C. R. Mecanique 343 (2015) 289–300

Contents lists available at ScienceDirect

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A gradient model for torsion of nanobeams

Francesco Marotti de Sciarra [a], Marko Canadija [b], Raffaele Barretta [a]

[a] Department of Structures for Engineering and Architecture, via Claudio 25, 80121 Naples, Italy
[b] Department of Engineering Mechanics, Faculty of Engineering University of Rijeka, Vukovarska 58, 51000 Rijeka, Croatia

ARTICLE INFO

Article history:
Received 28 November 2014
Accepted 16 February 2015
Available online 9 March 2015

Keywords:
Nanobeams
Size effects
Nonlocal thermodynamics
Nonlocal torsion
Analytical solutions

ABSTRACT

A first-order gradient model based on the Eringen nonlocal theory is presented. The variational formulation, the governing differential equation and both classical and non-classical boundary conditions of nonlocal nanobeams subjected to torsional loading distributions are derived using a thermodynamic approach, thus providing closed-form solutions. Nanocantilevers and fully campled nanobeams are considered to investigate the size-dependent static behavior of the proposed model in terms of torsional rotations and moments. The results are thus compared to those of the Eringen model, gradient elasticity theory and classical (local) model.

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1. Introduction

Micro- and nano-elements are particular structures whose characteristic size (thickness, diameter, etc.) is in the order of the micron and the sub-micron. These elements are widely used (see, e.g., Kahrobaiyan et al. [1], Li et al. [2], Tajalli et al. [3]) and it is well known that they are size dependent, see, e.g., Fleck et al. [4], Lam et al. [5], McFarland and Colton [6]. In particular, a size-dependent torsional behavior has been observed in [4] during micro-torsion tests on thin copper wires.

Nanobeams subjected to torsional loading conditions and/or prescribed torsional displacements are widely used in many kinds of micro- and nano-electromechanical systems (MEMS and NEMS) such as torsional microscanners (Arslan et al. [7]), torsional micromirrors (Huang et al. [8], Zhang et al. [9]), micro-gyroscopes (Maenaka et al. [10]) and torsional springs in NEMS oscillators (Papadakis et al. [11]).

Hence, the accurate modeling of torsion of nanobeams seems to be crucial in order to study the mechanical behavior of such systems.

The small-length scales involved in nanotechnology applications, such as the development of small actuators, are required to account for the effects of interatomic and intermolecular forces, commonly named size-dependent behavior (Arash and Wang [12], Rafiee and Moghadam [13], Marotti de Sciarra and Barretta [14]). Since classical continuum mechanics is incapable of capturing the size effect, some nonclassical continuum theories such as the nonlocal (Eringen [15,16], Barretta and Marotti de Sciarra [17], Canadija et al. [18], Marotti de Sciarra [19]), strain gradient (Aifantis [20,21], Peerlings et al. [22], Marotti de Sciarra [23], Askes and Aifantis [24], Pardoen and Massart [25], Xu et al. [26]) and couple stress (Lam et al. [5], Yang et al. [27], Asghari et al. [28]) theories have been introduced to investigate nanostructures.

In particular, some material parameters are considered in nonlocal models, in addition to the classical elastic constants in order to capture the size-dependent behavior. To determine the length-scale parameter for a specific material, some typical experiments such as micro-bend test, micro-torsion test and, recently, micro/nano indentation tests can be carried out (Fleck et al. [4], Paliwal et al. [29], Brcic et al. [30], Song et al. [31]).

E-mail addresses: marotti@unina.it (F. Marotti de Sciarra), marko.canadija@riteh.hr (M. Canadija), rabarret@unina.it (R. Barretta).

http://dx.doi.org/10.1016/j.crme.2015.02.004
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In this paper, a new constitutive model of torsion of nanobeams is formulated based on the nonlocal model of Eringen [15,16] and on the elasticity gradient theory (Aifantis [21], Xu et al. [26], Ru and Aifantis [32], Tenek and Aifantis [33]).

The proposed model (FTGEM) can be considered as a first-order gradient version of torsion based on Eringen nonlocal theory characterized by two length-scale parameters corresponding to the nonlocal behavior associated with the classical Eringen model (TEM) and with the gradient one (TGM).

