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Flexural vs. tensile strength in brittle materials

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ABSTRACT

The tests leading to the determination of the strength of brittle materials show a very wide scattering and a noticeable difference between flexural and tensile strengths. The corresponding statistics are usually described by the Weibull law, which only partly explained the observed difference. From a theoretical point of view, the coupled criterion reaches the same conclusion, the flexural strength is higher than the tensile one. It is shown that these two approaches complement to give a satisfying explanation of the difference between the flexural and tensile strengths. Moreover, according to the coupled criterion, the tensile strength appears to be the only material parameter.

1. Introduction

Determining the strength of a brittle material is based on tensile or flexural tests on unnotched specimens (Fig. 1) [1].

The role of randomly distributed defects is decisive, leading to a large scattering in the measurements. The statistics are usually described by the Weibull law [2]. It leads to a significant difference between the measurements made in bending and tension. The flexural strength is higher than the tensile one. Indeed, for two samples of the same size, only one half of the sample is stressed in bending while the whole is in tension, then fewer defects are involved in bending. Nevertheless, the Weibull law often underestimates the flexural strength.

From a theoretical point of view, in fracture mechanics, the crack nucleation cannot be predicted by the well-known Griffith law. This latter only allows deciding whether a *pre-existing* crack can grow or not. The problem of the initiation of a new crack in brittle materials has been the subject of many studies since the 70s. They enter into a theory baptized, since the end of the 90s, Finite Fracture Mechanics [3,4]. Among them, the coupled criterion proposed in 2002 [5] seems to be one of the most promising and has proven its effectiveness in particular to predict the failure of v-notched specimens. It is based on the simultaneous fulfilment of a stress and an energy conditions. It leads to a similar conclusion on the ratio between the flexural and the tensile strengths.

The aim of this work is to analyze both approach and to show that they are not contradictory. They even complement to give a satisfying explanation of the difference between the flexural and tensile strengths.

2. The statistical Weibull model

In brittle materials, the Weibull law [2,6,7] provides a statistical approach of the failure of a specimen of volume V undergoing a uniaxial stress field σ. It relies on the theory of the weakest link. Under a uniform tension $\sigma = \bar{\sigma} =$ Constant, the failure probability is

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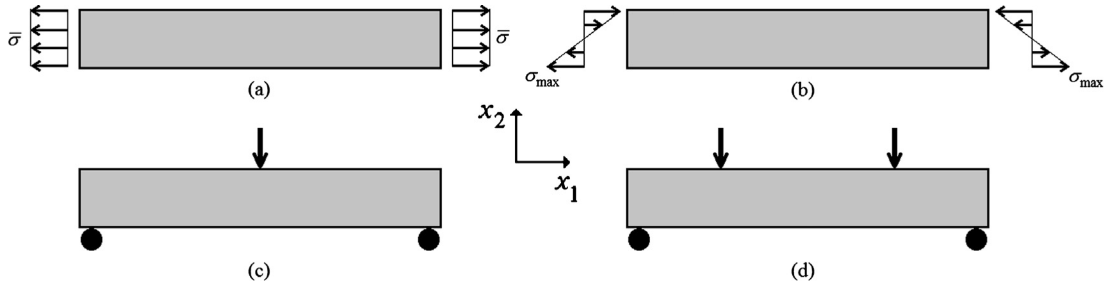

Fig. 1. Four different loadings to measure the strength of a brittle material: (a) uniform tension, (b) pure bending, (c) 3-point bending, (d) 4-point bending.

$$P_T(\sigma, V) = 1 - \exp\left(-\left(\frac{\bar{\sigma}}{\sigma_0}\right)^m \frac{V}{V_0}\right) \tag{1}$$

Where m is the Weibull modulus and σ_0, V_0 a pair of scaling parameters such that

$$P_T(\sigma_0, V_0) = 1 - \frac{1}{e} \simeq 0.632 \tag{2}$$

Here and further in the document, the indices "T, B, PB, 3P-B, 4P-B" hold respectively for tension, bending, pure bending, 3-point bending and 4-point bending.

For a uniaxial but non-uniform stress field σ, the Weibull law takes the following form

$$P(\sigma, V) = 1 - \exp\left(-\left(\frac{\sigma_W}{\sigma_0}\right)^m\right) \quad \text{with } \sigma_W = \frac{1}{V_0^{1/m}}\left[\int_{V'} \sigma^m \mathrm{d}V\right]^{1/m} \tag{3}$$

Where V' (involved in the integral) is the tested volume of the specimen.

