C. R. Mecanique 343 (2015) 210–218

Contents lists available at ScienceDirect

Comptes Rendus Mecanique

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The transition to turbulence in parallel flows: A personal view

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ARTICLE INFO

Article history:
Received 15 July 2014
Accepted 21 October 2014
Available online 19 December 2014

Keywords:
Fluid dynamics
Turbulence
Bifurcations

ABSTRACT

This is a discussion of the present understanding of transition to turbulence in parallel flows, based upon the idea that it arises from a subcritical instability. The result is a coupled set of equations, one amplitude equation in the direction of translational invariance of the geometry coupled with the standard Reynolds equation for the average transfer of momentum. It helps to understand a basic feature of the transition in parallel flows, namely that turbulence manifests itself in localised domains growing at a constant speed depending on the Reynolds number.

1. Historical introduction

The transition to turbulence is observed in flows when their speed with respect to the walls increases beyond a certain limit. This transition is generally attributed to the fact that, beyond this critical speed, the flow becomes unstable, an idea that can be traced back to a founding paper in fluid mechanics by Osborne Reynolds in 1883 [1]. How important it was, this paper was not written very clearly and its conclusions were somewhat ambiguous. Perhaps this explains why part of its message has been more or less forgotten over the years. To take an example, by reading a review on the transition in parallel flows [2] one sometimes finds the word "instability" in tentative explanations of the occurrence of localised structures, but without any clear definition of what is meant there. To explain that a fluid crosses the boundary of a turbulent spot in both the laminar-to-turbulent and turbulent-to-laminar directions Coles writes that there should be "some kind of strong local instability in the vorticity-bearing ambient flow", a rather wide (and unexplained) extrapolation of what is understood as an instability. This seems to imply that unstable fluctuations are carried by fluid velocity, which is incorrect for the flow considered in that paper (spirals in Taylor–Couette flows) at finite Reynolds numbers and where the advection of vorticity by the fluid is far from perfect, since such an advection exists in the inviscid limit only. Coles seems to imply that turbulence can grow only as the result of an instability, although I argue below that localised turbulent structures grow by a process of contamination and not by an instability, local or not. Therefore it seems pertinent to reconsider first what is meant by instability in the context of fluid mechanics and in parallel flows.

Stability theory is almost as old as Science as we know it and it kept a strong relationship to fluid mechanics from its very beginning. 2300 years ago Archimedes of Syracuse (Sicily) solved the problem of stability of what we would call 2D floating bodies with a parabolic cross section [3]. Using a geometrical method he proved that, if the floating body is a parabolic cylinder of uniform mass density cut horizontally above a certain height, its vertical equilibrium becomes unstable against tilting. Archimedes even found the new equilibrium positions. This was the beginning of studies of stability in fluid mechanics. As it is out of question to review here the whole history of the field, I jump to another very significant development of this idea of stability in fluid mechanics.

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http://dx.doi.org/10.1016/j.crme.2014.10.002

The next step we shall mention is the explanation of how wind excites waves on the surface of the sea. The theory of waves without wind begins in Newton's Principia, where it was shown that the wave speed is proportional to the square root of the wavelength. This result was established (without solving anything like a partial differential equation) by neglecting the air, windy or not, and by neglecting nonlinear effects, this being correct, as pointed by Newton, whenever the slope of the surface is small. This does not explain the obvious relationship between the wind strength and the wave height. A first link between water waves and wind was established by Kelvin and by Helmholtz almost two centuries after the Principia. In separate works they considered the dynamics of small-amplitude fluctuations of the sea surface under the effect of a wind blowing at uniform speed, all this done in the framework of inviscid fluid dynamics. Although the instability of the fluctuations derived in this way is weak, there is presently no good theory explaining how the linear Kelvin–Helmholtz instability is saturated by dissipation phenomena (mostly by wave breaking [4]).

This theory of Kelvin–Helmholtz instability in the linear approximation made the model of many subsequent studies of linear stability, culminating with the thesis by Werner Heisenberg in 1924 under the guidance of Sommerfeld. In this masterpiece of WKB analysis, Heisenberg [5] showed that, with viscosity included, the plane Poiseuille flow is linearly unstable above a critical Reynolds number, although it is always linearly stable without viscosity. In 1966, Iordanskii and Kulikovskii [6] showed that this flow is convectively, not absolutely, unstable. Although Heisenberg explained his paradoxical result (friction is responsible for instability), it met a strong opposition and various (incorrect) proofs of its "erroneous" character were published.

