset of the points. By definition the Hausdorff dimension requires the knowledge of the diameter of the covering sets to vanish. In general, real materials have a characteristic smallest length scale R_0 such as the radius of a particle such as an atom or a molecule. In fractal materials, the fractal structure cannot be observed on all scales, but only on those for which $R > R_0$, where R_0 is the characteristic scale of the particles. For real materials, a non-integer mass dimension can be used instead of the Hausdorff dimension. The mass dimension described how the mass of a medium region scales with the size of this region, where we assume an unchanged density. For many cases, we have an asymptotic relation between the mass $M(W)$ of a ball region W of material, and the radius R of this ball. The mass of fractal material satisfies a power-law relation $M(W) \sim R^D$. The parameter D is called the non-integer mass dimension of a fractal material. This parameter does not depend on the shape of the region W, or on whether the packing of sphere of radius R_0 is a close packing, a random packing or a porous packing with a uniform distribution of holes. Therefore a fractal material can be considered as a medium with non-integer mass dimension. Although the non-integer dimension does not reflect completely the geometrical and dynamical properties of the fractal materials, it nevertheless permits to draw a number of important conclusions about the behavior of the materials. It allows us to use effective models that take into account non-integer dimensions.

We can distinguish the following approaches to formulate models of fractal materials.

E-mail address: tarasov@theory.sinp.msu.ru.

http://dx.doi.org/10.1016/j.crme.2014.09.006

1) An approach based on methods of "analysis on fractals" [3–8] can be considered as the most rigorous approach to describe fractal materials. Unfortunately, a possibility of application of the "analysis on fractals" to solve real problems of fractal material is now very limited due to the weak development of this area of mathematics.

2) To describe a fractal material, we can apply a special continuum models suggested in [9–12] and then developed in the works [13–15,73]. These models can be called the fractional-integral continuum models. In this approach, we use integrations of non-integer orders, and two different notions such as the density of states and the distribution function [15]. The order of the fractional integrals is equal to the mass dimension of fractal materials. The kernels of these integrals are defined by the power-law type of the density of states.

3) Fractional derivatives of non-integer orders are used to describe some properties of fractal materials. This approach has been suggested in papers [16–18], where the so-called local fractional derivatives are used, and then developed in the works [19–25]. These models can be called the fractional-differential models.

4) Fractal materials can be described by using the theory of integration and differentiation for non-integer dimensional spaces [26–28]. Fractal materials are described as continuum in non-integer dimensional spaces. The dimensions of the spaces are equal to the mass dimensions of fractal materials.

Unfortunately, there are not enough differential equations that are solved for various problems for fractal materials in the framework of the fractional-differential model and by methods of "analysis on fractals".

The fractional-integral continuum models are used to solve differential equations for various problems of elasticity of fractal materials [29–33,14], and of thermoelasticity of fractal materials [34,35].

Continuum models with non-integer dimensional spaces are not currently used to describe the elasticity of fractal materials. In this paper, we consider an approach based on the non-integer dimensional space to describe the elasticity of isotropic fractal materials. The main difference of the continuum models with non-integer dimensional spaces and fractional-integral continuum models suggested in [9–12,15] may be reduced to the following: (a) the arbitrariness in the choice of the numerical factor in the density of states is fixed by the equation of the volume of non-integer dimensional ball region. (b) In the fractional-integral continuum models, the differentiations are integer orders, whereas the integrations are non-integer orders. In the continuum models with non-integer dimensional spaces, the integrations and differentiations are defined for the spaces with non-integer dimensions. The power law $M \sim R^D$ can be naturally derived by using the integrations in a non-integer dimensional space [26], whose dimension is equal to the mass dimension of the fractal material.

A vector calculus for non-integer dimensional space proposed in this paper allows us to use continuum models, which are based on non-integer dimensional space, to describe fractal materials. This is due to the fact that although the non-integer dimension does not reflect all geometrical and dynamical properties of the fractal materials, it nevertheless allows us to get important results about the behavior of fractal materials. Therefore continuum models with non-integer dimensional spaces can describe a wide class of fractal materials.

Integration over non-integer dimensional spaces is actively used in the theory of critical phenomena and phase transitions in statistical physics [36,37], and in the dimensional regularization of ultraviolet divergences in quantum field theory [38,39,26]. The axioms for integrations in a non-integer dimensional space are proposed in [40,27] and this type of integration is considered in the book by Collins [26] for rotationally covariant functions. In the paper [27], a mathematical basis of integration on a non-integer dimensional space is given, and a generalization of the Laplace operator for non-integer dimensional spaces is suggested. Using a product measure approach, the Stillinger's methods [27] have been generalized by Palmer and Stavrinou [28] for multiple variables case with different degrees of confinement in orthogonal directions. The scalar Laplace operators suggested by Stillinger in [27] and by Palmer, Stavrinou in [28] for non-integer dimensional spaces, have successfully been used for effective descriptions in physics and mechanics. The Stillinger's form of the Laplacian for the Schrödinger equation in a non-integer dimensional space is used by He [41–43] to describe a measure of the anisotropy and confinement by the effective non-integer dimensions. Quantum mechanical models with non-integer (fractional) dimensional space have been discussed in [27,28,44–48] and [49–52]. Recent progress in non-integer dimensional space approach also includes a description of the fractional diffusion processes in a non-integer dimensional space in [53], and the electromagnetic fields in a non-integer dimensional space in [54–56] and [57–60].

Unfortunately, in the articles [27,28], only second-order differential operators for scalar fields in the form of the scalar Laplacian for the non-integer dimensional space are proposed. A generalization of the vector Laplacian [61] for the non-integer dimensional space is not suggested. The first-order operators such as gradient, divergence, curl operators, and vector Laplacian are not considered in [27,28] also. In the work [62], the gradient, divergence, and curl operators are suggested only as approximations of the square of the Laplace operator. Consideration only the scalar Laplacian in non-integer dimensional space approach greatly restricts us in application of continuum models with non-integer dimensional space for fractal materials and material. For example, we cannot use the Stillinger's form of Laplacian for the displacement vector field $\mathbf{u}(\mathbf{r},t)$ in the theory of elasticity and thermoelasticity of fractal materials, for the velocity vector field $\mathbf{v}(\mathbf{r},t)$ in hydrodynamics of fractal fluids, for electric and magnetic vector fields in electrodynamics of fractal media in the framework of the non-integer dimensional space approach.

In this paper, we define the first- and second-order differential vector operations such as gradient, divergence, the scalar and vector Laplace operators for a non-integer dimensional space. In order to derive the vector differential operators in a non-integer dimensional space, we use the method of analytic continuation in dimension. For the sake of simplification, we consider rotationally covariant scalar and vector functions that are independent of angles. This allows us to reduce differential equations in non-integer dimensional space to ordinary differential equations with respect to r. The proposed operators allow us to describe fractal materials to describe processes in the framework of continuum models with non-integer dimensional spaces. In this paper, we solve elasticity problems for a fractal hollow ball with inside and outside pressures, for a cylindrical fractal elastic pipe with inside and outside pressures, and for a rotating cylindrical fractal pipe, for the gradient of elasticity and thermoelasticity of fractal materials.

2. Differential and integral operators in non-integer dimensional space

To derive equations for vector differential operators in a non-integer dimensional space, we use equations for the differential operators in the spherical (and cylindrical) coordinates in $\mathbb{R}^n$ for arbitrary n to highlight the explicit relations with dimension n. Then the vector differential operators for a non-integer dimension D can be defined by continuation in dimension from integer n to non-integer D. To simplify, we will consider only scalar fields φ and vector fields $\mathbf{u}$ that are independent of angles

$$\varphi(\mathbf{r}) = \varphi(r), \qquad \mathbf{u}(\mathbf{r}) = \mathbf{u}(r) = u_r \mathbf{e}_r$$

where $r = |\mathbf{r}|$ is the radial distance, $\mathbf{e}_r = \mathbf{r}/r$ is the local orthogonal unit vector in the directions of increasing r, and $u_r = u_r(r)$ is the radial component of $\mathbf{u}$. We will work with rotationally covariant functions only. This simplification is analogous to the simplification for definition of integration over the non-integer dimensional space described in Section 4 of the book [26].

2.1. Vector differential operators for spherical and cylindrical cases

Explicit definitions of differential operators for non-integer dimensional space can be obtained by using continuation from integer n to arbitrary non-integer D. We note that the same expressions can be obtained by using the integration in a non-integer dimensional space and the correspondent Gauss' theorem.

We define the differential vector operations such as gradient, divergence, the scalar and vector Laplacian for non-integer dimensional space. For simplifications, we assume that the vector field $\mathbf{u} = \mathbf{u}(\mathbf{r})$ is radially directed and the scalar and vector fields $\varphi(\mathbf{r})$, $\mathbf{u}(\mathbf{r})$ are not dependent on the angles.

The divergence in a non-integer dimensional space for the vector field $\mathbf{u} = \mathbf{u}(r)$ is:

$$\mathrm{div}_r^D \, \mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{D-1}{r} u_r \tag{1}$$

The gradient in non-integer dimensional space for the scalar field $\varphi = \varphi(r)$ is:

$$\mathrm{grad}_r^D \, \varphi = \frac{\partial \varphi}{\partial r} \mathbf{e}_r \tag{2}$$

The scalar Laplacian in a non-integer dimensional space for the scalar field $\varphi = \varphi(r)$ is:

$${}^S\Delta_r^D \varphi = \mathrm{div}_r^D \, \mathrm{grad}_r^D \, \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{D-1}{r} \frac{\partial \varphi}{\partial r} \tag{3}$$

The vector Laplacian in a non-integer dimensional space for the vector field $\mathbf{u} = u_r(r) \mathbf{e}_r$ is:

$${}^V\Delta_r^D \mathbf{u} = \mathrm{grad}_r^D \, \mathrm{div}_r^D \, \mathbf{u} = \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{D-1}{r} \frac{\partial u_r}{\partial r} - \frac{D-1}{r^2} u_r \right) \mathbf{e}_r \tag{4}$$

If $D = n$, Eqs. (1)–(4) give the well-known formulas for the integer dimensional space $\mathbb{R}^n$.

