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A Mori–Tanaka homogenization scheme for non-linear elasto-viscoplastic heterogeneous materials based on translated fields: An affine extension

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ABSTRACT

A Mori–Tanaka homogenization scheme based on the "translated fields" approach is extended to elasto-viscoplastic composites with non-linear viscoplasticity described by a first-order "affine"-type linearization. This extension leads to a new theoretical interaction law between mechanical average phase fields and overall ones. This interaction law contains the coupling between elastic- and viscoplastic- mechanical interactions and phase stress histories. In order to study and discuss the validity of the present approach, the results are reported for two-phase composites and are compared to other approaches.

1. Introduction

A large class of heterogeneous media (e.g., metals, polymers, geomaterials) exhibit time-dependent behaviors that, from the homogenization point of view, require the use of linear viscoelastic or of non-linear elasto-viscoplastic frameworks. In general, the coupling between viscoplastic and elastic deformations renders micromechanical homogenization particularly complex. For instance, it was shown that the overall behavior of aggregates with Maxwellian constituents is no longer Maxwellian as it contains a "long-memory effect" [1]. The origin of this long-memory effect is due to the differential nature of constitutive equations that involve different orders of time-derivation mechanical fields. In the past decades, several approaches were proposed to solve such a mechanical problem.

Stieltjes integral-based "hereditary approaches"—using Laplace–Carson transforms and the correspondence principle—were first developed in the case of the non-ageing viscoelastic inclusion problem [2] and further introduced in different homogenization schemes like those of Mori–Tanaka [3,4], Hashin–Shtrickman [5], or self-consistent [6,7] estimates. The "affine" formulation developed by [8] was developed for non-linear polycrystals in the framework of the self-consistent procedure. Comparisons between an "affine" formulation with the Mori–Tanaka scheme and finite elements calculations were performed in [9].

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A more efficient homogenization strategy is based on incremental internal variables approaches (IIVA) whereby the stress history is recorded through the time evolution of viscoplastic strains of the individual constituents. These approaches can also be applied to ageing materials. Several different types of IIVA have been proposed [10–18].

The "additive" interaction law for Maxwellian local behavior derived by [10,12] was applied to linear and non-linear viscoelastic materials using the "tangent" formulation for non-linear viscoplasticity. An IIVA was developed in the work of Lahellec and Suquet [15,16,19]—using two potentials (free energy density and dissipation potential)—with an implicit time discretization scheme based on a variational method. While linear viscoelastic composites were considered in [15], non-linear extensions of viscoplastic strain rates based on a "modified secant" method were adopted in [16] and applied to two-phase non-linear composites. A modification of the "second-order" procedure [20] was then developed in [19] to improve the results of the "modified secant" linearization scheme for the same two-phase composites. Fast Fourier Transform (FFT) results were also reported in [15,16,19] to obtain "exact" reference solutions. Recently, another variational approach was developed in [21] for non-linear two-phase elasto-viscoplastic composites using a "secant" linearization scheme and a Mori–Tanaka approximation.

A computationally appealing IIVA inspired from the collocation method was developed by Ricaud and Masson [17]. Interestingly, exact analytical results were provided in [17] in the case of two-phase elastically homogeneous linear viscoelastic composites. An extension of the previous method to the case of linear viscoelastic polycrystals with a self-consistent procedure was given in [18]. A self-consistent IIVA for two-phase isotropic viscoelastic composites was also proposed by [22] and successfully compared to the exact results of [7]. Recently, an IIVA based on a "sequential" linearization technique was developed [23] for linear and non-linear viscoelastic composites. This technique is based on the sequential resolution of purely elastic and purely viscoplastic interaction laws. Different "variants" of the technique were proposed for sequential linearization and the "additive" interaction law [12] was retrieved as a specific "variant" [23].

The last type of IIVA is based on the "translated fields" method. It was derived for linear and non-linear viscoelasticity with a "secant" linearization of the viscoplastic strain rates [11,13,14]. Starting from the integral equation of the heterogeneous elasto-viscoplastic problem, Paquin et al. [11] first kept the symmetry between elasticity and viscosity by introducing "translated fields" for both elastic and viscous parts. Thus, a very rich but complex interaction law was obtained for this first way. A simplified "translated fields" approach was then proposed in [13], in which the classic elastic Navier operator is still present, while a "translated fields" decomposition is used regarding the viscoplastic strain rates. Thus, simplifying the concentration rule, good predictions were found in comparison with the exact "1-site" self-consistent solution of [7]. However, it was reported in [24,25] that solely the "translated fields" approach developed in [11] is able to capture the exact solution to the Eshelby problem obtained by Hashin [2] when both constituents have Maxwellian linear incompressible isotropic behavior and inclusions have a spherical shape. Therefore, the first "translated fields" approach [11] was recently extended to treat linear viscoelastic composites with the Mori–Tanaka scheme [25].

To analyze the suitability of different IIVA methods, comparisons between both the "translated fields" and "additive" approaches were shown [25] in the case of two-phase linear compressible or incompressible viscoelastic composites. Even though both approaches generally give different interaction laws for the viscoelastic Eshelby problem, similar responses for monotonic as well as cyclic loadings were found when considering high mechanical phase contrasts.

This paper introduces a first-order "affine" extension of the "translated fields" method. Section 2 introduces the incremental field equations with local elasto-viscoplastic behavior in order to derive the integral equation. The integral equation is simplified using a "translated fields" decomposition in Section 3. Then, a new stress rate interaction law from the solution to the "translated fields" decomposition applied to the non-linear elasto-viscoplastic Eshelby problem is presented. In Section 4, the Mori–Tanaka scheme is applied to non-linear two-phase composites using the strain-rate concentration equations obtained from the Eshelby problem. It is first shown that in the case of linear compressible viscoelastic inclusion-reinforced composites, the phase and overall stress responses are in good agreement with the exact solutions presented in [17]. Then, the mechanical responses of non-linear elasto-viscoplastic fiber-reinforced or weakened composites are discussed in the light of the present formulation and in comparison with the variational and FFT approaches developed in [16,19].

Notations for a given quantity X are the following: $\underline{X}$ is a vector, $\underset{\sim}{X}$ is a second-order tensor, $\underset{\approx}{X}$ is a fourth-order tensor.

2. Field equations and integral equation

2.1. Field equations and affine approximation

Considering a heterogeneous medium V with Maxwellian constituents, local linear elastic moduli $\underset{\approx}{c}$ (elastic compliances $\underset{\approx}{s} = \underset{\approx}{c}^{-1}$) and viscoplastic strains $\dot{\underset{\sim}{\varepsilon}}^{\mathrm{vp}}$, the problem is treated in the framework of quasi-static equilibrium with infinitesimal strains and no volume forces. The field equations of the elasto-viscoplastic heterogeneous problem are given by:

$$\dot{\underset{\sim}{\varepsilon}} = \dot{\underset{\sim}{\varepsilon}}^{\mathrm{e}} + \underset{\sim}{g}(\underset{\sim}{\sigma})$$
$$\underline{\mathrm{div}}\,\dot{\underset{\sim}{\sigma}} = 0$$
$$\underline{\mathrm{div}}\,\underset{\sim}{\sigma} = 0$$
$$\dot{\underset{\sim}{\varepsilon}} = \nabla^{\mathrm{s}}\dot{\underline{u}}$$

$$\dot{\underline{u}}^{\mathrm{d}} = \dot{\underset{\sim}{E}} \cdot \underline{x} \tag{1}$$

where $\dot{\underset{\sim}{\varepsilon}}$ is the (linearized) total strain rate, $\underset{\sim}{\sigma}$ is the Cauchy stress, $\dot{\underset{\sim}{\varepsilon}}^{\mathrm{e}} = \underset{\approx}{s} : \dot{\underset{\sim}{\sigma}}$ (resp. $\dot{\underset{\sim}{\varepsilon}}^{\mathrm{vp}} = \underset{\sim}{g}(\underset{\sim}{\sigma})$) is the elastic (resp. viscoplastic) strain rate. The balance equation for unknown stress rate $\dot{\underset{\sim}{\sigma}}$ is incrementally solved in the time step $\mathrm{d}t$ (i.e. between t and $t + \mathrm{d}t$). At time t, the balance equation for the known stress $\underset{\sim}{\sigma}$ is verified. In the kinematic compatibility equation related to linearized total strain rates $\dot{\underset{\sim}{\varepsilon}}$, ∇^{s} denotes the symmetric part of the gradient. Here, a homogeneous velocity vector $\dot{\underline{u}}^{\mathrm{d}}$ is imposed on the boundary ∂V where $\dot{\underset{\sim}{E}}$ is prescribed. The solution consists in determining the strain and stress rates satisfying these field equations.