With the advent of artificial flight and motorcar industry, fluid mechanics became an applied science with many challenges to meet. Therefore the understanding of real flows at moderate to large Reynolds numbers became an urgent matter. Reynolds [1] himself set the stage by studying the experimental transition to turbulence in pipe flows. This led him to introduce what is now called the Reynolds number. He tried to show that the transition occurs at a well defined value of this number, a point hard to make in this case, because the transition is subcritical. As reported below, Reynolds, although he was not by far clear in his statement, seemed to make a distinction between subcritical and supercritical bifurcation. Moreover, he was well aware that fluctuations, if of sufficient amplitude, change the mean structure of the flow, introducing so a feedback between this mean flow (if driven by a constant pressure gradient), its Reynolds number, and the turbulent fluctuations. The Reynolds equation relates the mean velocity, the pressure and the Reynolds stress (which can be seen as the contribution of the turbulent fluctuations to the flux of momentum, another name for the stress—this has been rediscovered several times since in various forms.)

Somehow, Reynolds was first to consider the problem of the sub- or supercritical character of the bifurcation to turbulence in parallel flows. After reporting his experiments of transition to turbulence in a pipe he gave a hint that it could not be the result of a *linear* instability. He asked the question (the last one in a list of six): "Did the eddies make their first appearance as small and then increase gradually with the velocity, or did they come suddenly?"

His (unclear) answer was:

"The bearing of the last query may not be obvious; but, as will appear in the sequel, its importance was such that in spite of satisfactory answers to all the other queries, a negative answer to this in respect of one particular class of motion led to the reconsideration of the supposed cause of instability and eventually to the discovery of the instability caused by fluid friction."

2. From Reynolds to Landau: subcritical vs supercritical bifurcation

After Reynolds' work, many experimental studies put in evidence that parallel flows bifurcate to turbulence via a regime, at intermediate Reynolds number, such that turbulence is localised in well-separated domains having received various names. In careful studies Emmons [7] showed that a Blasius boundary layer shows beyond a range of Reynolds number what are called now "Emmons spots" growing with a well-defined arrowhead shape surrounded by laminar flow and turbulent inside (although with recognisable roll structures of axis in the streamwise direction).

This coexistence of laminar and turbulent domains (turbulent flashes in the words of Reynolds) was unexplained at the time of those observations. Such a coexistence can be stationary, in the Taylor–Couette case, for instance. In a paper presented at a Conference at Los Alamos in 1985 [8], I related this coexistence to the *subcritical* character of the bifurcation.

As this notion of a subcritical bifurcation is going to be central for the developments to come, it should be made more precise. A supercritical instability is an instability growing slowly near a threshold and saturating at a finite amplitude tending to zero as the threshold is reached from above, supposing that above the threshold there is a linear instability and none below. That the instability is sub- or supercritical depends on nonlinear effects. Qualitatively, "subcritical" means that the growth of the amplitude of the fluctuations tends to increase even more their rate of growth. On the contrary fluctuations of finite, even small amplitude have a negative effect on the rate of growth in the case of a "supercritical" bifurcation. However this addition of a nonlinear rate of instability to an already linearly unstable fluctuation does not exhaust all possibilities, because there are examples of flows that remain linearly stable for *all* values of the Reynolds number, like plane Couette flow or pipe Poiseuille flow. Nevertheless this kind of flow is subcritically unstable, like flows becoming linearly unstable at a given Reynolds number, this being the case of the plane Poiseuille flow. They can be considered as subcritical because for Reynolds number above a certain threshold, a turbulent state can exist with steady statistical properties, whereas below this threshold, only the laminar state can maintain itself forever. Somehow, in this case, the upper threshold is at infinite Reynolds number, a threshold defined as the Reynolds number where infinitesimally small perturbations allow a transition to the turbulent steady state.

In 1944, Landau [9] did introduce the distinction between sub- and supercritical bifurcation in fluid mechanics, limited to time-dependent perturbations only. My 1985 paper extended this to time- *and* space-dependent perturbations. The paper [8] explained that, in extended structures like parallel flows, subcritical transitions can lead to domains with a different behaviour, turbulent and laminar for example, like two different thermodynamic phases of the same substance, liquid and vapour can coexist at the same temperature and pressure. I developed various ideas connected to that. In a finite range of parameters, the fronts separating two different "phases" are pinned on the underlying lattice if one of the phase is a steady periodic pattern and the other the laminar state, as observed in Bénard–Marangoni thermal convection with hexagons. If the transition is from a laminar to a turbulent state, it belongs to the class of directed percolation at the onset. A thorough review of the present knowledge of the subject, both in experiments and in theory, has been given by Paul Manneville in two recent publications [10].