We can consider materials with axial symmetry, where the fields $\varphi(r)$ and $\mathbf{u}(r) = u_r(r) \mathbf{e}_r$ are also axially symmetric. Let the Z-axis be directed along the axis of symmetry. Therefore we use a cylindrical coordinate system.

The divergence in non-integer dimensional space for the vector field $\mathbf{u} = \mathbf{u}(r)$ is:

$$\mathrm{div}_r^D \, \mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{D-2}{r} u_r \tag{5}$$

The gradient in non-integer dimensional space for the scalar field $\varphi = \varphi(r)$ is:

$$\mathrm{grad}_r^D \, \varphi = \frac{\partial \varphi}{\partial r} \mathbf{e}_r \tag{6}$$

The scalar Laplacian in non-integer dimensional space for the scalar field $\varphi = \varphi(r)$ is:

$${}^S\Delta a_r^D \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{D-2}{r} \frac{\partial \varphi}{\partial r} \tag{7}$$

The vector Laplacian in non-integer dimensional space for the vector field $\mathbf{u} = v(r)\mathbf{e}_r$ is:

$$^{V}\Delta_r^D \mathbf{u} = \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{D-2}{r}\frac{\partial u_r}{\partial r} - \frac{D-2}{r^2} u_r \right) \mathbf{e}_r \tag{8}$$

Eqs. (5)–(8) can be easily generalized for the case $\varphi = \varphi(r,z)$ and $\mathbf{u}(r,z) = u_r(r,z)\mathbf{e}_r + u_r(r,z)\mathbf{e}_z$. In this case, the curl operator for $\mathbf{u}(r,z)$ is different from zero, and:

$$\mathrm{Curl}_r^D \, \mathbf{u} = \left(\frac{\partial u_r}{\partial z} - \frac{\partial v_z}{\partial r} \right) \mathbf{e}_\theta \tag{9}$$

Eqs. (1)–(8) with $D = 3$ and (9) give the well-known expressions for the gradient, divergence, curl operator, scalar, and vector Laplacian operators

The proposed operators for $0 < D < 3$ allow us to reduce the non-integer dimensional vector differentiations (1)–(4) and (5)–(8) to derivatives with respect to $r = |\mathbf{r}|$. It allows us to reduce partial differential equations for fields in non-integer dimensional space to ordinary differential equations with respect to r.

For a function $\varphi = \varphi(r,\theta)$ of radial distance r and related angle θ measured relative to an axis passing through the origin, the scalar Laplacian in a non-integer dimensional space proposed by Stillinger [27] is:

$$^{\mathrm{St}}\Delta^D = \frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin^{D-2}\theta}\frac{\partial}{\partial \theta}\left(\sin^{D-2}\theta \frac{\partial}{\partial \theta} \right) \tag{10}$$

where D is the dimension of space ($0 < D < 3$), and the variables $r \geq 0$, $0 \leq \theta \leq \pi$. Note that $(^{\mathrm{St}}\Delta^D)^2 \neq {}^{\mathrm{St}}\Delta^{2D}$. If the function depends on the radial distance r only ($\varphi = \varphi(r)$), then

$$^{\mathrm{St}}\Delta^D \varphi(r) = \frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial \varphi(r)}{\partial r} \right) = \frac{\partial^2 \varphi(r)}{\partial r^2} + \frac{D-1}{r}\frac{\partial \varphi(r)}{\partial r} \tag{11}$$

It is easy to see that the Stillinger's form of the Laplacian $^{\mathrm{St}}\Delta^D$ for radial scalar functions $\varphi(\mathbf{r}) = \varphi(r)$ coincides with the scalar Laplacian $^{S}\Delta_r^D$ defined by (3), i.e.,

$$^{\mathrm{St}}\Delta^D \varphi(r) = {}^{S}\Delta^D \varphi(r) \tag{12}$$

The Stillinger's Laplacian can be applied for scalar fields only. It cannot be used to describe vector fields $\mathbf{u} = u_r(r)\mathbf{e}_r$, because this Laplacian for $D = 3$ is not equal to the usual vector Laplacian for $\mathbb{R}^3$,

$$^{\mathrm{St}}\Delta^3 \mathbf{u}(r) \neq \Delta \mathbf{u}(r) = \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{2}{r}\frac{\partial u_r}{\partial r} - \frac{2}{r^2} u_r \right) \mathbf{e}_r \tag{13}$$

The gradient, divergence, curl operator and vector Laplacian are not considered by Stillinger in the paper [27].

2.2. Integration over a non-integer dimensional space

Integration for non-integer values of dimension D is defined by continuation in D [39,26]. The following properties suggested in [40] for integrals in D-dimensional space are necessary for applications [26].

a) Linearity:

$$\int \big(a f_1(\mathbf{r}) + b f_2(\mathbf{r})\big) \mathrm{d}^D \mathbf{r} = a \int f_1(\mathbf{r}) \mathrm{d}^D r + b \int f_2(\mathbf{r}) \mathrm{d}^D \mathbf{r} \tag{14}$$

where a and b are arbitrary real numbers.

b) Translational invariance:

$$\int f(\mathbf{r} + \mathbf{r}_0) \mathrm{d}^D \mathbf{r} = \int f(\mathbf{r}) \mathrm{d}^D \mathbf{r} \tag{15}$$

for any vector $\mathbf{r}_0$.

c) Scaling property:

$$\int f(\lambda \mathbf{r}) \mathrm{d}^D \mathbf{r} = \lambda^{-D} \int f(\mathbf{r}) \mathrm{d}^D \mathbf{r} \tag{16}$$

for any positive λ.

Note that linearity is true of any integration, while translation and rotation invariance are basic properties of a Euclidean space. The scaling property embodies the D-dimensionality. Not only the above three axioms are necessary, but they also ensure that integration is unique, aside from an overall normalization [40]. These properties must be used in order to have non-integer dimensional integrations [26]. These properties are natural in application of dimensional regularization to quantum field theory [26].

In general, we can consider any functions of the components of its vector argument $\mathbf{r}$. However, we do not know the meaning of the components of a vector in non-integer dimensions. In this paper, we will work with rotationally covariant functions for simplification. So we will assume that f is a scalar or vector function only of scalar products of vectors or of length of vectors. For example, in the elasticity theory, we consider the case, where the displacement vector $\mathbf{u}(\mathbf{r})$, is independent of the angles $\mathbf{u}(\mathbf{r}) = \mathbf{u}(r)$, where $r = |\mathbf{r}|$. The integration defined by Eq. (17) satisfies the properties (14)–(16).

The non-integer dimensional integration for scalar functions $f(\mathbf{r}) = f(|\mathbf{r}|)$ can be defined in terms of ordinary integration by the equation:

$$\int d^D\mathbf{r}\, f(\mathbf{r}) = \int_{\Omega_{D-1}} d\Omega_{D-1} \int_0^\infty dr\, r^{D-1} f(r) \tag{17}$$

where

$$\int_{\Omega_{D-1}} d\Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)} = S_{D-1} \tag{18}$$

Eq. (18) with integer $D = n$ gives the well-known area S_{n-1} of the $(n-1)$-sphere with unit radius.

As a result, we have [26] the explicit definition of the continuation of integration from integer n to arbitrary fractional D in the form

$$\int d^D\mathbf{r}\, f(|\mathbf{r}|) = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty dr\, r^{D-1} f(r) \tag{19}$$

This equation reduces a non-integer dimensional integration to an ordinary integration. Therefore, the linearity and translation invariance follow from linearity and translation invariance of ordinary integration. The scaling and rotation covariance are explicit properties of the definition.

As an example of applications of Eq. (19), we can consider non-integer dimensional integration for the function

$$f(\mathbf{r}^2) = \frac{\mathbf{r}^2 + a}{\mathbf{r}^2 + b} \tag{20}$$

where a and b are real numbers. The integral for (20) can be explicitly computed

$$\int d^D\mathbf{r}\, \frac{r^2 + a}{r^2 + b} = (\pi b)^{D/2}(a/b - 1)\Gamma(1 - D/2) \tag{21}$$

The other example of a non-integer dimensional integration is:

$$\int d^D\mathbf{r} \frac{r^{2\alpha}}{(r^2 + a^2)^\beta} = \frac{\Gamma(\alpha + D/2)\Gamma(\beta - \alpha - D/2)}{\Gamma(D/2)\Gamma(\beta)} \pi^{D/2} a^{D+2\alpha-2\beta} \tag{22}$$

where $r = |\mathbf{r}|$.

3. Mass and moment of inertia for fractal materials

3.1. Mass of fractal materials

Fractal materials can be characterized by the relation between the mass $M(W)$ of a region W of the fractal material, and the size R of the region containing this mass:

$$M_D(W) = M_0 \left(\frac{R}{R_0}\right)^D, \quad R/R_0 \gg 1 \tag{23}$$

The parameter D is called the non-integer mass dimension of the fractal material. The parameter D does not depend on the shape of the region W, or on whether the packing of a sphere of radius R_0 is a close packing, a random packing or a porous packing with a uniform distribution of holes. The cornerstone of fractal materials is the non-integer mass dimension. The mass dimension of real fractal materials can be measured by the box-counting method, which means drawing a box of size R and counting the mass inside.