In this time incremental internal variables approach, the viscoplastic strains in the individual constituents are internal variables that depend on their stress history. The local viscoplastic strain rate $\dot{\underset{\sim}{\varepsilon}}^{\mathrm{vp}} = \underset{\sim}{g}(\underset{\sim}{\sigma})$ in Eq. (1a) at a given reference deformation state (denoted by superscript $^{(\mathrm{ref})}$ can be approximated through first order Taylor expansion, also called "affine" linearization [26], in the vicinity of $\dot{\underset{\sim}{\varepsilon}}^{\mathrm{vp(ref)}}$

$$\dot{\underset{\sim}{\varepsilon}}^{\mathrm{vp}} = \underset{\approx}{m}^{\mathrm{t}} : \underset{\sim}{\sigma} + \underset{\sim}{\eta} \tag{2}$$

where $\underset{\approx}{m}^{\mathrm{t}}$ is the viscoplastic tangent compliance for a reference stress state $\underset{\sim}{\sigma}^{(\mathrm{ref})}$ and $\underset{\sim}{\eta}$ is a "back-extrapolated" strain. Both are defined as

$$\underset{\approx}{m}^{\mathrm{t}} = \left.\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{\sigma}}\right|_{\underset{\sim}{\sigma}^{(\mathrm{ref})}}$$
$$\underset{\sim}{\eta} = \underset{\sim}{g}\big(\underset{\sim}{\sigma}^{(\mathrm{ref})}\big) - \underset{\approx}{m}^{\mathrm{t}} : \underset{\sim}{\sigma}^{(\mathrm{ref})} \tag{3}$$

The viscoplastic tangent compliance $\underset{\approx}{m}^{\mathrm{t}}$ and modulus $\underset{\approx}{b}^{\mathrm{t}} = (\underset{\approx}{m}^{\mathrm{t}})^{-1}$ have both minor and major symmetries. The back-extrapolated stresses $\underset{\sim}{\tau}$ are defined as $\underset{\sim}{\tau} = -\underset{\approx}{b}^{\mathrm{t}} : \underset{\sim}{\eta}$. From Eq. (2)

$$\underset{\sim}{\sigma} = \underset{\approx}{b}^{\mathrm{t}} : \dot{\underset{\sim}{\varepsilon}}^{\mathrm{vp}} + \underset{\sim}{\tau} \tag{4}$$

2.2. Integral equation

At a given (reference) deformation state $^{(\mathrm{ref})}$ where an "affine" linearization step is applied, an infinite homogeneous reference medium with elastic and tangent viscoplastic compliances $\underset{\approx}{s}^{0}$ and $\underset{\approx}{m}^{t0}$ and with a homogeneous reference "back-extrapolated" strain $\underset{\sim}{\eta}^{0}$ is introduced. The corresponding heterogeneous elastic and tangent viscoplastic compliances $\underset{\approx}{s}$, $\underset{\approx}{m}^{\mathrm{t}}$ and "back-extrapolated" strain $\underset{\sim}{\eta}$ can be written in terms of spatial fluctuations $\delta\underset{\approx}{s}$, $\delta\underset{\approx}{b}^{\mathrm{t}}$ and $\delta\underset{\sim}{\eta}$ with respect to the infinite homogeneous reference medium as follows

$$\underset{\approx}{s} = \underset{\approx}{s}^{0} + \delta\underset{\approx}{s}$$
$$\underset{\approx}{m}^{\mathrm{t}} = \underset{\approx}{m}^{\mathrm{t0}} + \delta\underset{\approx}{m}^{\mathrm{t}}$$
$$\underset{\sim}{\eta} = \underset{\sim}{\eta}^{0} + \delta\underset{\sim}{\eta} \tag{5}$$

with $\underset{\approx}{s}^{0} = (\underset{\approx}{c}^{0})^{-1}$ and $\underset{\approx}{m}^{\mathrm{t0}} = (\underset{\approx}{b}^{\mathrm{t0}})^{-1}$. An elegant way to retrieve easily the integral equation is to use the "projection operator" algebra [11] on the statically admissible fields $\underset{\sim}{\sigma}$ and $\dot{\underset{\sim}{\sigma}}$ and on the compatible field $\dot{\underset{\sim}{\varepsilon}}$. The projection operators can be defined with reference to infinite homogeneous elastic and tangent viscoplastic moduli $\underset{\approx}{c}^{0}$ and $\underset{\approx}{b}^{t0}$ by the following relations

$$\underset{\approx}{\Pi}^{c^{0}}\big(\underline{x} - \underline{x}'\big) = \underset{\approx}{\Gamma}^{c^{0}}\big(\underline{x} - \underline{x}'\big) : \underset{\approx}{c}^{0}$$
$$\underset{\approx}{\Pi}^{b^{\mathrm{t0}}}\big(\underline{x} - \underline{x}'\big) = \underset{\approx}{\Gamma}^{b^{\mathrm{t0}}}\big(\underline{x} - \underline{x}'\big) : \underset{\approx}{b}^{\mathrm{t0}} \tag{6}$$

where $\underline{x}$ and $\underline{x}'$ are vector positions.

Here, $\underset{\approx}{\Gamma}^{c^{0}}$ and $\underset{\approx}{\Gamma}^{b^{\mathrm{t0}}}$ are the modified Green tensors respectively associated with $\underset{\approx}{c}^{0}$ and $\underset{\approx}{b}^{t0}$ and defined in index notations by

$$\Gamma^{*}_{ijkl} = -\frac{1}{2}\big(G^{*}_{ik,jl} + G^{*}_{jk,il}\big) \tag{7}$$

where $\underset{\sim}{G}^{*}$ are the Green tensors and $*$ corresponds to either $\underset{\approx}{c}^{0}$ or $\underset{\approx}{b}^{\mathrm{t0}}$. Furthermore, $\underset{\approx}{\Gamma}^{c^{0} \text{ or } b^{\mathrm{t0}}}$ (resp. $\underset{\approx}{\Pi}^{c^{0} \text{ or } b^{\mathrm{t0}}}$) split into a local part $\underset{\approx}{\Gamma}_{l}^{c^{0} \text{ or } b^{\mathrm{t0}}}$ (resp. $\underset{\approx}{\Pi}_{l}^{c^{0} \text{ or } b^{\mathrm{t0}}}$) and a non-local part $\underset{\approx}{\Gamma}_{nl}^{c^{0} \text{ ou } b^{\mathrm{t0}}}$ (resp. $\underset{\approx}{\Pi}_{nl}^{c^{0} \text{ ou } b^{\mathrm{t0}}}$). The non local part of the modified Green tensor (resp. the projection tensor) scales with $\frac{1}{|\underline{x}-\underline{x}'|^{3}}$. Balanced fields in Eqs. (1b) and (1c) yield

$$\underline{\text{div}}\dot{\underset{\sim}{\sigma}} = 0 \quad \Leftrightarrow \quad \underset{\approx}{\Pi}^{c^0} \star \underset{\approx}{s}^0 : \dot{\underset{\sim}{\sigma}} = 0$$
$$\underline{\text{div}}\underset{\sim}{\sigma} = 0 \quad \Leftrightarrow \quad \underset{\approx}{\Pi}^{b^{t0}} \star \underset{\approx}{m}^{t0} : \underset{\sim}{\sigma} = 0 \tag{8}$$

where $\star$ denotes spatial convolution.

Furthermore, Eqs. (1d) and (1e) yield:

$$\underset{\approx}{\Pi}^{c^0} \star \dot{\underset{\sim}{\varepsilon}} = \dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{E}} \tag{9}$$

These two properties (Eqs. (8) and (9)) can be demonstrated in the Fourier space [11]. Eqs. (8) and (9) are used to rewrite the set of field equations including the fluctuations (Eq. (5)) as follows:

$$\dot{\underset{\sim}{\varepsilon}} - \delta\underset{\approx}{s} : \dot{\underset{\sim}{\sigma}} - \delta\underset{\approx}{m}^{t} : \underset{\sim}{\sigma} - \delta\underset{\sim}{\eta} = \underset{\approx}{s}^0 : \dot{\underset{\sim}{\sigma}} + \underset{\approx}{m}^{t0} : \underset{\sim}{\sigma} + \underset{\sim}{\eta}^0$$
$$\underset{\approx}{\Pi}^{c^0} \star \underset{\approx}{s}^0 : \dot{\underset{\sim}{\sigma}} = 0$$
$$\underset{\approx}{\Pi}^{b^{t0}} \star \underset{\approx}{m}^{t0} : \underset{\sim}{\sigma} = 0$$
$$\underset{\approx}{\Pi}^{b^{t0}} \star \underset{\sim}{\eta}^0 = 0$$
$$\underset{\approx}{\Pi}^{c^0} \star \dot{\underset{\sim}{\varepsilon}} = \dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{E}} \tag{10}$$

Then, by applying $\underset{\approx}{\Pi}^{c^0} + \underset{\approx}{\Pi}^{b^{t0}}$ to the first equation in the set of Eqs. (10), the following integral equation is obtained:

$$\dot{\underset{\sim}{\varepsilon}} = \dot{\underset{\sim}{E}} + \underset{\approx}{\Pi}^{c^0} \star \delta\underset{\approx}{s} : \dot{\underset{\sim}{\sigma}} + \underset{\approx}{\Pi}^{b^{t0}} \star \left(\delta\underset{\approx}{m}^{t} : \underset{\sim}{\sigma} + \delta\underset{\sim}{\eta}\right) + \left(\underset{\approx}{\Pi}^{c^0} - \underset{\approx}{\Pi}^{b^{t0}}\right) \star \left(\underset{\approx}{m}^{t} : \underset{\sim}{\sigma} + \underset{\sim}{\eta}\right) \tag{11}$$

This integral equation is defined at a given reference deformation state where the viscoplastic strain is linearized using the "affine" formulation. The aim of the following section is to transform the integral equation (11) by introducing additional fluctuations on strains fields through the so-called "translated fields" approach.