Returning to the problem of defining as precisely as possible what is meant by a subcritical instability, one could take a heuristic view: from our point of view the most significant property of subcritical bifurcations is that in a range of parameters (the Reynolds number here) there is more than one solution to the fluid equations, usually the laminar one and the turbulent one. By "solution" here one implies steady solutions, like the uniform state and the hexagons in Bénerd–Marangoni convection, or solutions that are time dependent, but with stationary properties on (time) average, like in the coexistence of steady Couette flow and the inside of a turbulent spot, which is stationary only on average. The connection between this property and what is meant usually by stability is rather loose. One thinks to stability with respect to finite-amplitude perturbations, but this is not so well defined. For instance, the Couette flow is linearly stable for any Reynolds number, and—presumably—unstable against finite-amplitude perturbations of vanishingly small amplitude as the Reynolds number tends to infinity. Practically this property is irrelevant because it is experimentally impossible to reach Reynolds numbers of the order of a few thousand without making the Couette flow turbulent if not highly turbulent. A more significant property related to the "practical" onset of turbulence in Couette as well as in others flows with a subcritical transition to turbulence is the onset of growth of turbulent domains, which yields a well-defined criterion for the transition to turbulence, independent of the amplitude of the initial perturbation [10,11].

3. Turbulence and chaos theory

In principle turbulence is described by chaos theory. In its usual meaning, this theory concerns dynamical systems with a phase space of finite dimension, and so it does not help much to describe large-scale turbulent structures involving a phase space of very large dimension. When the oscillations are stable, periodic or quasiperiodic dynamics can synchronise by phase diffusion in large domains. The time needed for oscillations to get spatially in tune grows like the square of the size of the domain becoming synchronised, as verified in Marseilles [12] for the Bénard–von Kármán oscillations in the wake of a cylinder. This idea of synchronisation of interacting oscillating systems goes back to Huygens, who synchronised weakly interacting clocks. However, if the local dynamics is chaotic, no synchronisation does take place spontaneously. A large system, if chaotic in time and without external time dependent forcing, has to be chaotic in space too: the strength of the coupling coming from the large-distance interaction, measured by the inverse time needed to synchronise, decays like $1/L^2$ for L large (L size of the domain, this is the phase diffusion effect discussed by Kuramoto [13]), although the trajectories of a chaotic system are unstable with a constant, size-independent rate of growth. This excludes the synchronisation (same time dependence everywhere) of large chaotic systems. In more technical terms, the Lyapunov divergence of chaotic trajectories always wins over the synchronisation by next-neighbour interaction as L increases.

Before chaos theory, linear stability analysis dominated the subject of theoretical fluid mechanics. From the point of view of dynamical systems, it yields the first term in Landau's amplitude expansion (written below), an example of Poincaré normal form near the instability threshold of self-oscillating systems (Poincaré–Andronov bifurcation). An early attempt in 1944 to rationalise a nonlinear theory of fluid instabilities is due to Landau [9] but it had seemingly little impact in fluid mechanics and Poincaré was ignored. Landau defines clearly the notion of subcritical vs. supercritical instability (the wording we use in the present paper, words introduced years after Landau's paper who used instead "hard" for subcritical and "soft" for supercritical). Landau draws an instability diagram in control parameter/amplitude coordinates. Notice that the idea of subcritical instability is older and can be traced back to Poincaré's PhD thesis (1878) [14], in which he introduced the general notion (and the French word "nœud-col", translated later in reverse order as "saddle-node") for this kind of bifurcation.

Generally speaking, modern (and not so modern!) theory of dynamical systems was largely ignored in theoretical works on fluid mechanics. This lack of interest ended after Ruelle and Takens [15] suggested a (deep) connection between non-trivial results of dynamical systems theory and the transition to turbulence. Ruelle's theory dealt with systems of a few degrees of freedom, not the situation of parallel flows. In a pipe, for instance, one expects the number of degrees of freedom to grow (at large but finite Reynolds number) proportional to the pipe length (a point already made by Landau). Therefore the use of dynamical system theory is not so straightforward because of the a priori very large number of degrees of freedom at large Reynolds numbers.

4. Phase coexistence in thermodynamics and occurrence of turbulent spots in transition flows

Hopefully the previous developments have shown that one has to use ideas beyond chaos theory in its usual meaning to understand the transition to turbulence in parallel flows (and so in systems with many degrees of freedom). I suggested in [8] an analogy between thermodynamic phases and the coexisting laminar and turbulent state. This analogy should not hide deep differences however. At the deepest level (perhaps), one can say that thermodynamics, at least in its modern sense, is a way to understand and describe equilibrium properties of system of many atoms and molecules. Thanks to Boltzmann and Gibbs, we know how to make the connection between the underlying Newtonian dynamics of atoms and molecules and the macroscopic properties of matter via the Liouville invariant measure. Nothing like that is available for turbulence: no formal (or informal!) expression like the Liouville measure exists for the turbulent fluctuations related to the Navier–Stokes equation (equation which is the qualitative equivalent in turbulent flows of Newtonian dynamics in assemblies of many interacting particles). The only known "exact" method for connecting quantitatively the properties of a turbulent flow to the fundamental equations is the direct numerical solution to those equations at large Reynolds numbers, a numerical solution that is beyond our capabilities of today. However we expect that some qualitative features of turbulence can be captured by guesses like the one proposed in [8] for the transition to turbulence in parallel flows.