The fractality of a material means than the mass of the region $W \subset \mathbb{R}^3$ increases more slowly than the 3-dimensional volume of this region. For the ball region of the fractal medium, this property can be described by the power law $M_D(W) \sim R^D$, where R is the radius of the ball.

A fractal material is called homogeneous if the power law $M_D(W) \sim R^D$ does not depend on the translation of the region. The homogeneity property of the material can be formulated in the form: for all two regions W_1 and W_2 of the homogeneous fractal material with the equal volumes $V_D(W_1) = V_D(W_2)$, the masses of these regions are equal: $M_D(W_1) = M_D(W_2)$.

The power law (23) can be naturally derived by using the integration in the non-integer dimensional space such that the space dimensions is equal to the mass dimension of the material.

The mass of the region W of fractal material in W can be calculated by the integral in non-integer dimensional space:

$$M_D(W) = \int_W \rho(\mathbf{r})\, \mathrm{d}^D\mathbf{r} \tag{24}$$

where $\mathbf{r}$ is dimensionless vector variable. For a ball region W with radius R and density $\rho(\mathbf{r}) = \rho_0 = \text{const}$, we get the mass is defined by:

$$M_D(W) = \rho_0 V_D = \frac{\pi^{D/2}\rho_0}{\Gamma(D/2+1)} R^D \tag{25}$$

This equation define the mass of the fractal homogeneous ball. For $D = 3$, Eq. (25) gives the well-known equation for the mass of the non-fractal ball $M_3 = (4\rho_0\pi)/3R^3$ because $\Gamma(3/2) = \sqrt{\pi}/2$ and $\Gamma(z+1) = z\Gamma(z)$.

3.2. Moment of inertia of fractal materials

Let us consider a calculation of scalar moment of inertia $I(t)$, which is used when the axis of rotation is known. The scalar moment of inertia of a rigid body with density $\rho'(\mathbf{r}', t)$ with respect to a given axis is defined by the volume integral

$$I'(t) = \int_W \rho'\big(\mathbf{r}', t\big)\mathbf{r}'^2_\perp\, \mathrm{d}^3\mathbf{r}' \tag{26}$$

where $(\mathbf{r}')^2_\perp$ is the perpendicular distance from the axis of rotation, and $\mathrm{d}V'_3 = \mathrm{d}x'_1\mathrm{d}x'_2\mathrm{d}x'_3$. We note that SI units of I'_{kl} is $\mathrm{kg}\cdot\mathrm{m}^2$.

To generalize Eq. (26) for a non-integer dimensional space, we should represent this equation through the dimensionless coordinate variables. We can introduce the dimensionless values $x_k = x'_k/R_0$, $\mathbf{r} = \mathbf{r}'/R_0$, where R_0 is a characteristic scale, and the density $\rho(\mathbf{r}, t) = R_0^3\rho'(\mathbf{r}R_0, t)$. The SI unit of ρ is kg. We define the following moments of inertia $I(t) = R_0^{-2}I'(t)$. As a result, we obtain

$$I = \int_{W_3} \rho(\mathbf{r})\mathbf{r}^2_\perp\, \mathrm{d}^3\mathbf{r} \tag{27}$$

where x_k $(k = 1, 2, 3)$ and $\mathbf{r}$ are dimensionless. We note that the SI unit of I_{kl} is kg.

This representation allows us to generalize the equation of the scalar moment of inertia for a fractal material

$$I^{(D)}(t) = \int_W \rho(\mathbf{r}, t)\mathbf{r}^2_\perp\, \mathrm{d}V_D \tag{28}$$

where D is a mass dimension of the fractal material.

3.3. Moment of inertia of a fractal solid ball

Let us consider a fractal solid ball with radius R, and mass M. Note that the component of the radius perpendicular is

$$\mathbf{r}^2_\perp = (r\sin\theta)^2$$

Using the integration in a non-integer dimensional space, we have:

$$I^{(D)} = \int_W \mathrm{d}\mathbf{r}(r\sin\theta)^2\rho(r,\theta) = \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)}\int_0^\infty \mathrm{d}r\, r^{D-1}\int_0^\pi \mathrm{d}\theta\, \sin^{D-2}\theta(r\sin\theta)^2\rho(r,\theta) \tag{29}$$

For homogeneous materials $\rho(\mathbf{r}) = \rho_0$, we get

$$\begin{aligned} I^{(D)} &= \frac{2\pi^{(D-1)/2}\rho_0}{\Gamma((D-1)/2)}\int_0^\infty \mathrm{d}r\, r^{D+1}\int_0^\pi \mathrm{d}\theta\, \sin^D\theta = \frac{2\pi^{(D-1)/2}\rho_0}{\Gamma((D-1)/2)}\frac{\pi^{1/2}\Gamma(D/2)}{\Gamma(D/2+1)}\frac{R^{D+2}}{D+2} \\ &= \frac{2\pi^{D/2}\Gamma(D/2)\rho_0}{\Gamma(D/2-1/2)\Gamma(D/2+1)}\frac{R^{D+2}}{D+2} = \frac{\pi^{D/2}(D-1)\rho_0}{(D+2)\Gamma(D/2+1)}R^{D+2} \end{aligned} \tag{30}$$

where we use

$$\int_0^{\pi} d\theta \sin^D \theta = \frac{\pi^{1/2} \Gamma(D/2)}{\Gamma(D/2+1)} \tag{31}$$

Using the expression for mass (25), we can rewrite (30) as

$$I^{(D)} = \frac{2\Gamma(D/2)}{(D+2)\Gamma(D/2-1/2)} M_D R^2 = \frac{D-1}{D+2} M_D R^2 \tag{32}$$

where we use $\Gamma(z) = (z-1)\Gamma(z-1)$. For $D = 3$, Eq. (32) gives the well-known equation for the moment of inertia of a non-fractal ball $I^3 = (2/5)MR^2$.

4. Elasticity theory of a fractal material

4.1. Elasticity theory of a non-fractal material

The linear elastic constitutive relations for isotropic case is the well-known Hooke's law, which has the form

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \tag{33}$$

where λ and μ are the Lame coefficients, σ_{ij} is the stress, and ε_{kl} is the strain tensor. This expression determines the stress tensor in terms of the strain tensor for an isotropic material.

For homogenous and isotropic materials, the constitutive relation (33) gives the equation for the displacement vector fields $\mathbf{u} = \mathbf{u}(\mathbf{r}, t)$ in the form

$$\lambda \, \mathrm{grad} \, \mathrm{div} \, \mathbf{u} + 2\mu \Delta \mathbf{u} + \mathbf{f} = \rho D_t^2 \mathbf{u} \tag{34}$$

where $\mathbf{f} = \mathbf{u}(\mathbf{r}, t)$ is the vector field of the external force density.

If the deformation in the material is described by $\mathbf{u}(\mathbf{r}, t) = u(r, t)\mathbf{e}_r$, then Eq. (34) has the form:

$$(\lambda + 2\mu)\Delta \mathbf{u}(r, t) + \mathbf{f}(r, t) = \rho D_t^2 \mathbf{u}(r, t) \tag{35}$$

A formal generalization of Eqs. (35) for fractal material in the framework of non-integer dimensional models is:

$$(\lambda + 2\mu)^V \Delta_r^D \mathbf{u}(r, t) + \mathbf{f}(r, t) = \rho D_t^2 \mathbf{u}(r, t) \tag{36}$$

where ${}^V\Delta_r^D$ is defined by (4). Eq. (36) describes the dynamics of the displacement vector for fractal elastic materials.

4.2. Strain and stress in a non-integer dimensional space

Any deformation can be represented as the sum of a pure shear and a hydrostatic compression. To do so for fractal materials, we can use the identity:

$$\varepsilon_{kl} = \left(\varepsilon_{kl} - \frac{1}{D}\delta_{kl}\varepsilon_{ii}\right) + \frac{1}{D}\delta_{kl}\varepsilon_{ii} \tag{37}$$

The first term on the right is a pure shear, since the sum of diagonal terms is zero. Here we use the equation $\delta_{ii} = D$ for a non-integer dimensional space (for details see Property 4 in Section 4.3 of [26]). The second term is a hydrostatic compression. For $D = 3$, Eq. (37) has the well-known form:

$$\varepsilon_{kl} = \left(\varepsilon_{kl} - \frac{1}{3}\delta_{kl}\varepsilon_{ii}\right) + \frac{1}{3}\delta_{kl}\varepsilon_{ii} \tag{38}$$

where $\delta_{ii} = 3$ is used.