For a pure bending loading, the tensile stress linearly decreases through the thickness from a maximum value σ_{max} on the face of the specimen under tension to $-\sigma_{max}$ on the opposite face under compression

$$\sigma(x) = \sigma_{max}\left(1 - \frac{2x_2}{h}\right) \tag{4}$$

Where h is the thickness of the specimen.

Thus, in a pure bending loading, only one half of the specimen of volume V (see (1)) is under tension and V' is the volume defined by $0 \le x_2 \le h/2$, as a consequence (3) gives

$$\sigma_W = \frac{\sigma_{max}}{[2(m+1)]^{1/m}} \quad \text{and} \quad P_{PB}(\sigma) = 1 - \exp\left(-\left(\frac{\sigma_W}{\sigma_0}\right)^m\right) \tag{5}$$

Then the ratio R_{PB} between the flexural σ_c^{PB} and the tensile σ_c^T strengths for a pure bending can be obtained considering

$$P_T(\sigma, V) = P_{PB}(\sigma, V) \Rightarrow R_{PB} = \frac{\sigma_c^{PB}}{\sigma_c^T} = \frac{\sigma_{max}}{\bar{\sigma}} = \left[2(m+1)\right]^{1/m} \tag{6}$$

Similar relations can be derived for the 3- and 4-point bending tests [8]

$$R_{3P\text{-}B} = \left[2(m+1)^2\right]^{1/m}; \qquad R_{4P\text{-}B} = \left[\frac{6(m+1)^2}{m+3}\right]^{1/m} \tag{7}$$

Surprisingly, the ratio R in (6) and (7) does not depend on the specimen thickness h.

Fig. 2 shows a comparison between (6) and (7) for various Weibull modulus m. A noticeable property of these three curves is that $R \to 1$ as $m \to \infty$, i.e. as the scattering decreases and finally vanishes, the law becoming entirely deterministic.

Data from manufacturers are available for different materials (E is the Young modulus, ν the Poisson ratio, K_{Ic} the toughness of the material):

- Alumina AD998 [9], $E = 370$ GPa, $\nu = 0.22$, $K_{Ic} = 4.5$ MPa$\sqrt{}$m, $\sigma_c^T = 248$ MPa, $\sigma_c^B = 375$ MPa, $m = 21$ [10].
- Polymer PR520 [11,12], $E = 4$ GPa, $\nu = 0.4$, $K_{Ic} = 2.2$ MPa$\sqrt{}$m, $\sigma_c^T = 82.1$ MPa, $\sigma_c^B = 153.1$ MPa, $m = 16$ [8].
- Silicon carbide SA [13], $E = 430$ GPa, $\nu = 0.14$, $K_{Ic} = 4.6$ MPa$\sqrt{}$m, $\sigma_c^T = 234$ MPa, $\sigma_c^{4P\text{-}B} = 380$ MPa, $\sigma_c^{3P\text{-}B} = 550$ MPa, $m = 10$.

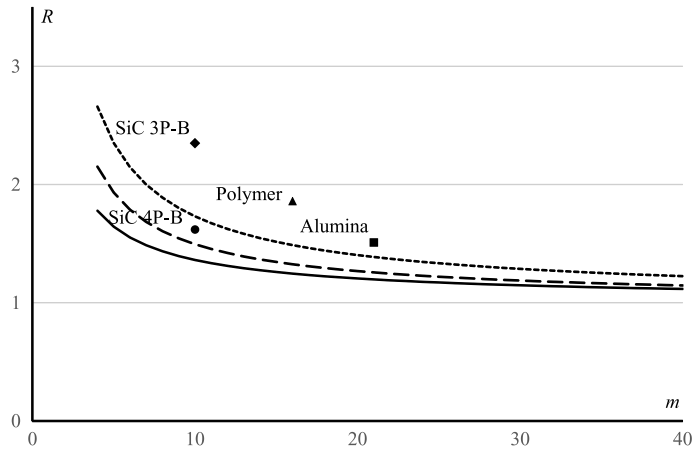

Fig. 2. The ratio R between the flexural strength and the tensile strength as a function of the Weibull modulus m: pure bending (solid line), 4-point bending (dashed line), 3-point bending (dotted line).