Compared to usual thermodynamics, the coarse graining involved by taking a turbulent phase as homogeneous requires averaging over length and time scales bigger than the one of the turbulent fluctuations. Even at quite large Reynolds numbers, the dominant scales in the turbulent fluctuations are still macroscopic, like the diameter of a pipe or the thickness of a boundary layer. This leads to significant differences in the transition process compared to thermodynamical phase transitions, where the coarse graining is made on invisibly small scales of space and time. Another difference is the possibility of *stable* Gibbs-like critical nuclei in non-equilibrium systems, as was pointed out first by Boris Malomed [16], although the Gibbs critical nucleus in thermodynamics (a drop of liquid in supercooled vapour for instance) is always unstable, as proved by Gibbs. Lastly, as I point out below, there is a fundamental difference between time-dependent fluctuations in turbulence and the ones in equilibrium thermodynamics, a difference related to the time-reversal symmetry in the sense of Onsager.

To describe the coexistence of domains with a different inner structure, like turbulent and laminar, one has to assume a space-dependent "order parameter" going continuously from a constant value deep in the laminar state to another one deep in the turbulent state. This order parameter could be (for instance) the mean square velocity of fluctuations. Although the transition layer from turbulent to laminar in experiments is of the order of the size of the turbulent structures (or vortices), the derivation of amplitude equations assumes that this transition occurs on a long space scale, likely not a crucial assumption. That it is not crucial follows from a comparison with transitions in equilibrium statistical mechanics: the liquid/vapour interface, outside of the neighbourhood of the critical point, is of order of the size of the molecules. Nevertheless, Maxwell and van der Waals [17] managed to understand its physical properties (particularly when deriving Laplace's pressure jump across a curved interface) by using a phase field model where the transition layer is assumed to be much thicker than molecular scales. This assumption of a thick transition layer helped to connect the theory of subcritical transitions to the amplitude equations with a slow space and time dependence as developed by Segel and by Newell–Whitehead for Rayleigh–Bénard thermal convection, a case of supercritical transition to a steady state. In this case, the amplitude equation is derived rigorously for a Rayleigh number slightly above the instability threshold, a limit with no possible equivalent in parallel flows, where either there is no threshold for linear stability or where there is no secondary parameter, like in Bénard–Marangoni instability, allowing one to derive the amplitude equation in a subcritical situation close to supercriticity. Somehow I assumed that this amplitude equation approach continues to work for subcritical transitions to time-dependent and even chaotic/turbulent states. This introduces a new ingredient in Landau's amplitude equation for fluid instabilities as he restricted himself to instabilities of uniform average amplitude in space, whereas I took into account the possibility of (slowly) space-dependent averages.

In Chapter 3 in Fluid Mechanics [18] Landau considers the instability of a steady flow against time periodic perturbations in the form $v = A(t) f(x, y, z)$ and he gives an equation for the amplitude $A(t) = \mathit{constant} \times \mathrm{e}^{\gamma t}\mathrm{e}^{\mathrm{i}\omega t}$ near the limit of linear stability, namely when γ is small.

The Poincaré normal form (or Landau equation) for the expansion of $\frac{\mathrm{d}|A|^2}{\mathrm{d}t}$ in powers of the amplitude reads:

$$\frac{\mathrm{d}|A|^2}{\mathrm{d}t} = 2\gamma |A|^2 - \alpha |A|^4 \tag{1}$$

It is not obvious to imagine a physical situation in a fluid such that Landau's theory applies without taking into account a possible space dependence of the amplitude. However such an amplitude equation in its original form applies when the extent in space of the unstable fluctuations is bounded (and so well defined near the threshold), like the oscillations in the wake of a sphere slightly above a critical Reynolds number of about 40, where a Poincaré–Andronov bifurcation takes place. The case of extended structures was considered by Segel and by Newell–Whitehead [19] in the same framework of amplitude equation for the Rayleigh–Bénard instability, where one has:

1) a supercritical instability,
2) a bifurcation from a steady state to another steady state like for instance in Euler's Elastica or Archimedes floating parabola. The frequency ω of the oscillations is set to zero in this case.

From Landau's equation (1), one may define what is meant by supercritical and subcritical instability for space-independent solutions.

The case of a supercritical instability is the one where:

1) the coefficient γ of the linear term (with respect to $|A|^2$) on the right-hand side of (1) is a smooth function of the Reynolds number crossing zero for a critical value R_c of this number and so becomes positive but small for R slightly larger than R_c;
2) α, called Landau's coefficient, is positive. In this case, the steady stable solution to (1) is $|A|^2 = \frac{2\gamma}{\alpha}$, which is small for γ small, which justifies that the right-hand side of (1) is limited to terms linear and quadratic with respect to $|A|^2$, because for γ small, the next-order cubic (with respect to $|A|^2$) term is negligible compared to the terms written explicitely.