The stress tensor can be represented as

$$\sigma_{kl} = K\varepsilon_{ii}\delta_{kl} + 2\mu\left(\varepsilon_{kl} - \frac{1}{D}\delta_{kl}\varepsilon_{ii}\right) \tag{39}$$

where the bulk modulus (modulus of hydrostatic compression) is related to the Lame coefficients by

$$K = \lambda + \frac{2\mu}{D} \tag{40}$$

In the hydrostatic compression of a body, the stress tensor is:

$$\sigma_{kl} = -p\delta_{kl} \tag{41}$$

Hence we have

$$\sigma_{kk} = -pD \tag{42}$$

If we use the Hooke's law for isotropic case in the form, then

$$\sigma_{ii} = (\lambda D + 2\mu)\varepsilon_{ii} \tag{43}$$

The components of the strain tensor is

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = \left(\mathbf{e}_r, \mathrm{grad}_r^D\, u_r\right) \tag{44}$$

Using (1), and the trace of the strain tensor, one has:

$$e(r) = Tr[\varepsilon_{kl}] = \varepsilon_{kk} = \mathrm{Div}_r^D\, \mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{D-1}{r} u_r \tag{45}$$

Note that we can consider

$$e(r) - \varepsilon_{rr}(r) = \mathrm{Div}_r^D\, \mathbf{u} - \left(\mathbf{e}_r, \mathrm{Grad}_r^D\, u_r\right) = \frac{D-1}{r} u_r$$

as a sum of the angle diagonal components in the spherical coordinates of the strain tensor. For $D = 3$, we have the well-known sum of the angle diagonal components in the spherical coordinates of the strain tensor

$$\varepsilon_{\theta\theta} + \varepsilon_{\varphi\varphi} = \frac{2}{r} u_r$$

When we consider the fractal medium distributed in three-dimensional space, we can define the effective value of the diagonal angular components of the strain tensor:

$$\varepsilon_{\theta\theta}^{\mathrm{eff}} = \varepsilon_{\varphi\varphi}^{\mathrm{eff}} = \frac{D-1}{2r} u_r \tag{46}$$

Using (44) and (45), the components of the stress tensor $\sigma_{kl} = \sigma_{kl}(r)$ in the spherical coordinates are:

$$\sigma_{rr}(r) = 2\mu\varepsilon_{rr}(r) + \lambda e(r) = (2\mu + \lambda)\frac{\partial u_r}{\partial r} + \lambda \frac{D-1}{r} u_r \tag{47}$$

It is well known that the diagonal angular components of the stress tensor for $D = 3$ in spherical coordinates are:

$$\sigma_{\theta\theta}(r) = 2\mu\varepsilon_{\theta\theta}(r) + \lambda e(r), \qquad \sigma_{\varphi\varphi}(r) = 2\mu\varepsilon_{\varphi\varphi}(r) + \lambda e(r) \tag{48}$$

For the fractal medium distributed in a three-dimensional space, we can define the effective value of the diagonal angular components of the stress tensor:

$$\sigma_{\theta\theta}^{\mathrm{eff}}(r) = 2\mu\varepsilon_{\theta\theta}^{\mathrm{eff}}(r) + \lambda e(r) \tag{49}$$

$$\sigma_{\varphi\varphi}^{\mathrm{eff}}(r) = 2\mu\varepsilon_{\varphi\varphi}^{\mathrm{eff}}(r) + \lambda e(r) \tag{50}$$

Using the form of the effective components (46), we get:

$$\sigma_{\theta\theta}^{\mathrm{eff}}(r) = \sigma_{\varphi\varphi}^{\mathrm{eff}}(r) = \lambda \frac{\partial u_r}{\partial r} + (\lambda + \mu)\frac{D-1}{r} u_r \tag{51}$$

This equation defines the diagonal angular components of the stress tensor for spherical coordinates.

4.3. Equilibrium equation for fractal materials

For static case, Eq. (36), we have:

$${}^{V}\Delta_r^D \mathbf{u}(r) + (\lambda + 2\mu)^{-1}\mathbf{f}(r) = 0 \tag{52}$$

where $\mathbf{u} = u_r \mathbf{e}_r$ and $\mathbf{f} = f(r)\mathbf{e}_r$. Here λ and μ are the Lame coefficients. Using (4), we represent Eq. (52) in the form:

$$\frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-1}{r}\frac{\partial u_r(r)}{\partial r} - \frac{D-1}{r^2} u_r(r) + (\lambda + 2\mu)^{-1} f(r) = 0 \tag{53}$$

The solution to Eq. (53) is:

$$u_r(r) = C_1 r + C_2 r^{1-D} + I_f(D,r) \tag{54}$$

where C_1 and C_2 are constants defined by boundary conditions, and

$$I_f(D,r) = \frac{r}{D(\lambda+2\mu)}\left(\frac{1}{r^D}\int \mathrm{d}r\, r^D f(r) - \int \mathrm{d}r\, r f(r)\right) \tag{55}$$

For $f(r) = f_0$, we get

$$I_f(D,r) = -\frac{f_0 r^2}{2(D+1)(\lambda+2\mu)} \tag{56}$$

and the displacement is

$$u_r(r) = C_1 r + C_2 r^{1-D} - \frac{f_0 r^2}{(D+1)(\lambda+2\mu)} \tag{57}$$

The components of the stress tensor $\sigma_{kl} = \sigma_{kl}(r)$ for the spherical coordinates can be calculated by Eqs. (47) and (51).

4.4. Elasticity of a fractal hollow ball with inside and outside pressures

Let us determine the deformation of a hollow fractal ball with internal radius R_1 and external radius R_2 with the pressure p_1 inside and the pressure p_2 outside.

We can use the spherical polar coordinates with the origin at the center of the ball. The displacement vector $\mathbf{u}$ is everywhere radial, and it is a function of $r = |\mathbf{r}|$ alone. Then the equilibrium equation for fractal ball is:

$$(\lambda+2\mu)^V \Delta_r^D \mathbf{u}(r) = 0 \tag{58}$$

where $\mathbf{u} = u_r \mathbf{e}_r$. Using (4), we represent Eq. (52) in the form

$$\frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-1}{r}\frac{\partial u_r(r)}{\partial r} - \frac{D-1}{r^2}u_r(r) = 0 \tag{59}$$

The solution to (53) is:

$$u_r(r) = C_1 r + C_2 r^{1-D} \tag{60}$$

The constants C_1 and C_2 are determined from the boundary conditions for radial stress:

$$\sigma_{rr}(R_1) = -p_1, \qquad \sigma_{rr}(R_2) = -p_2 \tag{61}$$

Using that the radial components of the stress is:

$$\sigma_{rr}(r) = (2\mu+\lambda)\frac{\partial u_r}{\partial r} + \lambda\frac{D-1}{r}u_r \tag{62}$$

we get:

$$\sigma_{rr}(r) = (2\mu + D\lambda)C_1 + 2(1-D)\mu C_2 r^{-D} \tag{63}$$

Then we have:

$$(2\mu + D\lambda)C_1 + 2(1-D)\mu R_1^{-D} C_2 = -p_1 \tag{64}$$

$$(2\mu + D\lambda)C_1 + 2(1-D)\mu R_2^{-D} C_2 = -p_2 \tag{65}$$

As a result, the coefficients have the form:

$$C_1 = \frac{-(p_1 R_2^{-D} - p_2 R_1^{-D})}{(2\mu+D\lambda)(R_2^{-D} - R_1^{-D})} = \frac{-(p_2 R_2^D - p_1 R_1^D)}{(2\mu+D\lambda)(R_2^D - R_1^D)} \tag{66}$$

$$C_2 = \frac{-(p_2 - p_1)(R_1 R_2)^D}{2(1-D)\mu(R_2^{-D} - R_1^{-D})} = \frac{p_2 - p_1}{2(1-D)\mu(R_2^D - R_1^D)} \tag{67}$$

Then the radial components of the stress is:

$$\sigma_{rr}(r) = \frac{-(p_2 R_2^D - p_1 R_1^D)}{R_2^D - R_1^D} + \frac{(p_2 - p_1)(R_1 R_2)^D}{R_2^D - R_1^D} r^{-D} \tag{68}$$

The stress distribution in a ball with pressure $p_1 = p$ inside and $p_2 = 0$ outside is given by:

$$\sigma_{rr}(r) = \frac{pR_1^D}{R_2^D - R_1^D} - \frac{p}{R_2^D - R_1^D} r^{-D} = \frac{pR_1^D}{R_2^D - R_1^D}\left(1 - \left(\frac{R_2}{r}\right)^D\right) \tag{69}$$

The stress distribution in an infinite elastic material with a spherical cavity of radius R subjected to hydrostatic compression is:

$$\sigma_{rr}(r) = -p\left(1 - \left(\frac{R}{r}\right)^D\right) \tag{70}$$

which can be obtained by putting $R_1 = R$, $R_2 \to \infty$, $p_1 = 0$ and $p_2 = p$ in Eq. (68).

5. Elasticity of fractal material with radial distribution in cylinder and pipe

5.1. Elasticity of a fractal pipe and cylinder

If we use the cylindrical coordinates with the z-axis and if the vector field $\mathbf{u}(\mathbf{r}, t)$ is a purely radial,

$$\mathbf{u} = u_r(r)\mathbf{e}_r \tag{71}$$

where $\mathbf{e}_r = \mathbf{r}/r$, then

$$^V\Delta_r^D \mathbf{u} = \left(\frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-2}{r}\frac{\partial u_r}{\partial r} - \frac{D-2}{r^2}u_r(r)\right)\mathbf{e}_r \tag{72}$$

Note that we have $(D-2)$ instead of $(D-1)$ in this equation. For $D = 3$, Eq. (72) gives the well-known equation for the elasticity of cylinder and pipe.

If $\mathbf{u} = u_r(r)\mathbf{e}_r$ is the displacement vector for non-fractal materials in the 3-dimensional case ($D = 3$), then the strain tensor $\varepsilon_{ij}(\mathbf{r})$ has the following nonzero components that can be defined by

$$\varepsilon_{rr} = (\mathbf{e}_r, \operatorname{grad} u_r) = \frac{\partial u_r}{\partial r} \tag{73}$$

$$\varepsilon_{\varphi\varphi} = \operatorname{div}\mathbf{u} - (\mathbf{e}_r, \operatorname{grad} u_r) = \frac{u_r}{r} \tag{74}$$

and the invariant

$$e = \varepsilon_{kk} = \operatorname{div}\mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} \tag{75}$$

These equations with divergence and gradient can be generalized for the non-integer dimensional case for $\mathbf{u} = u_r(r)\mathbf{e}_r$.