3. Translated fields decomposition

3.1. General framework

Here, the "translated fields" decomposition is introduced in the case of heterogeneous non-linear elasto-viscoplastic materials. Here, only the viscoplastic strain rates are non-linear (elastic properties are linear) and a first order "affine" approximation is used (Eq. (2)). The "translated field" decomposition is applied to the total strain-rate field:

$$\dot{\underset{\sim}{\varepsilon}} = \dot{\underset{\sim}{e}} + \dot{\tilde{\underset{\sim}{\varepsilon}}} = \dot{\underset{\sim}{e}}^{e} + \dot{\underset{\sim}{e}}^{vp} + \dot{\tilde{\underset{\sim}{\varepsilon}}}^{e} + \dot{\tilde{\underset{\sim}{\varepsilon}}}^{vp} \tag{12}$$

where $\dot{\underset{\sim}{e}}^{e}$, $\dot{\underset{\sim}{e}}^{vp}$ are pure linear elastic (resp. pure non-linear viscoplastic) compatible fields subjected to the same stress state as the elasto-viscoplastic heterogeneous material. Furthermore, $\dot{\tilde{\underset{\sim}{\varepsilon}}}^{e}$ and $\dot{\tilde{\underset{\sim}{\varepsilon}}}^{vp}$ are considered as "residual" linear elastic (resp. non-linear viscoplastic) strain-rate fields. The compatible fields $\dot{\underset{\sim}{e}}^{e}$ and $\dot{\underset{\sim}{e}}^{vp}$ can be written as follows

$$\dot{\underset{\sim}{e}}^{e} = \underset{\approx}{A}^{c^0} : \dot{\underset{\sim}{X}}$$
$$\dot{\underset{\sim}{e}}^{vp} = \underset{\approx}{A}^{b^{t0}} : \dot{\underset{\sim}{Y}} + \underset{\sim}{a} \tag{13}$$

In Eq. (13), $\dot{\underset{\sim}{X}}$ and $\dot{\underset{\sim}{Y}}$ are unknown uniform strain-rate tensors that are specified in this section, $\underset{\approx}{A}^{c^0}$ is a an elastic strain-rate concentration tensor for the purely linear elastic problem, and, $\underset{\approx}{A}^{b^{t0}}$ as well as $\underset{\sim}{a}$ are strain-rate concentration tensors associated with the purely non-linear viscoplastic problem. Using the "affine" formulation, the latter reduces to a stepwise linear thermoelastic problem [26]. The concentration tensor $\underset{\approx}{A}^{c^0}$ (resp. $\underset{\approx}{A}^{b^{t0}}$) depends on a homogeneous reference medium $\underset{\approx}{c}^0$ (resp. $\underset{\approx}{b}^{t0}$). Together with $\underset{\sim}{a}$, these tensors will depend on the adopted homogenization scheme. However, average theorems for both specific purely linear elastic and purely non-linear viscoplastic problems yield the following averaging rules for these tensors

$$\overline{\underset{\approx}{A}^{c^0}} = \underset{\approx}{I}$$
$$\overline{\underset{\approx}{A}^{b^{t0}}} = \underset{\approx}{I}$$
$$\bar{\underset{\sim}{a}} = 0 \tag{14}$$

where $\underset{\approx}{I}$ is the fourth-order unit tensor defined as $I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. The translated fields decomposition can also be written in terms of stress-rate decomposition for linear elasticity and stress decomposition for non-linear viscoplasticity as follows:

$$\dot{\underset{\sim}{\sigma}} = \underset{\approx}{c} : \dot{\underset{\sim}{\varepsilon}}^{\mathrm{e}} = \underset{\approx}{c} : \dot{\underset{\sim}{e}}^{\mathrm{e}} + \dot{\tilde{\underset{\sim}{\sigma}}}'$$
$$\underset{\sim}{\sigma} = \underset{\approx}{b}^{\mathrm{t}} : \dot{\underset{\sim}{\varepsilon}}^{\mathrm{vp}} + \underset{\sim}{\tau} = \underset{\approx}{b}^{\mathrm{t}} : \dot{\underset{\sim}{e}}^{\mathrm{vp}} + \underset{\sim}{\tau} + \tilde{\underset{\sim}{\sigma}}'' \tag{15}$$

where $\dot{\tilde{\underset{\sim}{\sigma}}}'$ and $\tilde{\underset{\sim}{\sigma}}''$ are respectively residual stress rates and residual stress associated with $\dot{\tilde{\underset{\sim}{\varepsilon}}}^{\mathrm{e}}$ and $\dot{\tilde{\underset{\sim}{\varepsilon}}}^{\mathrm{vp}}$ with $\dot{\tilde{\underset{\sim}{\sigma}}}' \neq \dot{\tilde{\underset{\sim}{\sigma}}}''$. After eliminating the "back-extrapolated" stress $\underset{\sim}{\tau}$ in the second equation of Eq. (15), the explicit expressions of $\dot{\tilde{\underset{\sim}{\sigma}}}'$ and $\tilde{\underset{\sim}{\sigma}}''$ are easily derived:

$$\dot{\tilde{\underset{\sim}{\sigma}}}' = \underset{\approx}{c} : \dot{\tilde{\underset{\sim}{\varepsilon}}}^{\mathrm{e}}$$
$$\tilde{\underset{\sim}{\sigma}}'' = \underset{\approx}{b}^{\mathrm{t}} : \dot{\tilde{\underset{\sim}{\varepsilon}}}^{\mathrm{vp}} \tag{16}$$

The volume averages of Eq. (1a) and Eq. (2) yield:

$$\overline{\dot{\underset{\sim}{\varepsilon}}} = \overline{\dot{\underset{\sim}{\varepsilon}}^{\mathrm{e}}} + \overline{\dot{\underset{\sim}{\varepsilon}}^{\mathrm{vp}}}$$
$$\overline{\dot{\underset{\sim}{\varepsilon}}^{\mathrm{e}}} = \overline{\underset{\approx}{s} : \dot{\underset{\sim}{\sigma}}}$$
$$\overline{\dot{\underset{\sim}{\varepsilon}}^{\mathrm{vp}}} = \overline{\underset{\approx}{m}^{\mathrm{t}} : \underset{\sim}{\sigma} + \underset{\sim}{\eta}} \tag{17}$$

Hence, Eq. (17) together with Eq. (13) and Eq. (14) yield:

$$\dot{\underset{\sim}{X}} = \overline{\underset{\approx}{s} : \dot{\underset{\sim}{\sigma}}}$$
$$\dot{\underset{\sim}{Y}} = \overline{\underset{\approx}{m}^{\mathrm{t}} : \underset{\sim}{\sigma} + \underset{\sim}{\eta}}$$
$$\overline{\dot{\tilde{\underset{\sim}{\varepsilon}}}^{\mathrm{e}}} = 0; \qquad \overline{\dot{\tilde{\underset{\sim}{\varepsilon}}}^{\mathrm{vp}}} = 0; \qquad \overline{\dot{\tilde{\underset{\sim}{\varepsilon}}}} = 0 \tag{18}$$

Because of the compatibility of $\dot{\underset{\sim}{e}}$, the total translated strain-rate field $\dot{\tilde{\underset{\sim}{\varepsilon}}}$ is also compatible from Eq. (12), then it verifies Eq. (9). Furthermore, from Eq. (15) the residual stress fields $\dot{\tilde{\underset{\sim}{\sigma}}}'$ and $\tilde{\underset{\sim}{\sigma}}''$ are balanced. Thus, both verify Eq. (8) and in the same way as Eqs. (10) and (11), the following integral equation of the translated problem is obtained:

$$\dot{\tilde{\underset{\sim}{\varepsilon}}} = \overline{\dot{\tilde{\underset{\sim}{\varepsilon}}}} + \underset{\approx}{\Pi}^{c^0} \star \delta\underset{\approx}{s} : \dot{\tilde{\underset{\sim}{\sigma}}}' + \underset{\approx}{\Pi}^{b^{\mathrm{t}0}} \star \delta\underset{\approx}{m}^{\mathrm{t}} : \tilde{\underset{\sim}{\sigma}}'' + \left(\underset{\approx}{\Pi}^{c^0} - \underset{\approx}{\Pi}^{b^{\mathrm{t}0}}\right) \star \underset{\approx}{m}^{\mathrm{t}} : \tilde{\underset{\sim}{\sigma}}'' \tag{19}$$

The strain-rate concentration equation for the total strain rate $\dot{\underset{\sim}{\varepsilon}}$ is obtained using Eq. (12).

In the following section, this translated problem is solved for the particular Eshelby ellipsoidal inclusion problem whereby a new stress interaction law is derived.