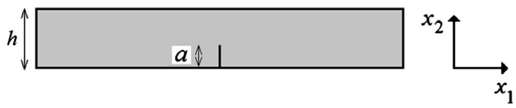

Fig. 3. Schematic view of a specimen embedding a crack with length a.

Note that it has not been possible to determine in the data sheets of the polymer and alumina cases, the type of bending test carried out to provide the flexural strength (index "B" alone). And even when 4-point bending is mentioned (SiC), two standards are available depending on the inner span length [14] and it is difficult to know which one is referred to, but it probably does not make a big difference.

The above values of R vs. m are plotted in Fig. 2, where it clearly appears that the experimentally measured values are greater than the theoretical ones predicted by the Weibull statistics.

3. The coupled criterion model

The well-known Griffith law cannot be used to predict any crack nucleation, it only allows to decide whether a *pre-existing* crack can grow or not.

Instead, the coupled criterion [5] can, it allows predicting the nucleation of a new crack in a sound structure made of a brittle material. In the general case the onset of such a crack occurs abruptly, the crack length jumps from 0 to a (Fig. 3). Two conditions must be fulfilled: (i) the tensile stress σ must exceed the tensile strength σ_c^T all along the expected crack path of length a, (ii) the incremental energy release rate G_{inc} must exceed the mode I toughness G_c of the material. These conditions can be written under the assumption of plane strain elasticity

$$\begin{aligned} &\sigma(x_2) \geq \sigma_c^T \quad \text{for } 0 \leq x_2 \leq a \\ &G_{inc}(a) \geq G_c \quad \text{where } G_{inc}(a) = \frac{1}{a}\int_0^a G(x_2)\,\mathrm{d}x_2 \end{aligned} \tag{8}$$

Here $G(x_2)$ denotes the energy release rate at the tip of a crack with length x_2, i.e. minus the derivative of the potential energy W^P with respect to the crack length

$$G(x_2) = -\frac{\partial W^P(x_2)}{\partial x_2} \Rightarrow G_{inc}(a) = \frac{W^P(0) - W^P(a)}{a} \tag{9}$$

In all cases, uniform tension, pure bending, 3-point and 4-point bending, the tensile stress $\sigma(x_2)$ is constant or decreasing with x_2, then the conditions (8) can be written

$$\frac{\sigma(a)}{\sigma_c^T} \geq 1; \qquad \frac{G_{inc}(a)}{G_c} \geq 1 \tag{10}$$

And it is convenient to plot the two conditions in the same graph (Figs. 4 and 5).

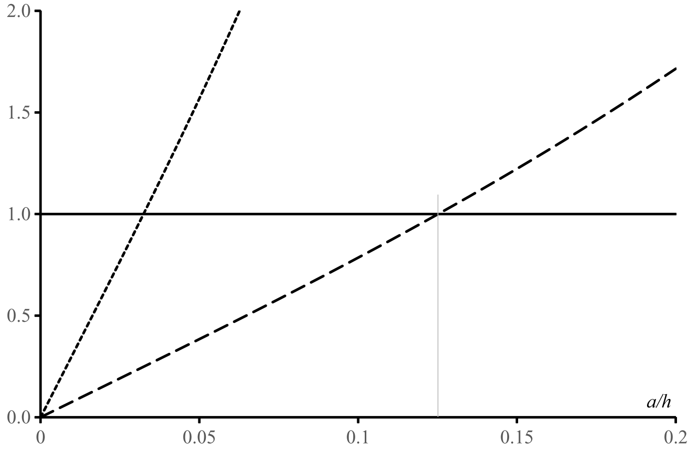

Fig. 4. The dimensionless energy release rate G/G_c (dotted line), the dimensionless incremental energy release rate G_{inc}/G_c (dashed line) and the dimensionless tensile stress σ/σ_c^T (solid line) function of the dimensionless crack length a/h, for $\bar{\sigma} = \sigma_c^T = 234$ MPa and $h = 3$ mm in the SiC case.