For the non-integer dimensional model of fractal materials, we can use the definitions for the components of displacement vector in non-integer dimensional space in the form:

$$\varepsilon_{rr} = \left(\mathbf{e}_r, \operatorname{grad}_r^D u_r\right) \tag{76}$$

$$\varepsilon_{\varphi\varphi} = \operatorname{div}_r^D \mathbf{u} - \left(\mathbf{e}_r, \operatorname{grad}_r^D u_r\right) \tag{77}$$

where we use the invariant:

$$e = \varepsilon_{kk} = \operatorname{div}_r^D \mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{D-2}{r}u_r \tag{78}$$

As a result, we get:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \tag{79}$$

$$\varepsilon_{\varphi\varphi} = \frac{D-2}{r}u_r \tag{80}$$

Let us assume that the elastic constitutive relations for fractal material in an isotropic case has the usual form

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{81}$$

In this case, the non-zero components of stress tensor are

$$\sigma_{rr} = 2\mu\varepsilon_{rr} + \lambda e = (2\mu + \lambda)\frac{\partial u_r}{\partial r} + \lambda\frac{D-2}{r}u_r \quad (82)$$

$$\sigma_{\varphi\varphi} = 2\mu\varepsilon_{\varphi\varphi} + \lambda e = \lambda\frac{\partial u_r}{\partial r} + (2\mu + \lambda)\frac{D-2}{r}u_r \quad (83)$$

$$\sigma_{zz} = 2\mu\varepsilon_{zz} + \lambda e = \lambda\frac{\partial u_r}{\partial r} + \lambda\frac{D-2}{r}u_r \quad (84)$$

where we use $\varepsilon_{zz} = 0$. For $D = 3$, we have the usual constitutive relations for the isotropic case in cylindrical coordinates.

5.2. Elasticity of cylindrical fractal pipe with inside and outside pressures

Let us consider the deformation of a fractal solid cylindrical pipe with internal radius R_1 and external radius R_2, with an inside pressure p_1 and outside pressure p_2. We use the cylindrical coordinates with the z-axis along the axis of the pipe. When the pressure is uniform along the pipe, the deformation is a purely radial displacement $\mathbf{u} = u_r(r)\mathbf{e}_r$, where $\mathbf{e}_r = \mathbf{r}/r$. The equation for the displacement $u_r(r)$ in fractal pine is:

$$\frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-2}{r}\frac{\partial u_r}{\partial r} - \frac{D-2}{r^2}u_r = 0 \quad (85)$$

where $0 < D \leq 3$. If $D = 3$, we get the usual (non-fractal) case.

The general solution to Eq. (85), where $D \neq 1$ and $D \neq 2$, has the form:

$$u_r(r) = C_1 r + C_2 r^{2-D} \quad (86)$$

Eqs. (85) with $D = 1$ has the general solution:

$$u_r(r) = C_1 r + C_2 r \ln(r) \quad (87)$$

For $D = 2$, Eq. (85) has the solution:

$$u_r(r) = C_1 + C_2 r \quad (88)$$

Note that the fractal dimension of the pipe material can be $D = 1$ or $D = 2$. These cases do not correspond to the distribution of matter along the line and surface. These fractal dimensions describe a distribution of matter in a 3-dimensional space (in the volume of pipe) such that the mass dimension is equal to D.

The constants C_1 and C_2 are determined by boundary conditions. Be the inside pressure p_1 and be the outside pressure p_2, we get the boundary condition in the form:

$$\sigma_{rr}(R_1) = -p_1, \sigma_{rr}(R_2) = -p_2 \quad (89)$$

Using (86), we get:

$$\frac{\partial u_r}{\partial r} = C_1 + (2-D)C_2 r^{1-D} \quad (90)$$

$$\frac{D-2}{r}u_r = (D-2)C_1 + (D-2)C_2 r^{1-D} \quad (91)$$

Then

$$\sigma_{rr} = (2\mu + \lambda)\frac{\partial u_r}{\partial r} + \lambda\frac{D-2}{r}u_r = \big(2\mu + \lambda(D-1)\big)C_1 - 2\mu(D-2)C_2 r^{1-D} \quad (92)$$

The boundary condition (89) has the form:

$$\big(2\mu + \lambda(D-1)\big)C_1 - 2\mu(D-2)C_2 R_1^{1-D} = -p_1 \quad (93)$$

$$\big(2\mu + \lambda(D-1)\big)C_1 - 2\mu(D-2)C_2 R_2^{1-D} = -p_2 \quad (94)$$

As a result, we have:

$$C_1 = -\frac{p_1 R_2^{1-D} - p_2 R_1^{1-D}}{(2\mu + \lambda(D-1))(R_2^{1-D} - R_1^{1-D})} \quad (95)$$

$$C_2 = \frac{p_2 - p_1}{2\mu(D-2)(R_2^{1-D} - R_1^{1-D})} \quad (96)$$

The stress is:

$$\sigma_{rr} = -\frac{p_1 R_2^{1-D} - p_2 R_1^{1-D}}{(R_2^{1-D} - R_1^{1-D})} - \frac{p_2 - p_1}{(R_2^{1-D} - R_1^{1-D})} r^{1-D} \tag{97}$$

If $2 < D < 3$ or $1 < D < 2$, then we can rewrite Eq. (97) under the form:

$$\sigma_{rr} = \frac{p_1 R_1^{D-1} - p_2 R_2^{D-1}}{(R_2^{D-1} - R_1^{D-1})} - \frac{p_2 - p_1}{(R_2^{D-1} - R_1^{D-1})} \left(\frac{R_1 R_2}{r}\right)^{D-1} \tag{98}$$

For the boundary conditions $\sigma_{rr}(R_2) = 0$ and $\sigma_{rr}(R_1) = -p$, i.e. $p_2 = 0$ and $p_1 = p$ for (98), the solution can be written under the form:

$$\sigma_{rr} = \frac{p R_1^{D-1}}{(R_2^{D-1} - R_1^{D-1})} \left(1 - \left(\frac{R_2}{r}\right)^{D-1}\right) \tag{99}$$

This is the deformation of cylindrical pipe with a pressure p inside and no pressure outside. For $D = 3$, we have:

$$\sigma_{rr} = \frac{p R_1^2}{(R_2^2 - R_1^2)} \left(1 - \left(\frac{R_2}{r}\right)^2\right) \tag{100}$$

which describes the stress in a non-fractal pipe material.

5.3. *Rotating cylindrical fractal pipe*

Let us consider the deformation of a fractal solid that is described by an equation with external force $f(r)$ for the displacement field $u_r(r)$ in a fractal pipe:

$$\frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-2}{r}\frac{\partial u_r}{\partial r} - \frac{D-2}{r^2} u_r + \frac{1}{\lambda + 2\mu} f(r) = 0 \tag{101}$$

where $D > 0$. The general solution to Eq. (101) has the form:

$$u_r(r) = C_1 r + C_2 r^{2-D} - \frac{1}{(D-1)(\lambda + 2\mu)} \left(\int_{R_1}^{R_2} f(r) r \mathrm{d}r - r^{2-D} \int_{R_1}^{R_2} f(r) r^{D-1} \, \mathrm{d}r \right) \tag{102}$$

Eq. (101) with $D = 1$ has the general solution:

$$u_r(r) = C_1 r + C_2 r \ln(r) + \frac{r}{\lambda + 2\mu} \left(\int_{R_1}^{R_2} f(r) \ln(r) \, \mathrm{d}r + \ln(r) \int_{R_1}^{R_2} f(r) \, \mathrm{d}r \right) \tag{103}$$

For $D = 2$, Eq. (101) has the solution

$$u_r(r) = C_1 + C_2 r - \frac{1}{\lambda + 2\mu} \left(\int_{R_1}^{R_2} f(r) r \, \mathrm{d}r - r \int_{R_1}^{R_2} f(r) \, \mathrm{d}r \right) \tag{104}$$

Let us consider the deformation of a fractal solid cylindrical pipe with internal radius R_1 and external radius R_2 rotating uniformly about its axis with angular velocity ω. Then the density of the centrifugal force is

$$f_r(r) = \rho_0 \omega^2 r \tag{105}$$

We use the cylindrical coordinates with the z-axis along the axis of the cylinder. When the pressure is uniform along the pipe, the deformation is a purely radial displacement $\mathbf{u} = u_r(r)\mathbf{e}_r$, where $\mathbf{e}_r = \mathbf{r}/r$. The equation for the displacement $u_r(r)$ in the fractal material is:

$$\frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-2}{r}\frac{\partial u_r}{\partial r} - \frac{D-2}{r^2} u_r = -\frac{\rho_0 \omega^2}{\lambda + 2\mu} r \tag{106}$$