3.2. New interaction law for non-linear elasto-viscoplastic Eshelby ellipsoidal inclusion problem

Volume V is now constituted of an ellipsoidal inclusion denoted V_I embedded in an infinite matrix denoted 0 subjected to $\dot{\underset{\sim}{\varepsilon}}^0$ at remote boundaries. The inclusion and the matrix follow a Maxwellian behavior with uniform respective elastic $\underset{\approx}{c}^I$, $\underset{\approx}{c}^0$ and tangent viscoplastic $\underset{\approx}{b}^{\mathrm{t}I}$, $\underset{\approx}{b}^{\mathrm{t}0}$ moduli (and conversely uniform respective elastic $\underset{\approx}{s}^I$, $\underset{\approx}{s}^0$ and tangent viscoplastic $\underset{\approx}{m}^{\mathrm{t}I}$, $\underset{\approx}{m}^{\mathrm{t}0}$ compliances), uniform back-extrapolated strains $\underset{\sim}{\eta}^I$, $\underset{\sim}{\eta}^0$ and stresses $\underset{\sim}{\tau}^I$, $\underset{\sim}{\tau}^0$. Thus, the spatial fluctuations of elastic and non-linear viscoplastic properties are given by $\delta\underset{\approx}{c}^I = \underset{\approx}{c}^I - \underset{\approx}{c}^0$, $\delta\underset{\approx}{s}^I = \underset{\approx}{s}^I - \underset{\approx}{s}^0$, $\delta\underset{\approx}{b}^{\mathrm{t}I} = \underset{\approx}{b}^{\mathrm{t}I} - \underset{\approx}{b}^{\mathrm{t}0}$, $\delta\underset{\approx}{m}^{\mathrm{t}I} = \underset{\approx}{m}^{\mathrm{t}I} - \underset{\approx}{m}^{\mathrm{t}0}$, $\delta\underset{\sim}{\eta}^I = \underset{\sim}{\eta}^I - \underset{\sim}{\eta}^0$, $\delta\underset{\sim}{\tau}^I = \underset{\sim}{\tau}^I - \underset{\sim}{\tau}^0$.

Following the translated fields decomposition introduced in Section 3, the non-linear viscoplastic strain rate in the inclusion $\dot{\underset{\sim}{\varepsilon}}^{\mathrm{vp}I}$ is "translated" with respect to an unknown compatible viscoplastic field $\dot{\underset{\sim}{e}}^{\mathrm{vp}I}$ as follows:

$$\dot{\underset{\sim}{\varepsilon}}^{\mathrm{vp}I} = \dot{\underset{\sim}{e}}^{\mathrm{vp}I} + \dot{\tilde{\underset{\sim}{\varepsilon}}}^{\mathrm{vp}I} \tag{20}$$

where $\dot{\tilde{\underset{\sim}{\varepsilon}}}^{\mathrm{vp}I}$ is the viscoplastic translated strain rate. The compatible viscoplastic field $\dot{\underset{\sim}{e}}^{\mathrm{vp}I}$ is also non linear and is chosen as the solution to an auxiliary non-linear viscoplastic Eshelby ellipsoidal inclusion problem. Using the "affine" linearization, this problem reduces to a linear thermoelastic-like problem at each deformation stage for which the problem was solved earlier by [26]. Here, the linear comparison thermoelastic-like formula are applied to the two-phase inclusion/matrix system to derive the expression of $\dot{\underset{\sim}{e}}^{\mathrm{vp}I}$ as follows:

$$\dot{\underset{\sim}{e}}^{\mathrm{vp}I} = \underset{\approx}{A}^{b^{\mathrm{t}0^I}} : \dot{\underset{\sim}{\varepsilon}}^{\mathrm{vp}0} + \underset{\sim}{a}^I \tag{21}$$

where

$$\underset{\sim}{a}^I = \left[\underset{\approx}{A}^{b^{\mathrm{t}0^I}} - \underset{\approx}{I}\right] : \left(\delta\underset{\approx}{b}^{\mathrm{t}I}\right)^{-1} : \delta\underset{\sim}{\tau}^I \tag{22}$$

with

$$\underset{\approx}{A}^{b^{t0^I}} = \left(\underset{\approx}{I} + \underset{\approx}{\Gamma}^{b^{t0^I}} : \delta\underset{\approx}{b}^{tI}\right)^{-1} \tag{23}$$

and $\underset{\approx}{\Gamma}^{b^{t0^I}}$ is defined for $\underline{x} \in V_I$ by

$$\underset{\approx}{\Gamma}^{b^{t0^I}} = \int_{V_I} \underset{\approx}{\Gamma}^{b^{t0}}\left(\underline{x} - \underline{x}'\right)\mathrm{d}V' \tag{24}$$

Introducing an unknown compatible elastic field $\dot{\underset{\sim}{e}}^{eI}$, the linear elastic strain rate in the inclusion $\dot{\underset{\sim}{\varepsilon}}^{eI}$ is decomposed as follows:

$$\dot{\underset{\sim}{\varepsilon}}^{eI} = \dot{\underset{\sim}{e}}^{eI} + \dot{\tilde{\underset{\sim}{\varepsilon}}}^{eI} = \underset{\approx}{A}^{c^{0^I}} : \dot{\underset{\sim}{\varepsilon}}^{e0} + \dot{\tilde{\underset{\sim}{\varepsilon}}}^{eI} \tag{25}$$

where $\dot{\tilde{\underset{\sim}{\varepsilon}}}^{eI}$ is the elastic translated strain rate and

$$\underset{\approx}{A}^{c^{0^I}} = \left(\underset{\approx}{I} + \underset{\approx}{\Gamma}^{c^{0^I}} : \delta\underset{\approx}{c}^{I}\right)^{-1} \tag{26}$$

with $\underset{\approx}{\Gamma}^{c^{0^I}}$ defined for $\underline{x} \in V_I$ by

$$\underset{\approx}{\Gamma}^{c^{0^I}} = \int_{V_I} \underset{\approx}{\Gamma}^{c^{0}}\left(\underline{x} - \underline{x}'\right)\mathrm{d}V' \tag{27}$$

In Eqs. (20) and (25), $\dot{\tilde{\underset{\sim}{\varepsilon}}}^{vpI}$ and $\dot{\tilde{\underset{\sim}{\varepsilon}}}^{eI}$ can be viewed as the "residual" viscoplastic and elastic strain rates such that the total strain rates in the inclusion read

$$\dot{\underset{\sim}{\varepsilon}}^{I} = \underset{\approx}{A}^{c^{0^I}} : \dot{\underset{\sim}{\varepsilon}}^{e0} + \underset{\approx}{A}^{b^{t0^I}} : \dot{\underset{\sim}{\varepsilon}}^{vp0} + \underset{\sim}{a}^{I} + \dot{\tilde{\underset{\sim}{\varepsilon}}}^{eI} + \dot{\tilde{\underset{\sim}{\varepsilon}}}^{vpI} \tag{28}$$

The total translated strain-rate field $\dot{\tilde{\underset{\sim}{\varepsilon}}}^{I} = \dot{\tilde{\underset{\sim}{\varepsilon}}}^{eI} + \dot{\tilde{\underset{\sim}{\varepsilon}}}^{vpI}$ is solved from Eq. (19) by simplifying the integral equation to the Eshelby ellipsoidal inclusion problem as follows:

$$\dot{\tilde{\underset{\sim}{\varepsilon}}}^{I} = \dot{\tilde{\underset{\sim}{\varepsilon}}}^{0} + \underset{\approx}{\Pi}^{c^{0^I}} : \delta\underset{\approx}{s}^{I} : \dot{\tilde{\underset{\sim}{\sigma}}}'^{I} + \underset{\approx}{\Pi}^{b^{t0^I}} : \delta\underset{\approx}{m}^{tI} : \tilde{\underset{\sim}{\sigma}}''^{I} + \left(\underset{\approx}{\Pi}^{c^{0^I}} - \underset{\approx}{\Pi}^{b^{t0^I}}\right) : \underset{\approx}{m}^{t} : \tilde{\underset{\sim}{\sigma}}''^{I} \tag{29}$$

where $\underset{\approx}{\Pi}^{c^{0^I}} = \underset{\approx}{\Gamma}^{c^{0^I}} : \underset{\approx}{c}^{0}$ and $\underset{\approx}{\Pi}^{b^{t0^I}} = \underset{\approx}{\Gamma}^{b^{t0^I}} : \underset{\approx}{b}^{t0}$.