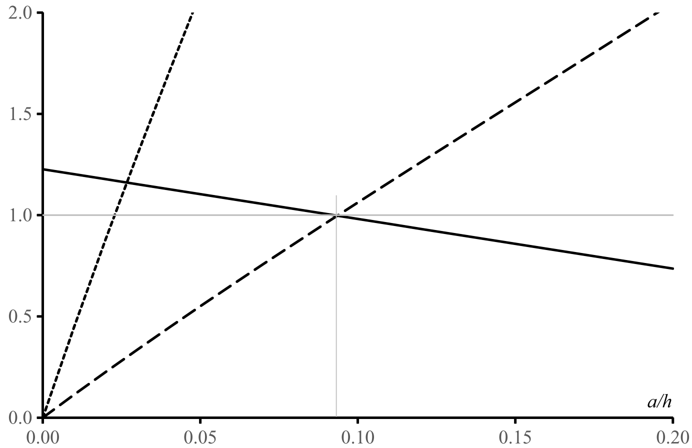

Fig. 5. The dimensionless energy release rate G/G_c (dotted line), the dimensionless incremental energy release rate G_{inc}/G_c (dashed line) and the dimensionless tensile stress σ/σ_c^T (solid line) function of the dimensionless crack length a/h, for $\sigma_{max} = 288$ MPa and $h = 3$ mm in the SiC case.

3.1. The uniform tension case

It is the simplest case, in a monotonic loading the first inequality (10) holds if $\sigma(a) = \bar{\sigma} = \sigma_c^T$, then the second one allows to determine a since $G_{inc}(a)$ is monotonically increasing with a. The stress intensity factor K_I at the tip of a crack of length a in an infinite strip of width h submitted to a uniform tension $\bar{\sigma}$ is given by

$$K_I = \bar{\sigma}\sqrt{\pi a} F_T\left(\frac{a}{h}\right) \tag{11}$$

Where the function F_T can be approximated by a polynomial of order 4 with a high accuracy provided $a/h < 0.6$ [15]. The energy release rate G can be derived from the Irwin formula

$$G = \frac{1-\nu^2}{E} K_I^2 \tag{12}$$

Then G_{inc} can be calculated from (8) and (12).

The conditions (10) are illustrated in Fig. 4, for $h = 3$ mm, in the SiC case. Note that the height $h = 3$ mm was chosen because it is the commonly used value in standards to carry out the tests shown in Fig. 1.

In such a tensile test, at initiation, the crack length jumps from 0 to $a_c^T = 0.125 \times h = 0.375$ mm. At this point it is clear, from Fig. 4, that $G > G_c$, then the crack goes on growing in an unstable manner up to the final failure of the specimen.

Typically the coupled criterion is reduced to the maximum tensile stress condition in this special case. A size effect (different from that caused by the statistical variations on the presence of defects [5,16]) could be observed for very short specimens but this is not the case for the sizes commonly used in this type of test ($3 \times 4 \times 40$ mm).

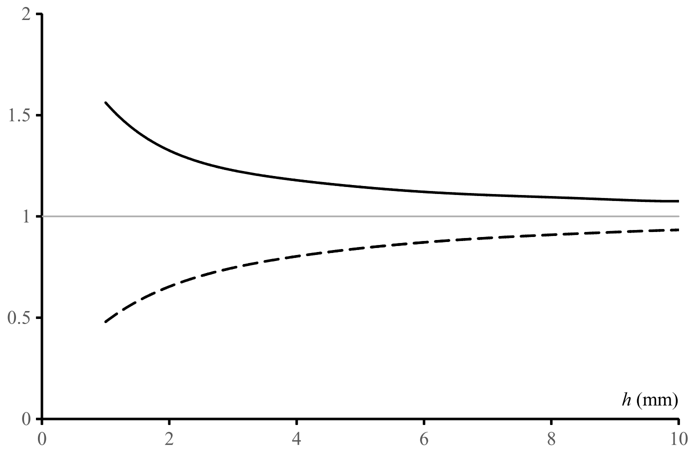

Fig. 6. The dimensionless load at failure $R_{PB} = \sigma_c^{PB}/\sigma_c^T$ (solid line) and the dimensionless crack jump a_c^{PB}/a_c^T (dashed line) during a pure bending loading, function of the thickness h, as a consequence of the coupled criterion.

3.2. The pure bending case

According to (4), $\sigma(x_2)$ is a decreasing function of x_2. Then it is clear that the condition $\sigma_{max} = \sigma_c^T$ is not sufficient to ensure the stress condition in (8) since $\sigma(x_2) < \sigma_c^T$ at any interior point $x_2 > 0$.