Nonlocal thermodynamics is utilized to obtain the variational formulation of the new nonlocal model. Then the governing differential equation of nonlocal torsion with classical and non-classical boundary conditions are derived in a straigthforward manner. As special cases the known nonlocal torsional theories based on the Eringen model (TEM) and on the gradient elasticity model (TGM) are recovered. As case studies, the exact solutions of nanocantilevers and fully clamped nanobeams modeled by the FTGEM are presented and the results are compared with those of the TEM, TGM and classical (local) theory.

2. Preliminaries

Let us consider a straight homogeneous isotropic nanobeam of length L occupying a domain $\mathcal{B}$ of the Euclidean space and let Ω be a two-dimensional domain representing the circular cross-section of the nanobeam. The x-axis is taken along the length of the nanobeam and is orthogonal to the plane of the cross-section containing the axes y and z.

The components of the displacement field for the torsion of the nanobeam are expressed as

$$s_x(x, y) = 0, \qquad s_y(x, y) = -\vartheta(x)\, z, \qquad s_z(x, y) = \vartheta(x)\, y \tag{1}$$

where s_x, s_y and s_z denote the displacements along the axes (x, y, z) and ϑ stands for the twist angle.

Cartesian components of shear strain vector $\boldsymbol{\gamma}$, position vector $\mathbf{r}$ and $\pi/2$ counterclockwise rotation $\mathbf{R}$ are respectively given by

$$\boldsymbol{\gamma} = \begin{bmatrix} \gamma_{yx} \\ \gamma_{zx} \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} y \\ z \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2}$$

Denoting by the apex $\bullet^{(n)}$ the n-derivative along the nanobeam's axis, the kinematically compatible deformation field is provided by the formulae

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{zy} = 0, \qquad \boldsymbol{\gamma} = \mathbf{Rr}\vartheta^{(1)} \tag{3}$$

Note that the vector $\mathbf{Rr} = [-z, y]^{\mathrm{T}}$ provides the $\pi/2$ counterclockwise rotated of $\mathbf{r}$.

Moreover, the circular cross-sectional area A and the polar moment of area J are given by

$$(A, J) = \int_{\Omega} (1, \mathbf{r} \cdot \mathbf{r})\, \mathrm{d}A \tag{4}$$

where the symbol $\cdot$ denotes the single (or double) index saturation.

The first law of thermodynamics for isothermal processes and for a nonlocal behavior can be formulated in a global form, while the second principle is expressed in its usual local form (see, e.g., [34–36]) so that the vanishing of the body energy dissipation for the considered nonlocal elastic model can be expressed as follows:

$$\int_{\mathcal{B}} \boldsymbol{\tau} \cdot \dot{\boldsymbol{\gamma}}\, \mathrm{d}V = \int_{\mathcal{B}} \dot{\psi}\, \mathrm{d}V \tag{5}$$

where ψ denotes the Helmholtz free energy of the nanobeam and $\boldsymbol{\tau} = \left(\tau_{yx}, \tau_{zx}\right)$ is the nonlocal shear stress vector. The superscript dot denotes differentiation with respect to time.

In order to derive a nonlocal model of torsion, a suitable definition of the free energy is given in the next section, so that the related nonlocal variational form can be provided and the corresponding differential equations with the required boundary conditions can then be consistently derived.

3. Eringen model

Let us preliminarily discuss the formulation of the elastic equilibrium of a nanobeam under torsion (TEM) with the following constitutive behavior according to the Eringen model [15,16]:

$$\boldsymbol{\tau} - c^2 \boldsymbol{\tau}^{(2)} = G\boldsymbol{\gamma} \tag{6}$$

with $c = e_0 l$, being l a length-scale parameter and e_0 a material constant, and G the shear modulus. Performing the inner product of Eq. (6) by $\mathbf{Rr}$ and integrating on the cross-section Ω, we get:

$$\int_{\Omega} \boldsymbol{\tau} \cdot \mathbf{Rr}\, \mathrm{d}A - c^2 \int_{\Omega} \boldsymbol{\tau}^{(2)} \cdot \mathbf{Rr}\, \mathrm{d}A = \int_{\Omega} G\boldsymbol{\gamma} \cdot \mathbf{Rr}\, \mathrm{d}A \tag{7}$$