The case of parallel flows is such that this situation is almost never met: Point one together with point two above are never true for a given parallel flow instability. Particularly, α is always negative. To alleviate that, one could think of carrying an expansion in powers of the amplitudes of all unstable modes. This is, I believe, a hopeless (and useless) task. Landau (section 28 of reference [18]) seems to imply that localised structures in flows above a certain Reynolds number are due to the fact that this flow is convectively unstable. This is debatable: a Poiseuille flow in a circular pipe is linearly stable at *all* Reynolds numbers. Although a plane Poiseuille flow is linearly unstable for Reynolds bigger than 5770, turbulent spots are convected and grow at smaller values of the Reynolds number, somewhere between 1000 and 2000, depending on the experimental conditions. Moreover, it is unclear if a concept borrowed from linear stability theory (convective vs. absolute instability) has any meaning in a strongly nonlinear system, with many interacting modes of finite amplitude. Lastly, there is a major difference between observations and the predictions based on linear theory: in this approach the amplitude of the fluctuations continues to grow with time in the convected region, although in the turbulent spots, the amplitude inside individual spots does not grow, only their *size* grows.

5. The reaction diffusion model with noise

The basic idea for taking into account a possible slow dependence of the amplitude of fluctuations with respect to space is derived from the theory of Segel–Newell–Whitehead. It amounts to add to the Landau amplitude equation a space-diffusion term, a convective term and a multiplicative noise with the result:

$$\frac{\partial E}{\partial t} = -\frac{\partial V}{\partial E} + \mathbf{U} \cdot \nabla E + D_{ij} \frac{\partial E}{\partial x_i \partial x_j} + \zeta(\mathbf{x}, t) E \tag{2}$$

where D_{ij} is a positive definite matrix, $\mathbf{U}$ a vector giving the average velocity of convection of turbulence by the mean flow. The trickiest points in this "general" formulation are the definition of the potential function $V(E)$ and the meaning of E (instead of $|A|^2$ in Landau's theory, but still a function of space and time), considered later. The writing of the term $-\frac{\partial V}{\partial E}$ is inspired by the right-hand side of Landau's equation (1): a somewhat "natural" extension of this equation is to assume this gradient flow structure of the deterministic and local part of the generalised amplitude equation. A possible form of $V(\cdot)$ is

$$\frac{\partial V}{\partial E} = v^2 E^2 (E - E_0)(E - E_1)$$

where the constants v and $E_{0,1}$ depend on the Reynolds number. With such a linear, quadratic and cubic power (with respect to E) in the Taylor expansion of $V(\cdot)$ near $E = 0$, one may have two minima of this potential, one at $E = 0$ representing the linearly stable laminar state (because of the positiveness of the coefficient v^2), the other at a finite value of E represents the turbulent state.

For the Bénard–Marangoni instability near threshold, the equation above without the noise term and without advection may be derived from the basic equations. This instability is known to be weakly subcritical and to be described by three coupled amplitude equations, one for each mode at an angle of $2\pi/3$ of the two others. In that case one finds a saddle-node bifurcation from the state without convection to two non-equivalent states with convection, one—stable—with an upward flow in the centre of the hexagons, another one—unstable—with a downward flow there and lastly the laminar state without hexagons at all, this one coexisting with the stable hexagons only in a finite range of values of the imposed temperature difference. One of the linearly stable states has a lower $V(\cdot)$ than the other and should be, in principle selected—for instance—by the direction of motion of an interface separating the two states (stable hexagons and no hexagons a all). However, as I predicted and as was observed [11], the front separating the two states is pinned on the hexagonal structure in a range of parameters.

Besides situations where this pinning is present, the interface separating the two phases (laminar and turbulent) moves generically at constant speed, a quite general property of solutions to the reaction–diffusion equations such as (2). This constant speed of the front is in good agreement with many observations on parallel flows, done in particular by Wygnanski et al. [20].

6. Remarks on the various terms in equation (2)

In the Landau theory, the amplitude A is well defined; it is the amplitude of the time-periodic solution representing unstable fluctuations of fluid velocity. In the far more complex case we have in mind, this cannot represent the amplitude of regular oscillations, but instead some quantity related to the amplitude of local turbulence. The simplest way to extend a definition of $|A|^2$ to turbulence is to take the mean square value of the velocity fluctuations as the amplitude. However this introduces an unwanted property: in a pipe flow this mean square value depends on the location in the turbulent region on the short scale, like the radial distance. We do not intend to describe this radial dependence, as the amplitude equation aims at describing the "large-scale" structure of the coexisting turbulent and laminar domains. For instance in the classical case of the Rayleigh–Bénard convection near onset, the amplitude equation is for an amplitude depending on the horizontal coordinates, not on the vertical coordinate. In the case of a pipe flow, the order parameter E should be understood as the mean square velocity fluctuations averaged in the radial and azimuthal directions, making it a function of the longitudinal distance only. This question is tightly related to the discussion below on the meaning of the coordinates x_i, x_j, etc., as they appear in Eq. (2).