The general solution to Eq. (106) has the form:

$$u_r(r) = C_1 r + C_2 r^{2-D} - A r^3 \tag{107}$$

where

$$A = \frac{\rho_0 \omega^2}{2(D+1)(\lambda + 2\mu)} \tag{108}$$

Using the condition that external forces do not act inside and outside the fractal pipe, we have the boundary condition:

$$\sigma_{rr}(R_1) = 0, \qquad \sigma_{rr}(R_2) = 0 \tag{109}$$

Using (107), we get:

$$\frac{\partial u_r}{\partial r} = C_1 + (2-D)C_2 r^{1-D} - 3Ar^2 \tag{110}$$

$$\frac{D-2}{r} u_r = (D-2)C_1 + (D-2)C_2 r^{1-D} - A(D-2)r^2 \tag{111}$$

Then

$$\begin{aligned}\sigma_{rr} &= (2\mu + \lambda)\frac{\partial u_r}{\partial r} + \lambda \frac{D-2}{r} u_r \\ &= \big(2\mu + \lambda(D-1)\big)C_1 - 2\mu(D-2)C_2 r^{1-D} - A\big(6\mu + \lambda(D+1)\big)r^2\end{aligned} \tag{112}$$

The boundary condition (109) has the form:

$$\big(2\mu + \lambda(D-1)\big)C_1 - 2\mu(D-2)C_2 R_1^{1-D} = A\big(6\mu + \lambda(D+1)\big)R_1^2 \tag{113}$$

$$\big(2\mu + \lambda(D-1)\big)C_1 - 2\mu(D-2)C_2 R_2^{1-D} = A\big(6\mu + \lambda(D+1)\big)R_2^2 \tag{114}$$

Then

$$C_1 = \frac{A(6\mu + \lambda(D+1))(R_2^{D+1} - R_1^{D+1})}{(2\mu + \lambda(D-1))(R_2^{D-1} - R_1^{D-1})} \tag{115}$$

$$C_2 = \frac{A(6\mu + \lambda(D+1))(R_2^2 - R_1^2)(R_1 R_2)^{D-1}}{2\mu(D-2)(R_2^{D-1} - R_1^{D-1})} \tag{116}$$

Substituting (115) and (116) into (107), we get:

$$\begin{aligned}u_r(r) &= \frac{A(6\mu + \lambda(D+1))(R_2^{D+1} - R_1^{D+1})}{(2\mu + \lambda(D-1))(R_2^{D-1} - R_1^{D-1})} r \\ &\quad + \frac{A(6\mu + \lambda(D+1))(R_2^2 - R_1^2)(R_1 R_2)^{D-1}}{2\mu(D-2)(R_2^{D-1} - R_1^{D-1})} r^{2-D} - Ar^3\end{aligned} \tag{117}$$

where A is defined by Eq. (108).

For the fractal cylinder ($R_1 = 0$, $R_2 = R$), we have:

$$u_r(r) = \frac{\rho_0 \omega^2}{2(D+1)(\lambda + 2\mu)} \left(\frac{6\mu + \lambda(D+1)}{2\mu + \lambda(D-1)} R^2 r - r^3 \right) \tag{118}$$

For $D = 3$, Eq. (118) gives

$$u_r(r) = \frac{\rho_0 \omega^2}{8(\lambda + 2\mu)} \left(\frac{3\mu + 2\lambda}{\mu + \lambda} R^2 r - r^3 \right) \tag{119}$$

Eq. (119) describes the displacement field for an elastic cylinder with a non-fractal material.

6. Gradient elasticity model for fractal materials

6.1. Gradient elasticity theory of non-fractal materials

Papers [63–65] suggested to generalize the constitutive relations (33) by the gradient modification that contains the Laplacian in the form:

$$\sigma_{ij} = (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}) - l_s^2 \Delta(\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}) \tag{120}$$

where l_s is the scale parameter [66].

For homogenous and isotropic materials equation for (120) has the form

$$\lambda\big(1 \pm l_s^2 \Delta\big)\, \mathrm{grad}\, \mathrm{div}\, \mathbf{u} + 2\mu\big(1 \pm l_s^2 \Delta\big)\Delta \mathbf{u} + \mathbf{f} = \rho D_t^2 \mathbf{u} \tag{121}$$

where $\mathbf{f}$ is the vector field of the external force density.

Using the relation

$$\mathrm{grad}\, \mathrm{div}\, \mathbf{u} = \mathrm{curl}\, \mathrm{curl}\, \mathbf{u} + \Delta \mathbf{u} \tag{122}$$

we can rewrite Eq. (121) as

$$\lambda\big(1 \pm l_s^2 \Delta\big)\, \mathrm{curl}\, \mathrm{curl}\, \mathbf{u} + (\lambda + 2\mu)\big(1 \pm l_s^2 \Delta\big)\Delta \mathbf{u} + \mathbf{f} = \rho D_t^2 \mathbf{u} \tag{123}$$

If we assume that the displacement vector $\mathbf{u}$ is everywhere radial and it is a function of $r = |\mathbf{r}|$ alone ($u_k = u_k(|\mathbf{r}|)$), then

$$\mathrm{curl}\, \mathbf{u} = 0$$

As a result, Eq. (123) has the form

$$(\lambda + 2\mu)\big(1 \pm l_s^2 \Delta\big)\Delta \mathbf{u} + \mathbf{f} = \rho D_t^2 \mathbf{u} \tag{124}$$

This is the gradient elasticity equation for homogenous and isotropic materials with spherical symmetry. For the non-gradient model ($l_s^2 = 0$), Eq. (124) for the displacement gives Eq. (35).

6.2. Gradient elasticity of fractal materials

A formal generalization of Eq. (124) for fractal materials in the framework of continuum models with non-integer dimensional space, where the displacement vector, $\mathbf{u} = \mathbf{u}(r, t)$, does not depend on the angles, has the form:

$$(\lambda + 2\mu)\big(1 \pm l_s^2(D)\, {}^V\Delta_r^D\big)\, {}^V\Delta_r^D \mathbf{u} + \mathbf{f} = \rho D_t^2 \mathbf{u} \tag{125}$$

This is the fractional gradient elasticity equation for homogenous and isotropic materials with spherical symmetry. Let us consider Eq. (125) for static case ($D_t^2 \mathbf{u} = 0$) with a minus in front of Laplacian, i.e. the GRADELA model for fractal materials [66],

$$(\lambda + 2\mu)\big(1 - l_s^2(D)\, {}^V\Delta_r^D\big)\, {}^V\Delta_r^D \mathbf{u} + \mathbf{f} = 0 \tag{126}$$

We can rewrite this equation as

$$\big({}^V\Delta_r^D\big)^2 \mathbf{u} - l_s^{-2}(D)\, {}^V\Delta_r^D \mathbf{u} - (\lambda + 2\mu)^{-1} l_s^{-2}(D)\mathbf{f} = 0 \tag{127}$$

Using the vector Laplacian (4), we have:

$${}^V\Delta_r^D \mathbf{u}(r) = \left(\frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-1}{r}\frac{\partial u_r(r)}{\partial r} - \frac{D-1}{r^2} u_r(r)\right)\mathbf{e}_r \tag{128}$$

and

$$\begin{aligned}\big({}^V\Delta_r^D\big)^2 \mathbf{u}(r) = \bigg(&\frac{\partial^4 u_r(r)}{\partial r^4} + \frac{2(D-1)}{r}\frac{\partial^3 u_r(r)}{\partial r^3} \\ &+ \frac{(D-1)(D-5)}{r^2}\frac{\partial^2 u_r(r)}{\partial r^2} - \frac{3(D-1)(D-3)}{r^3}\frac{\partial u_r(r)}{\partial r} + \frac{3(D-1)(D-3)}{r^4} u_r(r)\bigg)\mathbf{e}_r\end{aligned} \tag{129}$$

Using Eqs. (128) and (129) and $\mathbf{f}(r) = f(r)\mathbf{e}_r$, Eq. (127) gives:

$$\begin{aligned}&\frac{\partial^4 u_r(r)}{\partial r^4} + \frac{2(D-1)}{r}\frac{\partial^3 u_r(r)}{\partial r^3} + \left(\frac{(D-1)(D-5)}{r^2} - l_s^{-2}(D)\right)\frac{\partial^2 u_r(r)}{\partial r^2} \\ &\quad - \left(\frac{3(D-1)(D-3)}{r^3} + l_s^{-2}(D)\frac{D-1}{r}\right)\frac{\partial u_r(r)}{\partial r} \\ &\quad + \left(\frac{3(D-1)(D-3)}{r^4} + l_s^{-2}(D)\frac{D-1}{r^2}\right)u_r(r) - (\lambda + 2\mu)^{-1} l_s^{-2}(D) f(r) = 0\end{aligned} \tag{130}$$

The general solution for the case $f(r) = 0$ is

$$u_r(r) = C_1 r + C_2 r^{1-D} - C_3 I_I(D, r) - C_4 I_K(D, r) \tag{131}$$

where $I_I(D,r)$ and $I_K(D,r)$ are the integrals of the Bessel functions:

$$I_I(D,r) = Dr \int \mathrm{d}r\, r^{-D-1} \int \mathrm{d}r\, r^{D/2+1} I_{D/2}\big(r/l_s(D)\big) \tag{132}$$

$$I_K(D,r) = Dr \int \mathrm{d}r\, r^{-D-1} \int \mathrm{d}r\, r^{D/2+1} K_{D/2}\big(r/l_s(D)\big) \tag{133}$$

where $I_\alpha(x)$ and $K_\alpha(x)$ are Bessel functions of the first and second kinds.

7. Thermoelasticity of fractal materials

Let us consider a generalization of thermoelasticity [67–69] for a fractal material. In this section, we consider a non-integer-dimensional model of the thermoelasticity of a fractal material. Note that the thermoelasticity of fractal materials has been considered in the framework of the fractional-integral continuum model [10,9,15] by Ostoja-Starzewski in [34,35].