A similar relationship to (11) expresses the stress intensity at the tip of a crack of length a in an infinite strip of width h submitted to a pure bending (4)

$$K_I = \sigma_{max}\sqrt{\pi a} F_{PB}\left(\frac{a}{h}\right) \tag{13}$$

Again the function F_{PB} can be accurately approximated by a polynomial of order 4 [15] allowing an easy calculation of G and G_{inc}. Still in the SiC case, in order to fulfil the two conditions (10), the applied load must be increased so that $\sigma_{max} =$ 288 MPa (see Eq. (4)) as shown in Fig. 5, then (10) holds true at the single point $a/h = 0.093$. This value is retained as the flexural strength σ_c^{PB}.

At initiation, the crack length jumps from 0 to $a_c^{PB} = 0.093 \times h = 0.280$ mm. At this point, as before, it is clear (Fig. 5) that $G > G_c$, leading to a conclusion similar to that of the uniform tension case, the crack goes on growing up to the final failure of the specimen.

3.3. Influence of the specimen thickness on the flexural strength

The specimen thickness plays an important role when determining the bending strength σ_c^{PB} in Section 3.2 because it defines the slope of the tensile stress when moving away from the face in tension (see (4)). Thinner the specimen and higher σ_c^{PB} as shown in Fig. 6. The ratio R tends to 1 as h increases indefinitely.

Almost identical results are obtained in the 3-point and 4-point bending cases using specific formulas [17].

Despite a purely deterministic approach, the flexural strength does not equal the tensile strength ($R_{PB} > 1$) except for very thick specimens. Deterministic means that no scattering exists, thus it corresponds to $m \to \infty$ in the Weibull approach leading to conclude that $R \to 1$. There is no contradiction but one must be conscious that both mechanisms intervene when measuring the flexural strength.

3.4. Influence of the toughness on the flexural strength

The Weibull theory is based on a stress condition and does not involve the toughness of the material, thus a change in toughness should not have any influence on the Weibull modulus of the flexural strength whereas it is sometimes claimed that it can have. It is discussed at length in [18] without leading to a clear conclusion. It is expected that an increase of the toughness can cause an increase of the Weibull modulus. Nevertheless, it must be pointed out that most of the effects described in this paper are due to some microstructure changes, removal of large flaws or R-curve effects and that in absence of these phenomena the inverse is observed. It is reported, for instance, that for a toughened zirconia a reduction in Weibull modulus from 20 to 7.3 occurs on burnout and sintering while the toughness increases from 1.1 to 2 MPa $\sqrt{m}$.

Of course, since the coupled criterion needs an energy condition in addition to the stress one, it brings into evidence a dependency of the flexural strength on the material toughness. Fig. 7 shows how, according to the coupled criterion, the ratio R_{PB} depends on the toughness K_{Ic} for the SiC material. Such variation may be misinterpreted in terms of variation

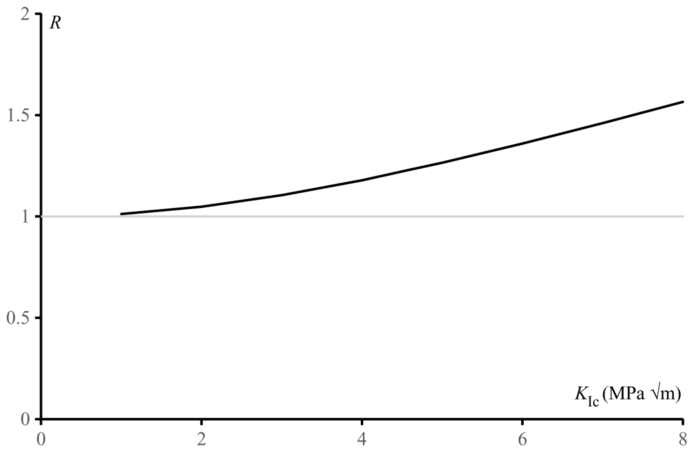

Fig. 7. The ratio R_{PB} function of the material toughness K_{Ic} for SiC as a consequence of the coupled criterion.

of the Weibull modulus, since, following (6), the ratio R_{PB} increases as m decreases. If the toughness evolves from 4.6 to 6 MPa $\sqrt{m}$ R_{PB} increases by 10% and then, using the only relationship (6), m should change from 10 to roughly 7, a decrease by a factor 0.7, to fit with the difference.