In Eq. (2), we left undefined what are the coordinates labelled x_i, x_j, etc. In the concrete situations we are thinking about, the transition to turbulence is there because of the friction of the fluid with the walls. For instance in the pipe flow of Reynolds experiments, this friction is with the cylindrical inner surface of the pipe, in plane Poiseuille flow this is with the two parallel plates, etc. In those situations there are two geometrical elements to be accounted for: first there is a length scale, the diameter of the pipe, the gap between the plates, the thickness of the Blasius boundary layer, etc. next there is one or two directions of translational invariance, the pipe length, the spanwise and streamwise directions in plane Poiseuille flow, etc. As the amplitude equation concerns the large-scale features of the turbulence, the intrinsic length scale cannot appear in the amplitude equation which has not the resolution power necessary for this intrinsic scale. Therefore, the coordinates x_i, x_j, etc. which appear in the equation above are the coordinates in the direction of translational invariance (or quasi-invariance in Blasius boundary layers) like along the pipe in circular Poiseuille flow. However, this program fails in one respect, because the thickness of the transition layer is not much bigger than the intrinsic length. It is actually of the same order of magnitude. A similar situation is met in thermodynamics: the vapour–liquid transition layer has a thickness of the order of the size of molecules, except close to the critical point. Nevertheless, it is possible to find the equilibrium shape of an interface between the two phases by using macroscopic concepts like the Laplace equation for the capillary drop of pressure across the interface and by writing the constraint of uniform pressure in each of the two phases. The equivalent program for finding the evolution of localised turbulent structures in parallel flows is to find first the local Reynolds number and the local amplitude E of the turbulent fluctuations by looking at their relation at the minima of the potential $V(\cdot)$. This gives the information needed to determine the local speed of displacement of the interface between the laminar and turbulent domains: this speed is defined by a local solution to Eq. (2). In general, the local minima of $V(\cdot)$ do not satisfy the constraint that $V(\cdot)$ takes there the same value in the turbulent phase and in the laminar one. This difference in the values of $V(\cdot)$ on both sides yields the speed of displacement of the interface. In the laminar and turbulent phase, one has to solve the Reynolds equation for the momentum balance, including Reynolds stress on the turbulent side. This makes sense in the limit where the turbulent spot is far bigger than the intrinsic length scale (radius of the pipe in pipe flows, gap thickness in plane Poiseuille flow, etc.). In this limit one has to solve Darcy-like (for plane Poiseuille flow) equation on both sides with an imposed pressure gradient at infinity and a condition of continuity of the velocity across the laminar/turbulent interface. This yields at the end a well-defined method for finding in a concrete way the time dependence of a turbulent domain. However this program does not work so simply for reasons outlined below.

All the discussion above neglected the noise term in the equation. One can expect that it becomes relevant when the difference of potential between the two "phases" is small and that it is washed out when this difference becomes bigger. I suggested [8] that, because of this sensitivity to noise near threshold, the propagation of the front near the transition between receding and advancing fronts belongs to the class of directed percolation, something that is at least qualitatively true because near this transition there is a tendency for the growing turbulent domain to split spontaneously into disjoint pieces, as it would follow from this scenario of directed percolation. This addition of a noise term to the diffusion equation is implicit in [8]: such a noise is necessary to the scenario of directed percolation discussed there. Chaté and Manneville [21] have studied a model equation where this noise is intrinsic, the equations being deterministic, but with a "steady" stochastic/chaotic part with steady average properties. This deterministic noise plays the same role as a random noise given from outside and, in this model, one observes a transition belonging to the directed percolation universality class, believed to be a generic scenario (independent of the details) with universal exponents [22]. Such a noise term has been also added [23] explicitely in models aimed at describing this transition in deterministic systems without added noise.