In Eq. (29), $\tilde{\underset{\sim}{\sigma}}'^{I}$ and $\dot{\tilde{\underset{\sim}{\sigma}}}'^{I}$ are derived from Eqs. (20), (21) and (25):

$$\begin{aligned}\dot{\tilde{\underset{\sim}{\sigma}}}'^{I} &= \underset{\approx}{c}^{I} : \left(\dot{\underset{\sim}{\varepsilon}}^{eI} - \underset{\approx}{A}^{c^{0^I}} : \dot{\underset{\sim}{\varepsilon}}^{e0}\right)\\ \tilde{\underset{\sim}{\sigma}}''^{I} &= \underset{\approx}{b}^{tI} : \left(\dot{\underset{\sim}{\varepsilon}}^{vpI} - \underset{\approx}{A}^{b^{t0^I}} : \dot{\underset{\sim}{\varepsilon}}^{vp0} - \underset{\sim}{a}^{I}\right)\end{aligned} \tag{30}$$

As a consequence of Eqs. (20) to (26) (see also Eq. (18)), the translated strain field at remote boundaries $\dot{\tilde{\underset{\sim}{\varepsilon}}}^{0}$ is zero. After straightforward algebraic manipulations, Eq. (29) simplifies to

$$\dot{\tilde{\underset{\sim}{\varepsilon}}}^{I} = \underset{\approx}{A}^{c^{0^I}} : \underset{\approx}{\Gamma}^{c^{0^I}} : \delta\underset{\approx}{c}^{I} : \underset{\approx}{m}^{t} : \tilde{\underset{\sim}{\sigma}}''^{I} - \underset{\approx}{A}^{c^{0^I}} : \underset{\approx}{\Gamma}^{b^{t0^I}} : \delta\underset{\approx}{b}^{tI} : \underset{\approx}{m}^{t} : \tilde{\underset{\sim}{\sigma}}''^{I} + \underset{\approx}{A}^{c^{0^I}} : \left(\underset{\approx}{\Pi}^{c^{0^I}} - \underset{\approx}{\Pi}^{b^{t0^I}}\right) : \underset{\approx}{m}^{t} : \tilde{\underset{\sim}{\sigma}}''^{I} \tag{31}$$

From Eq. (31), the strain-rate concentration rule is obtained by going back to the total strain-rate field of the elasto-viscoplastic Eshelby inclusion problem using Eq. (28). Using the expression of $\tilde{\underset{\sim}{\sigma}}''^{I}$ defined in Eq. (30), and after further simplifications using Eqs. (23) and (26), Eq. (31) leads to the following strain-rate concentration equation:

$$\begin{aligned}\dot{\underset{\sim}{\varepsilon}}^{I} = {}& \underset{\approx}{A}^{c^{0^I}} : \dot{\underset{\sim}{\varepsilon}}^{0} + \dot{\underset{\sim}{\varepsilon}}^{vpI} - \underset{\approx}{A}^{c^{0^I}} : \left(\underset{\approx}{A}^{b^{t0^I}}\right)^{-1} : \left(\dot{\underset{\sim}{\varepsilon}}^{vpI} - \underset{\sim}{a}^{I}\right)\\ &+ \underset{\approx}{A}^{c^{0^I}} : \left(\underset{\approx}{\Gamma}^{c^{0^I}} : \underset{\approx}{c}^{0} - \underset{\approx}{\Gamma}^{b^{t0^I}} : \underset{\approx}{b}^{t0}\right) : \left(\dot{\underset{\sim}{\varepsilon}}^{vpI} - \underset{\approx}{A}^{b^{t0^I}} : \dot{\underset{\sim}{\varepsilon}}^{vp0} - \underset{\sim}{a}^{I}\right)\end{aligned} \tag{32}$$

The new stress rate interaction law for the elasto-viscoplastic Eshelby inclusion V_I embedded in an infinite matrix 0 is determined from Eq. (32) and from the elastic Hooke's law:

$$\begin{aligned}\dot{\underset{\sim}{\sigma}}^{I} - \dot{\underset{\sim}{\sigma}}^{0} = {}& \left(\underset{\approx}{c}^{0} - \left(\underset{\approx}{\Gamma}^{c^{0^I}}\right)^{-1}\right) : \left(\dot{\underset{\sim}{\varepsilon}}^{I} - \dot{\underset{\sim}{\varepsilon}}^{0}\right) - \underset{\approx}{c}^{0} : \left(\dot{\underset{\sim}{\varepsilon}}^{vpI} - \dot{\underset{\sim}{\varepsilon}}^{vp0}\right)\\ &- \left(\underset{\approx}{\Gamma}^{c^{0^I}}\right)^{-1} : \underset{\approx}{\Gamma}^{b^{t0^I}} : \delta\underset{\approx}{b}^{tI} : \dot{\underset{\sim}{\varepsilon}}^{vpI} + \left(\underset{\approx}{\Gamma}^{c^{0^I}}\right)^{-1} : \left(\underset{\approx}{A}^{b^{t0^I}}\right)^{-1} : \underset{\sim}{a}^{I}\\ &+ \left(\underset{\approx}{c}^{0} - \left(\underset{\approx}{\Gamma}^{c^{0^I}}\right)^{-1} : \underset{\approx}{\Gamma}^{b^{t0^I}} : \underset{\approx}{b}^{t0}\right) : \left(\dot{\underset{\sim}{\varepsilon}}^{vpI} - \underset{\approx}{A}^{b^{t0^I}} : \dot{\underset{\sim}{\varepsilon}}^{vp0} - \underset{\sim}{a}^{I}\right)\end{aligned} \tag{33}$$

The first term contains the classic elastic Hill's constraint tensor, which solves the purely elastic interaction problem. The other terms represent the complex elastic/viscoplastic interactions between inclusion and matrix at the origin of the long-memory effect. The present interaction law is different from the "additive law" with non-linear viscoplasticity [10,12] regarding the additional terms that contains coupled elastic/non-linear viscoplastic interactions. In comparison with the linear version of the translated field approach derived in Eq. (17) of [25], the present non-linear interaction law contains the term $\underset{\sim}{a}^I$ and the tangent viscoplastic moduli $\underset{\approx}{b}^{tI}$, $\underset{\approx}{b}^{t0}$, which result from the local piecewise uniform "affine" approximation. Some specific solutions can be derived from the new interaction law. First of all, linear viscoelasticity can be retrieved easily considering the linear viscoplastic moduli instead of tangent viscoplastic moduli and by setting $\underset{\sim}{a}^I$ to zero in Eq. (33). In the particular case of isotropic incompressible linear viscoelastic matrix and inclusion, it was highlighted in a recent contributions [25] using Laplace–Carson transforms that the interaction law given by Eq. (33) provides the same result as the exact solution. For compressible linear viscoelasticity, successful comparisons with the "additive" interaction law were also recently provided in [25].

For small physical times, i.e. when viscoplastic strain rates are negligible: $\dot{\underset{\sim}{\varepsilon}}^I \to \dot{\underset{\sim}{\varepsilon}}^{eI}$, $\dot{\underset{\sim}{\varepsilon}}^0 \to \dot{\underset{\sim}{\varepsilon}}^{e0}$, Eq. (32) reduces to the linear elastic stress interaction rule:

$$\dot{\underset{\sim}{\sigma}}^I - \dot{\underset{\sim}{\sigma}}^0 = \left(\underset{\approx}{c}^0 - \left(\underset{\approx}{\Gamma}^{c^{0I}}\right)^{-1}\right) : \left(\dot{\underset{\sim}{\varepsilon}}^{eI} - \dot{\underset{\sim}{\varepsilon}}^{e0}\right) \tag{34}$$

At large physical times, Eq. (33) can be simplified by considering the steady state given by $\dot{\underset{\sim}{\sigma}}^I = \dot{\underset{\sim}{\sigma}}^0 \to \underset{\sim}{0}$, $\dot{\underset{\sim}{\varepsilon}}^I \to \dot{\underset{\sim}{\varepsilon}}^{vpI}$, $\dot{\underset{\sim}{\varepsilon}}^0 \to \dot{\underset{\sim}{\varepsilon}}^{vp0}$ and $\dot{\underset{\sim}{\varepsilon}}^{vpI} \to \underset{\approx}{A}^{b^{t0I}} : \dot{\underset{\sim}{\varepsilon}}^{vp0} + \underset{\sim}{a}^I$ (see Eqs. (20) and (21)). After simplifications, the asymptotic non-linear viscoplastic interaction law reads

$$\underset{\sim}{\sigma}^I - \underset{\sim}{\sigma}^0 = \left(\underset{\approx}{b}^{t0} - \left(\underset{\approx}{\Gamma}^{b^{t0I}}\right)^{-1}\right) : \left(\dot{\underset{\sim}{\varepsilon}}^{vpI} - \dot{\underset{\sim}{\varepsilon}}^{vp0}\right) \tag{35}$$

which is the same as the one derived in [27] for the "tangent" formulation of the non-linear Eshelby inclusion problem.

4. Application to a homogenization Mori–Tanaka scheme for two-phase non-linear elasto-viscoplastic composites

4.1. Explicit concentration equations for two-phase composite materials

In the case of two-phase composite materials, the Representative Volume Element (or "RVE") is constituted of one inclusion phase I with volume fraction f embedded in a matrix phase M with volume fraction $(1-f)$. The RVE is subjected at its boundary ∂V to the homogeneous macroscopic strain rate denoted $\dot{\underset{\sim}{E}}$ satisfying

$$f\dot{\underset{\sim}{\varepsilon}}^I + (1-f)\dot{\underset{\sim}{\varepsilon}}^M = \dot{\underset{\sim}{E}} \tag{36}$$

The Mori–Tanaka procedure [28] is well suited to estimate the effective properties of two-phase composites with an isotropic distribution of phases provided the volume fraction f of inclusions is not large (generally not larger than 0.2). Thus, the strain-rate concentration equations for the Mori–Tanaka estimate can be directly found from Eq. (32) together with Eq. (36). Thus, the following strain-rate concentration equation is obtained for the matrix phase M as follows

$$\begin{aligned}\dot{\underset{\sim}{\varepsilon}}^M = {} & \underset{\approx}{T}^{c^{MI}} : \dot{\underset{\sim}{E}} - f\underset{\approx}{T}^{c^{MI}} : \dot{\underset{\sim}{\varepsilon}}^{vpI} + f\underset{\approx}{T}^{c^{MI}} : \underset{\approx}{A}^{c^{MI}} : \left(\underset{\approx}{A}^{b^{tMI}}\right)^{-1} : \left(\dot{\underset{\sim}{\varepsilon}}^{vpI} - \underset{\sim}{a}^I\right) \\ & - f\underset{\approx}{T}^{c^{MI}} : \underset{\approx}{A}^{c^{MI}} : \left(\underset{\approx}{\Gamma}^{c^{MI}} : \underset{\approx}{c}^M - \underset{\approx}{\Gamma}^{b^{tMI}} : \underset{\approx}{b}^{tM}\right) : \left(\dot{\underset{\sim}{\varepsilon}}^{vpI} - \underset{\approx}{A}^{b^{tMI}} : \dot{\underset{\sim}{\varepsilon}}^{vpM} - \underset{\sim}{a}^I\right)\end{aligned} \tag{37}$$

where $\underset{\approx}{T}^{c^{MI}}$ writes:

$$\underset{\approx}{T}^{c^{MI}} = \left((1-f)\underset{\approx}{I} + f\underset{\approx}{A}^{c^{MI}}\right)^{-1} \tag{38}$$

For the inclusion phase I, the strain-rate concentration equation follows from Eq. (36) and Eq. (32) with $0 = M$.