7.1. Thermoelastic constitutive relation for fractal materials

If the isotropic material is non-uniformly heated, then the constitutive relation for a thermoelastic material must include [70] the term

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - K\alpha(T - T_0)\delta_{ij} \tag{134}$$

where λ and μ are the Lame coefficients, K is the bulk modulus or modulus of compression. The third term in Eq. (134) gives the additional stresses caused by the change in temperature.

The bulk modulus for fractal materials is related to the Lame coefficients by:

$$K = \lambda + \frac{2}{D}\mu \tag{135}$$

In this formula, we use the dimension D instead of 3 because $\delta_{kk} = D$ for a non-integer dimensional space (see Property 4 in Section 4.3 of [26]).

If external forces are absent, then the stress is equal to zero $\sigma_{ij} = 0$ and we have a free thermal expansion. Using $\sigma_{ij} = 0$, Eq. (134) gives:

$$\lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - K\alpha(T - T_0)\delta_{ij} = 0 \tag{136}$$

Using $\delta_{kk} = D$ and (135), we obtain:

$$\varepsilon_{kk} = \alpha(T - T_0) \tag{137}$$

Because the function $e = \varepsilon_{kk}$ describes the relative change of volume caused by deformation, then α is the thermal expansion coefficient of the material [70].

The constitutive relation (134) for a fractal material can be represented under the form:

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - (D\lambda + 2\mu)\alpha_T(T - T_0)\delta_{ij} \tag{138}$$

where we use the dimension D instead of 3 since $K = (D\lambda + 2\mu)/D$.

7.2. Thermoelastic equations for fractal material

For homogenous and isotropic non-fractal materials, we have the thermoelasticity equation:

$$\lambda\, \mathrm{grad}\,\mathrm{div}\,\mathbf{u} + 2\mu\Delta\mathbf{u} + \mathbf{f} - (3\lambda + 2\mu)\alpha\, \mathrm{grad}\, T = \rho D_t^2\mathbf{u} \tag{139}$$

where $\mathbf{f}$ is the external force density vector field. For the case $\mathbf{u} = \mathbf{u}(r,t)$ and $\mathbf{f} = \mathbf{f}(r,t)$, Eq. (139) gives:

$$(\lambda + 2\mu)\Delta\mathbf{u} + \mathbf{f} - (3\lambda + 2\mu)\alpha\, \mathrm{grad}\, T = \rho D_t^2\mathbf{u} \tag{140}$$

The thermoelasticity equation for fractal materials in the framework of models with non-integer dimensional spaces has the form:

$$(\lambda + 2\mu)^V\Delta_r^D\mathbf{u}(r,t) + \mathbf{f}(r,t) - (D\lambda + 2\mu)\alpha\, \mathrm{Grad}_r^D\, T(r,t) = \rho D_t^2\mathbf{u}(r,t) \tag{141}$$

where we assume $\mathbf{f} = f_r(r,t)\mathbf{e}_r$ and $\mathbf{u} = u_r(r,t)\mathbf{e}_r$. Using (4), Eq. (141) can be written under the form:

$$\frac{\partial^2 u_r(r,t)}{\partial r^2} + \frac{D-1}{r}\frac{\partial u_r(r,t)}{\partial r} - \frac{D-1}{r^2}u_r(r,t) + \frac{1}{\lambda+2\mu}f_r(r,t) - \frac{\alpha(D\lambda+2\mu)}{\lambda+2\mu}\frac{\partial T(r,t)}{\partial r} = \frac{\rho}{\lambda+2\mu}D_t^2 u_r(r,t) \quad (142)$$

If the fractal material is non-uniformly heated, then the equation of equilibrium has the form:

$$\frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-1}{r}\frac{\partial u_r(r)}{\partial r} - \frac{D-1}{r^2}u_r(r) = \frac{(D\lambda+2\mu)\alpha}{\lambda+2\mu}\frac{\partial T(r)}{\partial r} \quad (143)$$

This is the thermoelasticity equation in spherical coordinates for the pure radial deformation of fractal materials with fractal dimension D. For $D = 3$, we get the usual equation for the thermoelasticity of a solid ball [70].

The general solution to Eq. (143) has the form

$$u_r(r) = C_1 r + \frac{c_2}{r^{D-1}} + \frac{(D\lambda+2\mu)\alpha}{D(\lambda+2\mu)}\left(rT(r) - \frac{D}{r^{D-1}}\int r^D T(r)\,\mathrm{d}r\right) \quad (144)$$

where C_1 and C_2 are defined by boundary conditions.

8. Conclusion

In this paper, continuum models with non-integer dimensional space are suggested to describe isotropic fractal materials. A generalization of differential vector operators for a non-integer dimensional space is proposed to describe the elasticity of fractal materials in the framework of continuum models. The differential operators of first and second orders for non-integer dimensional spaces are suggested for rotationally covariant scalar and vector functions. We consider applications for the elasticity theory in the case of spherical and axial symmetries of the fractal material. Elastic properties of fractal hollow ball and cylindrical fractal pipe with inside and outside pressures, rotating cylindrical fractal elastic pipe are described. Equations for thermoelasticity and gradient elasticity of fractal materials are solved.

Although the non-integer dimension does not reflect all the properties of fractal materials, the suggested models with non-integer dimensional space nevertheless allow us to derive a number of important conclusions about the behavior of fractal materials. Therefore continuum models with non-integer dimensional spaces can be successfully used to describe the elasticity and the thermoelasticity of fractal materials.

The proposed continuum models of fractal materials can be extended to more complex fractal materials: (1) we assume that continuum models with non-integer dimensional space can be generalized for anisotropic fractal materials [74]; (2) the non-integer dimensional models of fractal elastic materials can easily be generalized for the boundary dimensions $d \neq D-1$ [75]; (3) we also assume that differential and integral operators of fractional orders can also be defined for non-integer dimensional spaces to take into account the non-locality of fractal materials. Note that the dimensional continuation of the Riesz fractional integrals and derivatives [71] to generalize differentials and integrals of fractional orders for non-integer dimensional spaces has been considered in [72].

References

[1] K.F. Falconer, The Geometry of Fractal Sets, Cambridge University Press, 1985.
[2] J. Feder, Fractals, Plenum Press, New York, 1988.
[3] J. Kugami, Analysis on Fractals, Cambridge University Press, 2001.
[4] R.S. Strichartz, Differential Equations on Fractals, Princeton Univ. Press, Princeton and Oxford, 2006, 170 pp.
[5] R.S. Strichartz, Analysis on fractals, Not. Am. Math. Soc. 46 (10) (1999) 1199–1208.
[6] J. Harrison, Flux across nonsmooth boundaries and fractal Gauss/Green/Stokes' theorems, J. Phys. A 32 (28) (1999) 5317–5328.
[7] T. Kumagai, Recent developments of analysis on fractals, Transl. Am. Math. Soc. 223 (202) (2008) 81–96.
[8] G. Derfel, P. Grabner, F. Vogl, Laplace operators on fractals and related functional equations (topical review), J. Phys. A 45 (46) (2012) 463001, 34 pp., arXiv:1206.1211.
[9] V.E. Tarasov, Continuous medium model for fractal media, Phys. Lett. A 336 (2–3) (2005) 167–174, arXiv:cond-mat/0506137.
[10] V.E. Tarasov, Fractional hydrodynamic equations for fractal media, Ann. Phys. 318 (2) (2005) 286–307, arXiv:physics/0602096.
[11] V.E. Tarasov, Dynamics of fractal solid, Int. J. Mod. Phys. B 19 (27) (2005) 4103–4114, arXiv:0710.0787.
[12] V.E. Tarasov, Wave equation for fractal solid string, Mod. Phys. Lett. B 19 (15) (2005) 721–728, arXiv:physics/0605006.
[13] M. Ostoja-Starzewski, Continuum mechanics models of fractal porous media: integral relations and extremum principles, J. Mech. Mater. Struct. 4 (5) (2009) 901–912.
[14] H. Joumaa, M. Ostoja-Starzewski, Acoustic-elastodynamic interaction in isotropic fractal media, Eur. Phys. J. Spec. Top. 222 (8) (2013) 1951–1960.
[15] V.E. Tarasov, Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media, Springer, New York, 2011.
[16] A. Carpinteri, B. Chiaia, P. Cornetti, Static-kinematic duality and the principle of virtual work in the mechanics of fractal media, Comput. Methods Appl. Mech. Eng. 191 (1–2) (2001) 3–19.
[17] A. Carpinteri, P. Cornetti, A fractional calculus approach to the description of stress and strain localization in fractal media, Chaos Solitons Fractals 13 (1) (2002) 85–94.
[18] A. Carpinteri, B. Chiaia, P. Cornetti, On the mechanics of quasi-brittle materials with a fractal microstructure, Eng. Fract. Mech. 70 (15) (2003) 2321–2349.
[19] A. Carpinteri, P. Cornetti, K.M. Kolwankar, Calculation of the tensile and flexural strength of disordered materials using fractional calculus, Chaos Solitons Fractals 21 (3) (2004) 623–632.