Another question concerns the derivation of a reaction–diffusion equation like (2) from the basic fluid equations. This question deserves of course to be considered, but with a grain of salt. We all know, thanks to the effort of generations of scientists, that fluid mechanics is actually described by a well-defined partial differential equation, the Navier–Stokes equation. One should point out however that, even though this equation is well known, it does not help much to compute simple turbulent flows, either in the situation of fully developed turbulence or with the more modest (?) aim to understand how a simple flow like plane Poiseuille flow goes from laminar to turbulent. This implies that one has to introduce in one way or another some statistical ideas to obtain results in this direction. The model for such a statistical theory is, of course, the very successful theory of statistical physics relying on Boltzmann ergodic assumption. This theory avoids the impossible task of solving the equation of motion of 10^{24} particles to understand how a droplet of water evaporate or condense the vapour around. Unfortunately there is no such a thing like the ergodic assumption for nonequilibrium systems like a fluid at Reynolds number above a few thousands. This leaves us with the possibility of using various analogies to understand what is observed in real fluids. I used in [8] and above in the present paper the idea of a reaction-diffusion system with a potential depending on the Reynolds number and on the amplitude of the turbulent fluctuations. This can be at best a metaphor of the real properties of solutions vis-à-vis the Navier–Stokes equation, well known to have no variational structure (and so to be without potential function). After all one needs two things to understand the existence of turbulent spots. First there has to be for the same value of the Reynolds number two possible statistically steady solutions: the laminar one and the turbulent steady one. The existence of a potential function $V(\cdot)$ is not really needed to have this property of multiple solutions (in this sense) at the same Reynolds number. The potential function is necessary only to decide which state (laminar or turbulent) invades the other one if the two states share a common interface. This last point may have an answer without comparing two potential values. One can imagine to solve numerically the time-dependent Navier–Stokes equations with a laminar phase in a half-space and the turbulent state in the other half-space. In principle the numerical study could determine the direction of motion of the interface without having to compare the value of a potential $V(\cdot)$ on both sides. If the two states on each side are infinitely extended, in the long time the front will move at constant speed, unless the Reynolds number is exactly at a critical value.

The last term on the right-hand side of Eq. (2) is the noise term. This makes the equation for the growth (or decay) of turbulent structures different from a standard (= noiseless) reaction-diffusion equation. It is explained by the fact that, in the domain where E is not zero, there is turbulence although there is none in the laminar domain where E is zero. Therefore, in order to represent this turbulent noise, the simplest assumption is to take a noise term proportional to E, as done in the equation above. Moreover, the noise source $\zeta(\mathbf{x}, t)$ can be taken as white Gaussian, likely an idealisation of what is this noise in concrete situations of fluid mechanics. However imperfect is this choice of noise, this is a way to take into account the randomness in space and time of turbulent fluctuations, something needed if one wants to describe real turbulent flows. Said in another way, a purely deterministic picture of turbulence is in principle possible, because—after all—turbulent flows obey the deterministic Navier–Stokes equation, but it is very hard to put in evidence with this equation the randomness existing in real turbulence: as already pointed out, in turbulence there is nothing like an explicit Liouville invariant measure like in classical mechanics.

The introduction of this noise term has non-trivial consequences. Equation (2) without the noise term describes a gradient flow, when it is written as

$$\frac{\partial E}{\partial t} = -\frac{\delta F}{\delta E} \tag{3}$$

where $\frac{\delta}{\delta}$ is a Fréchet derivative, and where the functional F is such that:

$$F(a) = \int \mathrm{d}\mathbf{x}\left(\frac{1}{2} D_{ij}\frac{\partial a}{\partial x_i}\frac{\partial a}{\partial x_j} + V(a)\right)$$

As shown in [24], because of this gradient flow structure, the fluctuations term *without* the multiplicative factor E yields a probabilistic set of solutions to Eq. (2) that are time-reversible, namely such that time correlations satisfy the equality

$$\left\langle f\big(E(t_1)\big) g\big(E(t_2)\big)\right\rangle = \left\langle f\big(E(t_2)\big) g\big(E(t_1)\big)\right\rangle$$

with $f(\cdot)$ and $g(\cdot)$ different smooth functions. Other similar equalities hold for multitime correlations. If those equalities are satisfied, by analysing a solution of Eq. (2) with the noise not multiplied by E, there is no way of deciding what is the direction of time. On the contrary, one expects that the fluctuations due to turbulence do *not* have this property of time reversal symmetry. This is exactly what happens with the prefactor E in front of the noise: because of it the time-reversal symmetry is broken. This remark is significant because the scenario of transition to turbulence by directed percolation (time being one of the directions) is not compatible with time-reversal symmetry. Another consequence of the existence of this noise term and of its dependence on the amplitude E is the observed property (particularly obvious for Emmons spots) that the structure of the spots lacks completely symmetry between their upstream and downstream part: the cross flow across the turbulence/laminar boundary makes the boundary different if the flow goes from turbulent to laminar or the reverse. This is an indirect manifestation of the lack of time-reversal symmetry.