Introducing the classical shear stress $\boldsymbol{\tau}_0$ and the stress resultant moments $(M_\mathrm{t}, M_{0\mathrm{t}})$ given by

$$\begin{aligned}\boldsymbol{\tau}_0 &= G\boldsymbol{\gamma} = G\vartheta^{(1)}\mathbf{Rr}\\ (M_\mathrm{t}, M_{0\mathrm{t}}) &= \int_\Omega (\boldsymbol{\tau}\cdot\mathbf{Rr}, \boldsymbol{\tau}_0\cdot\mathbf{Rr})\,\mathrm{d}A\end{aligned} \tag{8}$$

and recalling the differential equilibrium equation $M_\mathrm{t}^{(1)} = -m_\mathrm{t}$, being m_t the distributed torque-per-unit-length about the x-axis, we get the relation

$$M_\mathrm{t} = -c^2 m_\mathrm{t}^{(1)} + M_{0\mathrm{t}} \tag{9}$$

The expression of the torsional curvature takes then the form

$$\vartheta^{(1)} = \frac{M_\mathrm{t}}{GJ} + \frac{c^2 m_\mathrm{t}^{(1)}}{GJ} \tag{10}$$

Setting $\vartheta^e_{\mathrm{ER}} = \dfrac{M_\mathrm{t}}{GJ}$ and $\bar{\vartheta}_{\mathrm{ER}} = \dfrac{c^2 m_\mathrm{t}^{(1)}}{GJ}$, Eq. (10) can be interpreted as an additive decomposition formula of the torsional curvature in an elastic and inelastic (imposed) part.

Accordingly, the torsional rotation field in Eringen's model can be obtained by considering an equivalent classical (local) nanobeam, having the same kinematic constraints, loading conditions and local elastic parameters as the nonlocal nanobeam, which is subjected to a prescribed torsional curvature $\bar{\vartheta}_{\mathrm{ER}}$ describing nonlocality effects.

Finally, multiplying Eq. (9) by the torsional curvature $\dot{\vartheta}^{(1)}$ and integrating along the beam axis x, we get the variational formulation of the TEM:

$$\int_0^L M_\mathrm{t}\dot{\vartheta}^{(1)}\mathrm{d}x = \int_0^L M_{0\mathrm{t}}\dot{\vartheta}^{(1)}\mathrm{d}x - \int_0^L c^2 m_\mathrm{t}^{(1)}\dot{\vartheta}^{(1)}\mathrm{d}x \tag{11}$$

4. Governing equations for nonlocal torsion

Following the analogy provided in Section 3, a first-order gradient version of the Eringen constitutive model for the torsion of nanobeams (FTEGM) can be obtained by considering an equivalent classical (local) nanobeam, having the same kinematic constraints, loading conditions and local elastic parameters as the nonlocal nanobeam, which is subjected to a prescribed strain $\bar{\boldsymbol{\gamma}}$ describing the nonlocal effects.

Accordingly, the shear strain $\boldsymbol{\gamma}$ of the nonlocal nanobeam is assumed to be additively decomposed into an elastic part $\boldsymbol{\gamma}^e$ and into a prescribed shear strain $\bar{\boldsymbol{\gamma}}$

$$\boldsymbol{\gamma} = \boldsymbol{\gamma}^e + \bar{\boldsymbol{\gamma}} \tag{12}$$

Hence the FTGEM can be formulated by assuming the following expression of the free energy ψ

$$\psi\left(\boldsymbol{\gamma}, \boldsymbol{\gamma}^{(1)}\right) = \frac{1}{2}G(\boldsymbol{\gamma} - \bar{\boldsymbol{\gamma}})\cdot(\boldsymbol{\gamma} - \bar{\boldsymbol{\gamma}}) + \frac{1}{2}c_1^2 G\left(\boldsymbol{\gamma}^{(1)} - \bar{\boldsymbol{\gamma}}^{(1)}\right)\cdot\left(\boldsymbol{\gamma}^{(1)} - \bar{\boldsymbol{\gamma}}^{(1)}\right) \tag{13}$$

where the coefficient $c_1 = e_0 l_1$ incorporates the small-scale effect, l_1 the material's length scale and e_0 a material constant [16].