[20] A. Carpinteri, B. Chiaia, P. Cornetti, A fractal theory for the mechanics of elastic materials, Mater. Sci. Eng. A 365 (1–2) (2004) 235–240.
[21] A. Carpinteri, B. Chiaia, P. Cornetti, A disordered microstructure material model based on fractal geometry and fractional calculus, Z. Angew. Math. Mech. 84 (2) (2004) 128–135.
[22] A. Carpinteri, B. Chiaia, P. Cornetti, The elastic problem for fractal media: basic theory and finite element formulation, Comput. Struct. 82 (6) (2004) 499–508.
[23] A. Carpinteri, B. Chiaia, P. Cornetti, Numerical modelization of disordered media via fractional calculus, Comput. Mater. Sci. 30 (1–2) (2004) 155–162.
[24] A. Carpinteri, P. Cornetti, A. Sapora, M. Di Paola, M. Zingales, Fractional calculus in solid mechanics: local versus non-local approach, Phys. Scr. T 136 (2009) 14003.
[25] A. Carpinteri, P. Cornetti, A. Sapora, Static-kinematic fractional operators for fractal and non-local solids, Z. Angew. Math. Mech. 89 (3) (2009) 207–217.
[26] J.C. Collins, Renormalization, Cambridge University Press, Cambridge, 1984.
[27] F.H. Stillinger, Axiomatic basis for spaces with noninteger dimensions, J. Math. Phys. 18 (6) (1977) 1224–1234.
[28] C. Palmer, P.N. Stavrinou, Equations of motion in a non-integer-dimensional space, J. Phys. A 37 (27) (2004) 6987–7003.
[29] M. Ostoja-Starzewski, J. Li, Fractal materials, beams and fracture mechanics, Z. Angew. Math. Phys. 60 (6) (2009) 1194–1205.
[30] J. Li, M. Ostoja-Starzewski, Fractal solids, product measures and fractional wave equations, Proc. R. Soc. A 465 (2108) (2009) 2521–2536;
J. Li, M. Ostoja-Starzewski, Correction to Li and Ostoja-Starzewski 465 (2108) 2521, Proc. R. Soc. A 467 (2128) (2011) 1214 (1 p.).
[31] P.N. Demmie, M. Ostoja-Starzewski, Waves in fractal media, J. Elast. 104 (1–2) (2011) 187–204.
[32] J. Li, M. Ostoja-Starzewski, Micropolar continuum mechanics of fractal media, Int. J. Eng. Sci. 49 (12) (2011) 1302–1310.
[33] M. Ostoja-Starzewski, J. Li, H. Joumaa, P.N. Demmie, From fractal media to continuum mechanics, Z. Angew. Math. Mech. (J. Appl. Math. Mech.) 94 (5) (2014) 373–401.
[34] M. Ostoja-Starzewski, Towards thermoelasticity of fractal media, J. Therm. Stresses 30 (9–10) (2007) 889–896.
[35] M. Ostoja-Starzewski, Towards thermomechanics of fractal media, Z. Angew. Math. Phys. 58 (6) (2007) 1085–1096.
[36] K.G. Wilson, M.E. Fisher, Critical exponents in 3.99 dimensions, Phys. Rev. Lett. 28 (4) (1972) 240–243.
[37] K.G. Wilson, J. Kogut, The renormalization group and the ϵ expansion, Phys. Rep. 12 (2) (1974) 75–199.
[38] G. 't Hooft, M. Veltman, Regularization and renormalization of gauge fields, Nucl. Phys. B 44 (1) (1972) 189–213.
[39] G. Leibbrandt, Introduction to the technique of dimensional regularization, Rev. Mod. Phys. 47 (4) (1975) 849–876.
[40] K.G. Wilson, Quantum field — theory models in less than 4 dimensions, Phys. Rev. D 7 (10) (1973) 2911–2926.
[41] X.-F. He, Anisotropy and isotropy: a model of fraction-dimensional space, Solid State Commun. 75 (2) (1990) 111–114.
[42] X.-F. He, Fractional dimensionality and fractional derivative spectra of interband optical transitions, Phys. Rev. B 42 (18) (1990) 11751–11756.
[43] X.-F. He, Excitons in anisotropic solids: the model of fractional-dimensional space, Phys. Rev. B 43 (3) (1991) 2063–2069.
[44] A. Thilagam, Exciton–phonon interaction in fractional dimensional space, Phys. Rev. B 56 (15) (1997) 9798–9804.
[45] A. Matos-Abiague, Deformation of quantum mechanics in fractional-dimensional space, J. Phys. A 34 (49) (2001) 11059–11068, arXiv:quant-ph/0107062.
[46] A. Matos-Abiague, Bose-like oscillator in fractional-dimensional space, J. Phys. A 34 (14) (2001) 3125–3138.
[47] M.A. Lohe, A. Thilagam, Quantum mechanical models in fractional dimensions, J. Phys. A 37 (23) (2004) 6181–6199.
[48] R. Eid, S.I. Muslih, D. Baleanu, E. Rabei, On fractional Schrödinger equation in α-dimensional fractional space, Nonlinear Anal., Real World Appl. 10 (3) (2009) 1299–1304.
[49] S.I. Muslih, Solutions of a particle with fractional δ-potential in a fractional dimensional space, Int. J. Theor. Phys. 49 (9) (2010) 2095–2104, arXiv:1001.4352v2.
[50] S.I. Muslih, O.P. Agrawal, Schrödinger equation in fractional space, in: Fractional Dynamics and Control, Springer, New York, 2012, pp. 209–215.
[51] G. Calcagni, G. Nardelli, M. Scalisi, Quantum mechanics in fractional and other anomalous spacetimes, J. Math. Phys. 53 (10) (2012) 102110, arXiv:1207.4473.
[52] J. Martins, H.V. Ribeiro, L.R. Evangelista, L.R. da Silva, E.K. Lenzi, Fractional Schrödinger equation with noninteger dimensions, Appl. Math. Comput. 219 (4) (2012) 2313–2319.
[53] L.S. Lucena, L.R. da Silva, A.A. Tateishi, M.K. Lenzi, H.V. Ribeiro, E.K. Lenzi, Solutions for a fractional diffusion equation with noninteger dimensions, Nonlinear Anal., Real World Appl. 13 (4) (2012) 1955–1960.
[54] S.I. Muslih, D. Baleanu, Fractional multipoles in fractional space, Nonlinear Anal., Real World Appl. 8 (1) (2007) 198–203.
[55] D. Baleanu, A.K. Golmankhaneh, A.K. Golmankhaneh, On electromagnetic field in fractional space, Nonlinear Anal., Real World Appl. 11 (1) (2010) 288–292.
[56] S.I. Muslih, M. Saddallah, D. Baleanu, E. Rabei, Lagrangian formulation of Maxwell's field in fractional D dimensional space–time, Rom. Rep. Phys. 55 (7–8) (2010) 659–663.
[57] M. Zubair, M.J. Mughal, Q.A. Naqvi, The wave equation and general plane wave solutions in fractional space, Prog. Electromagn. Res. Lett. 19 (2010) 137–146.
[58] M. Zubair, M.J. Mughal, Q.A. Naqvi, On electromagnetic wave propagation in fractional space, Nonlinear Anal., Real World Appl. 12 (5) (2011) 2844–2850.
[59] M. Zubair, M.J. Mughal, Q.A. Naqvi, An exact solution of the spherical wave equation in D-dimensional fractional space, J. Electromagn. Waves Appl. 25 (10) (2011) 1481–1491.
[60] M. Zubair, M.J. Mughal, Q.A. Naqvi, An exact solution of cylindrical wave equation for electromagnetic field in fractional dimensional space, Prog. Electromagn. Res. 114 (2011) 443–455.
[61] P. Moon, D.E. Spencer, The meaning of the vector Laplacian, J. Franklin Inst. 256 (6) (1953) 551–558.
[62] M. Zubair, M.J. Mughal, Q.A. Naqvi, Electromagnetic Fields and Waves in Fractional Dimensional Space, Springer, Berlin, 2012.
[63] E.Ñ. Aifantis, On the role of gradients in the localization of deformation and fracture, Int. J. Eng. Sci. 30 (10) (1992) 1279–1299.
[64] B.S. Altan, E.C. Aifantis, On the structure of the mode-Ill crack-tip in gradient elasticity, Scr. Metall. Mater. 26 (2) (1992) 319–324.
[65] C.Q. Ru, E.C. Aifantis, A simple approach to solve boundary-value problems in gradient elasticity, Acta Mech. 101 (1) (1993) 59–68.
[66] H. Askes, E.C. Aifantis, Gradient elasticity in statics and dynamics: an overview of formulations, length scale identification procedures, finite element implementations and new results, Int. J. Solids Struct. 48 (13) (2011) 1962–1990.
[67] H. Parkus, Thermoelasticity, second ed., Springer-Verlag, Vien, New York, 1976.
[68] A.D. Kovalenko, Basiñs of Thermoelasticity, Naukova Dumka, Kiev, 1970, in Russian.
[69] J. Ignacza, M. Ostoja-Starzewski, Thermoelasticity with Finite Wave Speeds, Oxford, 2009.
[70] L.D. Landau, E.M. Lifshitz, Theory of Elasticity, Oxford, 1986.
[71] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006, 353 pp.
[72] S.I. Muslih, O.P. Agrawal, Riesz fractional derivatives and fractional dimensional space, Int. J. Theor. Phys. 49 (2) (2010) 270–275.
[73] M. Ostoja-Starzewski, Extremum and variational principles for elastic and inelastic media with fractal geometries, Acta Mech. 205 (1–4) (2009) 161–170.
[74] V.E. Tarasov, Anisotropic fractal media by vector calculus in non-integer dimensional space, J. Math. Phys. 55 (8) (2014) 083510.
[75] V.E. Tarasov, Vector calculus in non-integer dimensional space and its applications to fractal media, Commun. Nonlinear Sci. Numer. Simul. 20 (2) (2015) 360–374.