In thermodynamics, the growth of a phase at the expense of another is not ruled by the difference between the local values of the thermodynamic parameters and their value at the steady coexistence. In real thermodynamics, this growth is ruled by the release of conserved quantities (energy and mass) across the transition layer. This can be the (latent) heat release in a solid growing in a melt, or the release of one chemical species when a solid grows in a liquid mixture. So it seems of interest to look at the case of the growth (or decay) or turbulent domains in the light of what happens in thermodynamics. The equations for the growth or decay of a thermodynamic phase in another phase are the Fourier heat equation or the Fick equation for the diffusion of concentration. Both equations are to be solved with a boundary condition on the interface between the two phases, the Stefan condition expressing the conservation of energy or the conservation of mass of the chemical species. The Stefan condition for the energy states that the velocity of the front multiplied by the latent heat is exactly the difference between the heat fluxes across the interface. In the case of a growing or receding interface between two states, laminar and turbulent, a boundary condition exists also. This is the problem considered below.

In the case of expanding (or receding) turbulent domains, the Reynolds number is defined locally by the solution to the equation for the pressure and the average horizontal fluid velocity. This equation averaged on time becomes the Reynolds equation including a part proportional to the square of the velocity fluctuations, the Reynolds stress. This indicates how to define the "order parameter" E: it has to be related to the Reynolds stress, namely proportional to the pair correlations of the turbulent velocity fluctuations. The part of this tensor relevant for the Reynolds equation for the large-scale flow around a turbulent spot is the one with indices of coordinates in the direction normal to the short (vertical) scale. The simplest choice is to take for E the trace of the velocity–velocity correlation in the horizontal direction and averaged on the vertical direction. On the other hand, the potential $V(\cdot)$ is also dependent on the Reynolds number, and so on the average flow velocity in the turbulent domain. Instead of a single reaction–diffusion equation for the amplitude one finds a set of coupled equations for the function E and for the mean flow, including the Reynolds equation for the average pressure and Reynolds stress. If one does not consider the inner structure of the transition region between laminar and turbulent, the only information pertinent for the structure of the turbulent domain is the relationship between the local Reynolds number and the local value of E, which is given in principle, if one knows the potential $V(\cdot)$, by minimising this one at the fixed local Reynolds number. This Reynolds number is determined itself by solving Reynolds equation with the value of the Reynolds stress derived from E. Such a program was carried out for explaining steady spirals observed in the Taylor–Couette experiment [25]. It explained how the spirals stop growing when their widths reach a value which brings to zero the azimuthal velocity of expansion of the spiral width because of the feedback between turbulence and the properties of the mean flow. In this case the geometry is such that the feedback between the mean azimuthal flow and the growing width of the spiral is simple to describe analytically. In the more complex situation of, for instance, Emmons spots growing in a boundary layer, the coupling between the mean flow and the spot is more difficult to describe mathematically.

Let us make a few comments on the large-scale flow produced by a growing turbulent spot in a parallel flow. This makes sense at scales much larger than the intrinsic length of the problem: fundamentally the large-scale flow extends inside the turbulent spots and outside with an extension of the order of magnitude of the size of the turbulent spot. If the flow outside of the turbulent spot is described by Darcy's equation, it cannot show any vortex, as solutions of Darcy's equation are potential velocity fields. However, it could be that the real situation is more complex because the size of the turbulent spot is not much bigger than the small length scale of the problem, namely the gap between the two plates in a Poiseuille or Couette flow. In this respect it could be of interest to look at the "large-scale" flow outside of an Emmons spot, because the horizontal (in-plane) size of a Blasius boundary layer can be much bigger than its vertical (normal to the plate) thickness.

Another significant difference between the coexistence of two phases in thermodynamics and in parallel flows is that, in thermodynamics, the localised steady solution of a phase in the other (the Gibbs nucleus) is always unstable, although one observes in parallel flows stable localised structures non expanding and filled with turbulent fluctuations. This sharp difference can be explained in two different ways (not necessarily incompatible): first in non-variational systems [16] with a subcritical bifurcation one may have a stable nucleus of one phase within the other. Next, in flows, there is a possible stabilisation of the critical nucleus by the large scale feedback flow as was observed and predicted for Taylor–Couette spirals [25]. In this respect it would be of interest to know if such localised turbulent structures have actually (as one can believe) a finite number of degrees of freedom by staying localised in space: the existing models [16] of stable localised structures imply that this structure has time periodic oscillations, not a chaotic dynamics.

7. Conclusion

To conclude this exposition of various points related to the transition to turbulence in parallel flows, one can only hope that a rational theory based on the idea of subcritical transition and of the growth or decay by contamination of turbulent domains may account quantitatively for observations going back, for some, to Reynolds. It would be also of interest to look at the possible analytical representation of this transition by using the available models of turbulence based on various modellings of the turbulent transfer relying on assumed relations between the Reynolds stress and the average properties of the flow. In this respect it would be very important to have multiple steady solutions for given parallel flows in such turbulent models. The next step would be to look at the coexistence of different steady solutions in a given geometry, like downstream and upstream in a pipe.

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