Further, the prescribed strain $\bar{\boldsymbol{\gamma}}$ has the following expression:

$$\bar{\boldsymbol{\gamma}} = \frac{c^2 m_\mathrm{t}^{(1)}}{GJ}\mathbf{Rr} \tag{14}$$

where $c = e_0 l$; l is a length scale parameter and m_t represents the distributed torque-per-unit-length about the x-axis.

Remark 1. The free energy expression (13) can be straightforwardly extended to include higher-order gradient models of nanobeams by considering the following expression

$$\begin{aligned}\psi\left(\boldsymbol{\gamma}, \boldsymbol{\gamma}^{(1)}, \ldots, \boldsymbol{\gamma}^{(n)}\right) &= \frac{1}{2}G(\boldsymbol{\gamma} - \bar{\boldsymbol{\gamma}})\cdot(\boldsymbol{\gamma} - \bar{\boldsymbol{\gamma}})\\ &\quad + \frac{1}{2}G\sum_{i=1}^{n} c_i^2\left(\boldsymbol{\gamma}^{(i)} - \bar{\boldsymbol{\gamma}}^{(i)}\right)\cdot\left(\boldsymbol{\gamma}^{(i)} - \bar{\boldsymbol{\gamma}}^{(i)}\right)\end{aligned} \tag{15}$$

where c_i is the i-length scale parameter.

The free-energy time rate is then given by

$$\dot{\psi}(\boldsymbol{\gamma}) = G(\boldsymbol{\gamma} - \bar{\boldsymbol{\gamma}}) \cdot \dot{\boldsymbol{\gamma}} + c_1^2 G\left(\boldsymbol{\gamma}^{(1)} - \bar{\boldsymbol{\gamma}}^{(1)}\right) \cdot \dot{\boldsymbol{\gamma}}^{(1)} \tag{16}$$

being $\dot{\bar{\boldsymbol{\gamma}}} = \mathbf{0}$, so that using Eqs. (3)$_2$, (14) and (16), we have:

$$\begin{aligned}
&\int_{\mathcal{B}} \boldsymbol{\tau} \cdot \dot{\boldsymbol{\gamma}}\, \mathrm{d}V = \int_{\mathcal{B}} \boldsymbol{\tau} \cdot \mathbf{Rr}\dot{\vartheta}^{(1)} \mathrm{d}V = \int_0^L M_\mathrm{t} \dot{\vartheta}^{(1)} \mathrm{d}x \\
&\int_{\mathcal{B}} \dot{\psi}\, \mathrm{d}V = \int_{\mathcal{B}} G(\boldsymbol{\gamma} - \bar{\boldsymbol{\gamma}}) \cdot \dot{\boldsymbol{\gamma}}\, \mathrm{d}V + \int_{\mathcal{B}} c_1^2 G\left(\boldsymbol{\gamma}^{(1)} - \bar{\boldsymbol{\gamma}}^{(1)}\right) \cdot \dot{\boldsymbol{\gamma}}^{(1)} \mathrm{d}V \\
&\qquad = \int_{\mathcal{B}} \boldsymbol{\tau}_0 \cdot \mathbf{Rr}\dot{\vartheta}^{(1)} \mathrm{d}V - \int_{\mathcal{B}} G\bar{\boldsymbol{\gamma}} \cdot \mathbf{Rr}\dot{\vartheta}^{(1)} \mathrm{d}V + \int_{\mathcal{B}} c_1^2 \boldsymbol{\tau}_0^{(1)} \cdot \mathbf{Rr}\dot{\vartheta}^{(2)} \mathrm{d}V \\
&\qquad\quad - \int_{\mathcal{B}} c_1^2 G\bar{\boldsymbol{\gamma}}^{(1)} \cdot \mathbf{Rr}\dot{\vartheta}^{(2)} \mathrm{d}V \\
&\qquad = \int_0^L M_{0\mathrm{t}} \dot{\vartheta}^{(1)} \mathrm{d}x - \int_0^L c^2 m_\mathrm{t}^{(1)} \dot{\vartheta}^{(1)} \mathrm{d}x + \int_0^L c_1^2 M_{0\mathrm{t}}^{(1)} \dot{\vartheta}^{(2)} \mathrm{d}x \\
&\qquad\quad - \int_0^L c^2 c_1^2 m_\mathrm{t}^{(2)} \dot{\vartheta}^{(2)} \mathrm{d}x
\end{aligned} \tag{17}$$