4.2. Results and discussion

Results are reported for two-phase linear and non-linear composites. The "translated fields" approach is denoted "TF" and the Mori–Tanaka scheme is denoted "MT". In the case of non-linear elastoviscoplasticity, the "affine" linearization is denoted "AFF". For comparisons, a "secant" linearization is denoted "SEC".

4.2.1. Linear viscoelasticity

In the case of linear Maxwellian compressible or incompressible two-phase composites, exact solutions were recently obtained for the Mori–Tanaka approximation by Ricaud and Masson [17]. In order to compare the present "MT + TF" approach to their exact solutions, their internal variables formulation for non-ageing isotropic linear two-phase composites are time-integrated during a tension–compression loading path for different viscous moduli contrast between matrix and

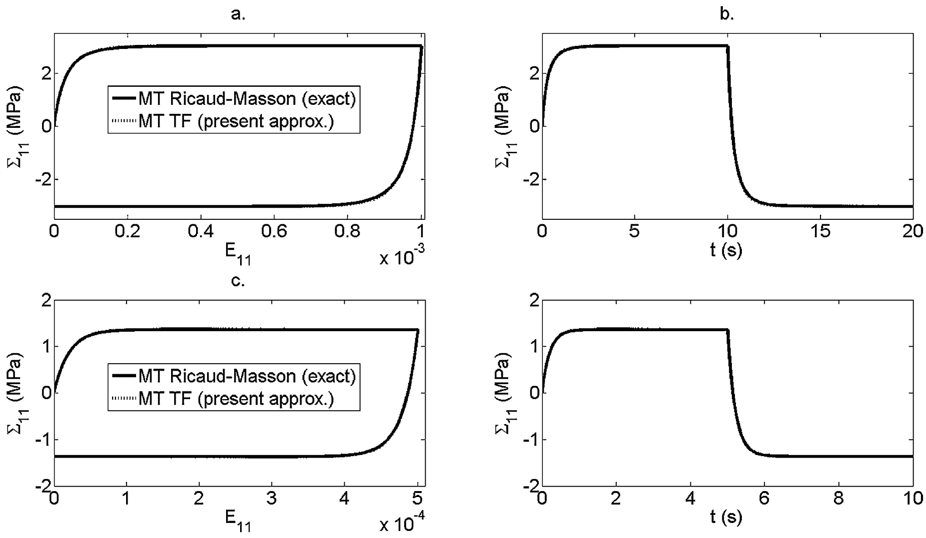

Fig. 1. Overall tension-compression uniaxial stress Σ_{11} responses predicted by the linear "MT + TF" approach as functions of the applied macroscopic strain E_{11} (a and c) and of physical time t (b and d), and comparisons with the exact solutions [17]. Different viscous moduli constrasts $k_I{}^{\mathrm{v}}/k_M{}^{\mathrm{v}} = \mu_I{}^{\mathrm{v}}/\mu_M{}^{\mathrm{v}}$ are considered between the inclusion and the matrix phases: 100 (a, b) and 1/100 (c, d).

inclusions (see Eqs. (10), (18) and Table 1 in [17]). For comparisons, the linear viscoelastic properties of both phases are isotropic and compressible. Eq. (1a) reduces to

$$\dot{\underset{\sim}{\varepsilon}} = \underset{\approx}{s} : \dot{\underset{\sim}{\sigma}} + \underset{\approx}{m} : \underset{\sim}{\sigma} \tag{39}$$

where

$$\begin{aligned} \underset{\approx}{s} &= \frac{1}{3k^{\mathrm{e}}}\underset{\approx}{J} + \frac{1}{2\mu^{\mathrm{e}}}\underset{\approx}{K} \\ \underset{\approx}{m} &= \frac{1}{3k^{\mathrm{v}}}\underset{\approx}{J} + \frac{1}{2\mu^{\mathrm{v}}}\underset{\approx}{K} \end{aligned} \tag{40}$$

where k^{e}, k^{v} are the elastic and viscous bulk moduli, μ^{e}, μ^{v} are the elastic and viscous shear moduli. $\underset{\approx}{J}$ and $\underset{\approx}{K}$ are respectively defined by $J_{ijkl} = \frac{1}{3}\delta_{ij}\delta_{kl}$ and $K_{ijkl} = I_{ijkl} - J_{ijkl}$. To be consistent with [17], both phases have same elastic properties. In the case of a linear incompressible viscoelastic behavior, the exact results of Hashin [2] are retrieved with the "MT + TF" approach, see [25]. In the case of linear compressible viscoelasticity, some illustrations are given for $k_I{}^{\mathrm{e}} = k_M{}^{\mathrm{e}} = 38\,890$ MPa, $\mu_I{}^{\mathrm{e}} = \mu_M{}^{\mathrm{e}} = 50\,000$ MPa (i.e. the elastic Poisson ratio is 0.05). Different inelastic moduli with a constant viscous Poisson ratio of 0.3 are considered for the matrix and inclusion phases and two different mechanical contrasts are chosen for the following matrix viscous properties: $k_M{}^{\mathrm{v}} = 21\,666.66$ MPa s and $\mu_M{}^{\mathrm{v}} = 10\,000$ MPa s. A first contrast of 100 is chosen with $k_I{}^{\mathrm{v}} = 2\,166\,666$ MPa s, $\mu_I{}^{\mathrm{v}} = 1\,000\,000$ MPa s (i.e. hard inclusions), and, a second one of 1/100 with $k_I{}^{\mathrm{v}} = 216.6666$ MPa s, $\mu_I{}^{\mathrm{v}} = 100$ MPa s (i.e. soft inclusions). The inclusions are supposed spherical and their volume fraction f is set to 0.2. A uniaxial tension–compression test is simulated with the following prescribed (macroscopic) strain rate

$$\dot{\underset{\sim}{E}} = \dot{E}_{11}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -0.5 \end{pmatrix}$$

with $\dot{E}_{11} = 10^{-4}$ s^{-1} for tension and $\dot{E}_{11} = -10^{-4}$ s^{-1} for compression.

Fig. 1 shows the overall stress cyclic response up to $t = 20$ s for hard inclusions and up to $t = 10$ s for soft inclusions. For a given contrast, Fig. 1a,c and Fig. 1b,d respectively show the evolutions of the overall uniaxial stress Σ_{11} as a function of imposed strain E_{11} and physical time t. For both contrasts, it is observed that the responses between the present "MT + TF" approach and the exact solution of [17] are almost superimposed, even in the transient regime between tension and compression loadings.

In order to study the differences between both approaches, the evolutions of phase average stresses given by the "MT + TF" approach are reported in Fig. 2 and compared to the exact approach of [17]. For a constrast of 100 (hard inclusions), there is a little difference for the inclusion phase stress response with the exact solution of [17] (Fig. 2a). A slight difference is also observed for the contrast of 1/100 (see Fig. 2b) regarding the matrix stress response. These illustrations show the accuracy of the "MT + TF" approach in its linear form.

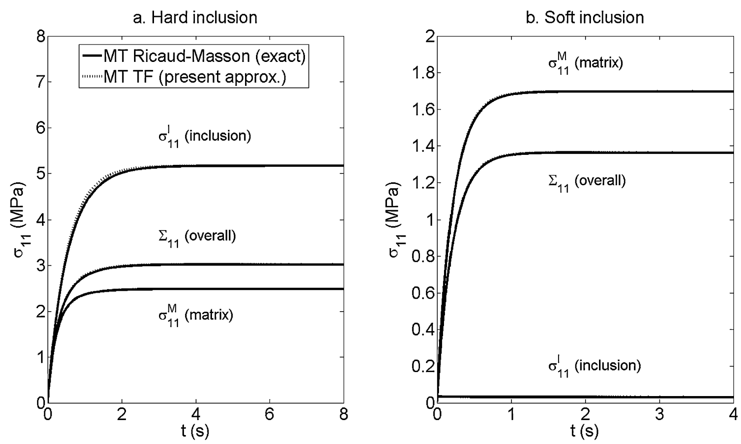

Fig. 2. Overall uniaxial stress Σ_{11} and phase average stress σ_{11} responses as functions of time t predicted by the linear "MT+TF" approach and comparisons with the exact solutions [17]. Different viscous moduli constrasts $k_I{}^{v}/k_M{}^{v} = \mu_I{}^{v}/\mu_M{}^{v}$ are considered between the inclusion and the matrix phases: 100 (a) and 1/100 (b).