It must be emphasized that the only role of the toughness in the tensile test is to change the length of the crack jump at initiation: the highest the toughness, the largest the crack jump length, as can be derived from Fig. 4.

As a conclusion of the two above subsections, according to the coupled criterion, the flexural strength measured during a bending test depends on the geometry of the specimen (its thickness, Section 3.3). Moreover flexural strength is also clearly linked to the value of the toughness (Section 3.4). Whereas both the thickness and the toughness have strictly no influence on the tensile strength measured in a pure tension test (Section 3.1). Amongst the two strength parameters, the tensile strength appears to be the only material parameter.

4. Combining the Weibull approach with the coupled criterion

On the one hand, according to the statistical Weibull theory, the flexural strength is higher than the tensile strength, the ratio R depends on the scattering measured by the Weibull modulus m but does not depend on the specimen size (Section 2). In technical sheets, knowing the flexural strength, the tensile strength is sometimes derived without new measurements using this ratio, since bending experiments are easier to carry out for brittle materials [1].

On the other hand the coupled criterion comes to a similar conclusion on the ratio R but with a different explanation (Section 3). It is deterministic, both energy and stress are involved to predict crack initiation leading to the final failure. The governing parameter is the thickness h of the specimens which plays a key role in the behavior of the tensile component of the stress tensor during a bending test.

Obviously, using either one or the other correction to the flexural strength leads to overestimate the tensile strength. Then, since the same above conclusions are due to totally independent reasons, the idea is to combine them, applying successively the two corrections.

Fig. 8 reuses Fig. 2, the measured values are in light gray, a first correction associated with the coupled criterion is brought and results are plotted in medium gray. Obviously the corrected values remain above the expected ones ($R = 1$). Then a second correction relying on the Weibull law is added leading to the black symbols. Obviously combining these two corrections provides satisfying estimates of the tensile strength (i.e. R close to 1).

5. Conclusion

There has been a controversy about energy approaches to crack initiation. In particular, the often cited paper by Parvizi et al. [19], published in 1978, has been much discussed by the upholders of the statistical approach. They studied the transverse cracking in cross-ply laminates and observed that the critical load increases with the thinness of the inner transverse layer. This increase was correctly described by the energy-based approach they proposed. But, of course, Weibull's arguments point in the same direction. Kelly [20], in 1988, responded to critics, trying to prove that Weibull type statistics was unlikely in the case analyzed by Parvizi et al. Finally we reconcile everyone by showing that it is necessary to combine the two approaches, at least in the case studied herein and probably in most cases, and that there are experimental evidences to confirm this.

Agreeing with this statement, the most important conclusion to draw from the coupled criterion is that, amongst the two strength parameters, the tensile strength seems to be the only one to be a material parameter.

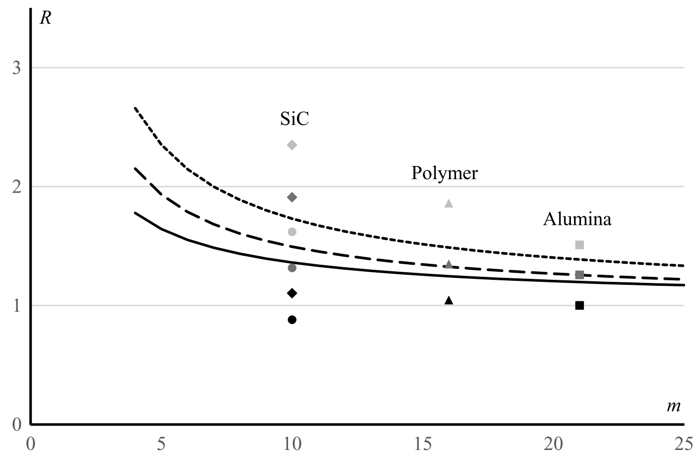

Fig. 8. The corrected ratio R. The light gray points are the measured values (for SiC diamonds correspond to 3P-B and circles to 4P-B), the medium gray ones are corrected by the coupled criterion ($h = 3$ mm) and the black ones are corrected by both the coupled criterion and the Weibull theory. Refer to Fig. 2 for the curves.

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