4.2.2. Non-linear elastoviscoplasticity

To study the effect of non-linear viscosity on the overall composite responses, the present Mori–Tanaka approach in its non-linear form (see Eq. (37)) is labeled as "MT + TF (AFF)". Here, the "MT + TF (AFF)" approach is applied to two-phase composites ($r = I, M$) constituted of a matrix phase ($r = M$) and long cylindrical fibers ($r = I$). All the fibers are aligned in the same direction ($x3$) and are perpendicular to the ($x1, x2$) plane. These microstructures were extensively studied in [16,19] using a time incremental variational method with a "Composite Cylinder Assemblage" (CCA) microstructure and Hashin–Shtrikman (HS) estimates. This approximation leads to a lower bound of the effective behavior for the composite for hard fibers and an upper bound for the case of soft fibers. The FFT method was also used in [16,19] to obtain "exact" reference solutions for these fiber-reinforced or weakened composites with respective fiber volume fractions of $f = 0.21$ and $f = 0.41$. Here, both phases are elasto-viscoplastic with non-linear viscoplasticity given by a power law (Norton's model):

$$\dot{\underset{\sim}{\varepsilon}} = \underset{\approx}{s} : \dot{\underset{\sim}{\sigma}} + \frac{3}{2}\dot{\varepsilon}_0 \left(\frac{\sigma_{\mathrm{eq}}}{\sigma_0^{(r)}}\right)^{1/m} \frac{\underset{\sim}{s}}{\sigma_{\mathrm{eq}}} \tag{41}$$

where m is the strain-rate sensitivity, which is the same for both phases, $\underset{\sim}{s}$ is the deviatoric stress, σ_{eq} is the equivalent Von Mises stress, $\dot{\varepsilon}_0$ is a reference strain rate and $\sigma_0^{(r)}$ is the flow stress for each phase (r). The materials' parameters used for the simulations are the same as in [16,19]. For the matrix, $\sigma_0^{(M)} = 1$ GPa and the inclusions are "hard fibers" when $\sigma_0^{(I)} = 5\sigma_0^{(M)}$ or "soft fibers" when $\sigma_0^{(I)} = 0.2\sigma_0^{(M)}$. The elastic properties are given for both phases by elastic Young's modulus $E = 100$ GPa and Poisson ratio $\nu = 0.45$. The other viscoplastic common parameters are $\dot{\varepsilon}_0 = 1\ \mathrm{s}^{-1}$ and $m = 0.2$. The linear case ($m = 1$) is also reproduced and compared with the FFT exact result. Inclusions are axisymmetric ellipsoidal inclusions elongated in the principal fiber direction with aspect ratio $c/a = 1/100$ (a is the semi-axis length of the fibers along the ($x3$) direction and c is their radius). The applied strain rate $\dot{\underset{\sim}{E}}$ to the RVE is isochoric and corresponds to in-plane shear loading at 45° with respect to axis ($x1$):

$$\dot{\underset{\sim}{E}} = \dot{E}_{11} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $\dot{E}_{11} = 0.5\ \mathrm{s}^{-1}$. The following results display (i) the evolution of overall stress Σ_{11} as a function of the macroscopic applied strain E_{11}, (ii) the evolutions of mean phase stresses $\sigma_{11}^{(r=I,M)}$ (i.e. the first moment of stresses) as functions of mean phase strains $\varepsilon_{11}^{(r=1,2)}$ (i.e. the first moment of strains).

Regarding "hard fibers", the results of the "MT + TF (AFF)" approach are displayed in Fig. 3. For $f = 0.21$, the predicted responses from the "MT + TF (AFF)" approach method are first compared to the results given by the "SEC (EIV + HS)" and the FFT approaches reported in [16] in Fig. 3a for $m = 1$ and in Fig. 3b for $m = 0.2$. Based on a variational approach, the "SEC (EIV + HS)" approach consists in an incremental effective potential approach where the internal variables are the effective viscoplastic strain rates (i.e. "EIV" for "Effective Internal Variable") using a "modified secant" approach to derive the phase viscoplastic moduli. The effective properties of the thermoelastic-like homogeneous equivalent medium are obtained by HS estimates for a CCA microstructure. For $m = 1$ (linear case), Fig. 3a shows that the overall response given by the full-field FFT technique is reproduced by the "MT + TF (AFF)" approach. For $m = 0.2$ (non-linear case), some differences

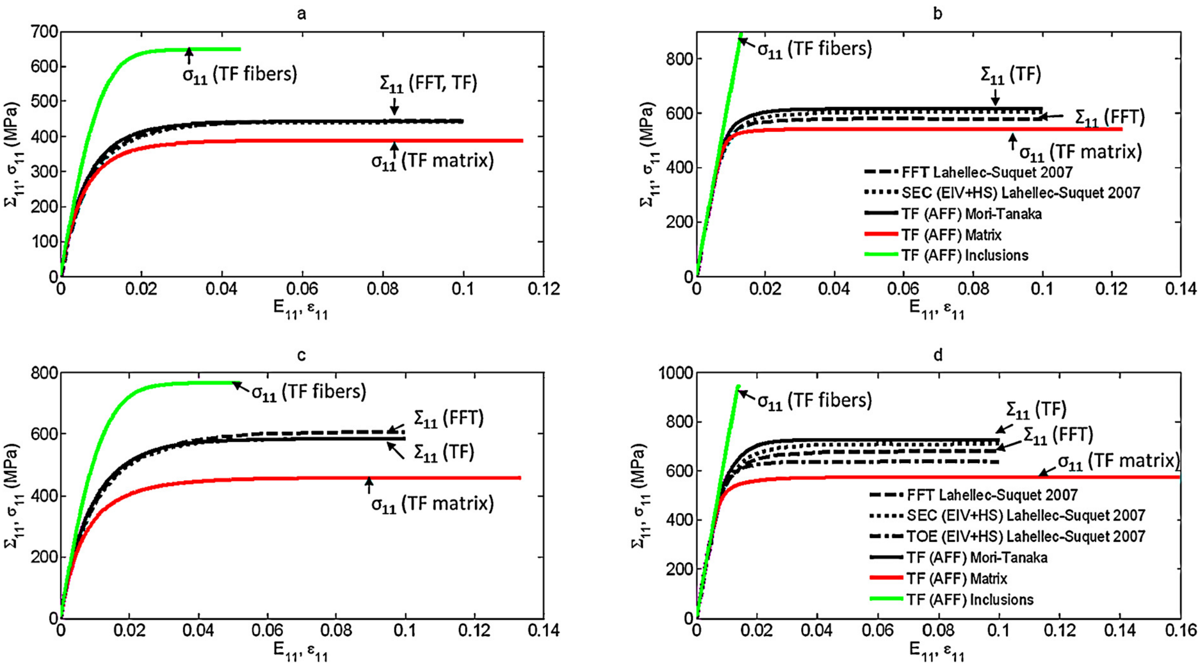

Fig. 3. (Color online.) Overall and phase average stress monotonic responses of two-phase non-linear elasto-viscoplastic fiber-reinforced composites (case referred to in the text as "hard fibers") subjected to in-plane shear loading perpendicular to the fibers for different fiber volume fractions and different non-linearities: (a) $f = 0.21$ and $m = 1$, (b) $f = 0.21$ and $m = 0.2$, (c) $f = 0.41$ and $m = 1$, (d) $f = 0.41$ and $m = 0.2$.

occur in Fig. 3b with the FFT response, especially at large physical times when the overall behavior tends to the viscoplastic asymptotic state. The transient regime predicted with the present interaction law is seen to be shorter than for $m = 1$. This trend is also observed with the Lahellec–Suquet "SEC (EIV + HS)" or FFT approaches. The estimate given by the "MT + TF (AFF)" approach is very close to the "SEC (EIV + HS)" approach, while the phase behaviors show large different mechanical responses compared to the linear case (see Fig. 3b). For $f = 0.41$, Fig. 3 compares the results of the present approach with the "SEC (EIV + HS)", "TOE (EIV + HS)" and FFT approaches [19] in Fig. 3c for $m = 1$ and Fig. 3d for $m = 0.2$. The Lahellec and Suquet "TOE (EIV + HS)" approach is a complex modified second-order procedure based on a third-order Taylor expansion of the phase viscoplastic potential. The present results for $m = 0.2$ still remain close to the "SEC (EIV + HS)" developed in [16] (Fig. 3d). For $f = 0.41$, the Mori–Tanaka estimate (or the HS bound [16]) becomes less efficient for this higher volume fraction due to the influence of spatial interactions between fibers.

Regarding "soft fibers", the results of the "MT + TF (AFF)" approach reported in Fig. 4 show that for $m = 1$, the FFT results are well reproduced (see Figs. 4a and 4c). There is still a slight difference on the asymptotic value obtained at largest strains in Fig. 4c due to the Mori–Tanaka method, which is less adapted for $f = 0.41$ than for $f = 0.21$ (same prediction as the HS bound used in [16]). For $m = 0.2$ (Figs. 4b and 4d) the "MT + TF (AFF)" approach leads to an overall stress close to the results of the "SEC (EIV + HS)". It is also observed that phase behaviors predicted with the "MT + TF (AFF)" approach show large different mechanical responses compared to the linear case. However, the discrepancy regarding the asymptotic overall stress with the FFT results are larger than for "hard fibers". This was also observed with the "SEC (EIV + HS)" even though a "modified secant" scheme was used. The "modified second-order" Lahellec and Suquet "TOE (EIV + HS)" approach [19] gives the closest predictions with respect to the exact FFT results for "soft fibers" (see Figs. 4b and 4d).

It is interesting to discuss the results in comparisons with the "secant" extension adopted in [11] as functions of the strain-rate sensitivity parameter m. This initial linearization scheme adapted to the Mori–Tanaka method is denoted "MT + TF (SEC)". In Fig. 5, the asymptotic overall stresses Σ_{11}^{∞} normalized with the asymptotic stresses obtained in the matrix alone (i.e. the homogeneous matrix phase without fiber) $\Sigma_{11}^{M\infty}$ are reported as functions of m for "hard fibers" with $f = 0.21$ in Fig. 5a, with $f = 0.41$ in Fig. 5b, and, for "soft fibers" with $f = 0.21$ in Fig. 5c, with $f = 0.41$ in Fig. 5d. The estimate of $\Sigma_{11}^{\infty}/\Sigma_{11}^{M\infty}$ shows that the present "MT + TF (AFF)" (solid lines) approach drastically reduces the stiffness observed with the "MT + TF (SEC)" approach (dashed lines) for these fiber-reinforced composites (see Figs. 5a and 5b). Even for high non-linear viscoplasticity linearity ($m = 0.1$), the error with respect to the FFT is not so high and the same trend as the "SEC (EIV + HS)" approach is found with "MT + TF (AFF)" for the case of "hard fibers". For "soft fibers", the differences between the "MT + TF (SEC)" (dashed lines) and "MT + TF (AFF)" (solid lines) approaches are not so high for all strain-rate sensitivities (see Figs. 5c and 5d). The discrepancies with the FFT results are increased when $m = 0.1$ and $f = 0.41$ for the fiber-weakened composites. Thus, the present "MT + TF (AFF)" approach tends to overestimate the effective behavior of fiber-weakened composites with respect to exact results.

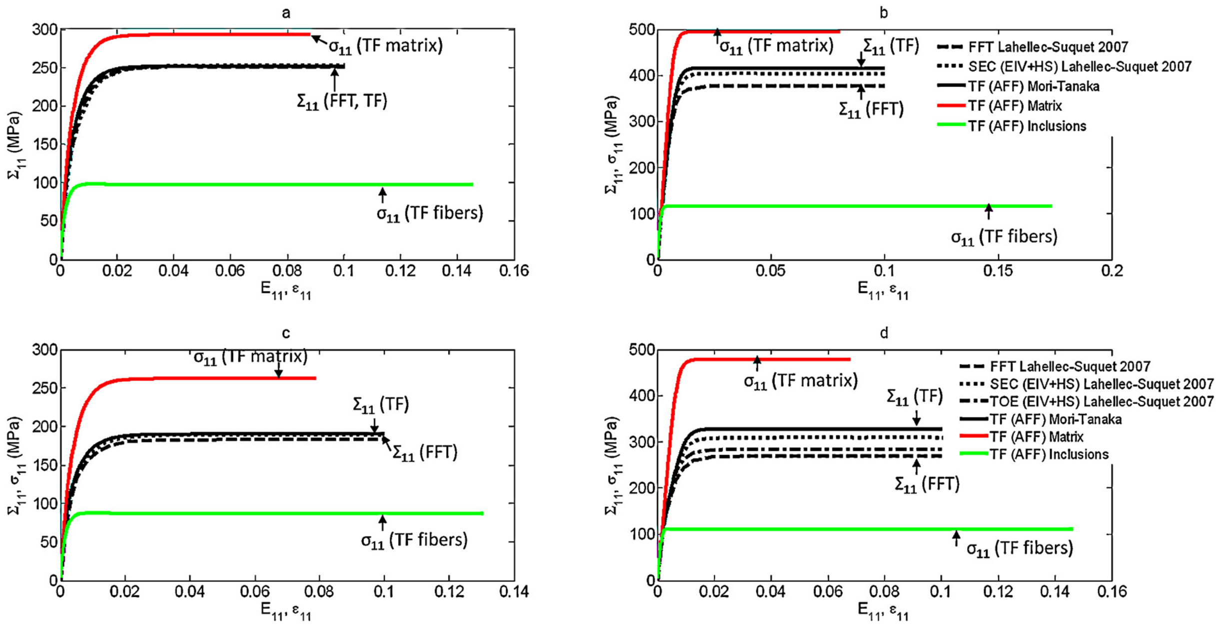

Fig. 4. (Color online.) Overall and phase average stress monotonic responses of two-phase non-linear elasto-viscoplastic fiber-weakened composites (case referred to in the text as “soft fibers”) subjected to in-plane shear loading perpendicular to the fibers for different fiber volume fractions and different non-linearities: (a) $f = 0.21$ and $m = 1$, (b) $f = 0.21$ and $m = 0.2$, (c) $f = 0.41$ and $m = 1$, (d) $f = 0.41$ and $m = 0.2$.

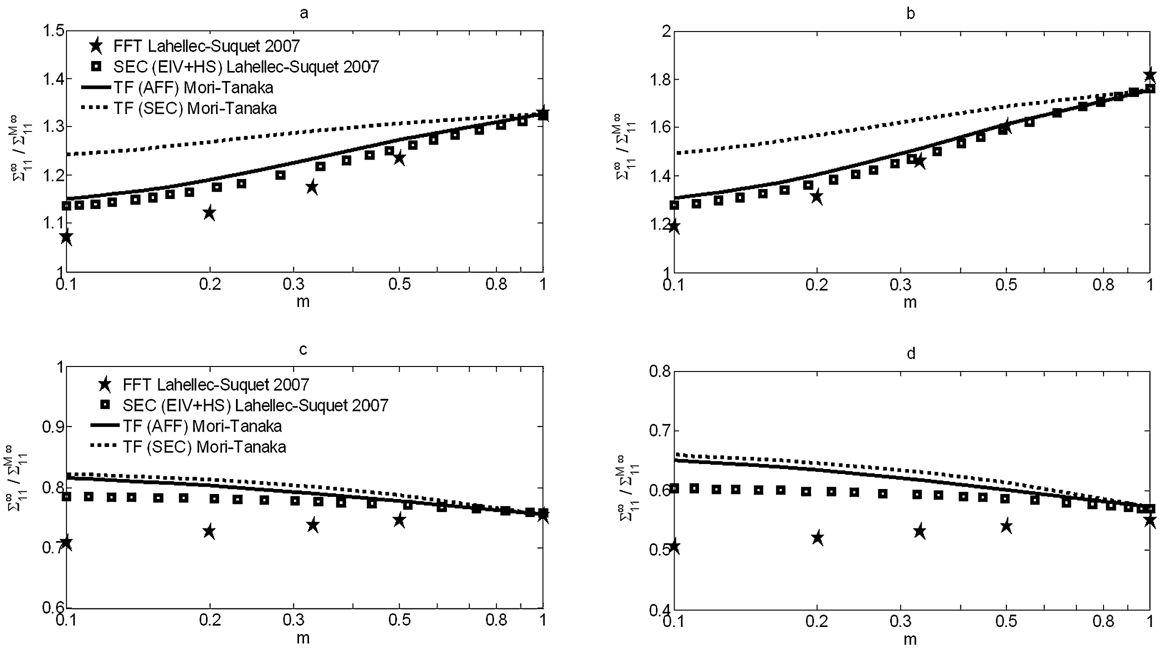

Fig. 5. Asymptotic overall stresses Σ_{11}^{∞} normalized with the asymptotic stresses obtained in the matrix alone (without fibers) $\Sigma_{11}^{M\infty}$ plotted as functions of m: for “hard fibers” with (a) $f = 0.21$, (b) $f = 0.41$, and, for “soft fibers” with (c) $f = 0.21$, (d) $f = 0.41$.

5. Conclusions

A first-order “affine” extension of the “translated fields” (TF) approach is proposed and embedded in a Mori–Tanaka homogenization scheme. It is first shown that the model is in very good agreement with the exact solutions of Ricaud and Masson [17] in the case of two-phase linear compressible viscoelastic inclusion-reinforced composites.

For non-linear elasto-viscoplastic composites, the main conclusions are the following:

- the present "affine" extension of the "translated fields" (the so-called "MT + TF (AFF)") approach gives reasonable results compared to the variational approaches developed by Lahellec and Suquet and FFT exact results [16,19]. The predictions are found to be closer to the FFT results when "hard fibers" are considered;
- the "affine" TF approach gives better estimates (i.e. softer) than existing "secant" linearization versions of the TF method developed in [11];
- the new interaction law derived in this paper still only contains first moments of stresses and strains, but remains very simple to handle compared to variational approaches.

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