where $\boldsymbol{\tau}_0$ is the classical shear stress and $(M_\mathrm{t}, M_{0\mathrm{t}})$ are the stress resultant moments reported in Eqs. (8).

Hence, the variational condition for the nonlocal FTGEM can be obtained from the thermodynamic condition (5) in the following form:

$$\begin{aligned}
\int_0^L M_\mathrm{t} \dot{\vartheta}^{(1)} \mathrm{d}x = &\int_0^L M_{0\mathrm{t}} \dot{\vartheta}^{(1)} \mathrm{d}x - \int_0^L c^2 m_\mathrm{t}^{(1)} \dot{\vartheta}^{(1)} \mathrm{d}x + \int_0^L c_1^2 M_{0\mathrm{t}}^{(1)} \dot{\vartheta}^{(2)} \mathrm{d}x \\
&- \int_0^L c^2 c_1^2 m_\mathrm{t}^{(2)} \dot{\vartheta}^{(2)} \mathrm{d}x
\end{aligned} \tag{18}$$

Remark 2. The nonlocal variational formulation (18) of the FTGEM can be specialized to the variational formulation of the nonlocal torsional model of the Eringen type (TEM) by setting $c_1 = 0$ and to the one corresponding to the nonlocal torsional model of the gradient type (TGM) by setting $c = 0$.

The differential equilibrium relation can be recovered by integrating by parts the l.h.s. of Eq. (18) and, imposing the equality with the external virtual power, we get $M_\mathrm{t}^{(1)} = -m_\mathrm{t}$. Moreover the boundary conditions at $x = \{0, L\}$ are $M_\mathrm{t} = \mathcal{M}_\mathrm{t}$ where $\mathcal{M}_\mathrm{t}$ denotes the classical torque acting on the end sections of the nanobeam.

4.1. Strong form

The differential equation and the relevant boundary conditions for the FTGEM, corresponding to the variational condition reported in Eq. (18), can be obtained by using a standard localization procedure based on the Green formula of integration by parts.

Accordingly, the nonlocal differential equation of the FTGEM is given by

$$-c_1^2 M_{0\mathrm{t}}^{(3)} + M_{0\mathrm{t}}^{(1)} = -m_\mathrm{t} + c^2 m_\mathrm{t}^{(2)} - c^2 c_1^2 m_\mathrm{t}^{(4)} \tag{19}$$

and the boundary conditions are

$$\begin{aligned}
&\text{specify } \vartheta \quad \text{or} \quad -c_1^2 M_{0\mathrm{t}}^{(2)} + M_{0\mathrm{t}} = M_\mathrm{t} + c^2 m_\mathrm{t}^{(1)} - c^2 c_1^2 m_\mathrm{t}^{(3)} \\
&\text{specify } \vartheta^{(1)} \quad \text{or} \quad c_1^2 M_{0\mathrm{t}}^{(1)} = c^2 c_1^2 m_\mathrm{t}^{(2)}
\end{aligned} \tag{20}$$

The nonlocal differential equation (19) and the relevant boundary condition (20) can be expressed in terms of the twist angle ϑ of the nanotube about the x-axis by noting that the stress resultant moment $M_{0\mathrm{t}}$ can be recovered from Eqs. (8) in the following form:

$$M_{0\mathrm{t}} = GJ\vartheta^{(1)} \tag{21}$$

Hence the nonlocal differential equilibrium equation of the FTGEM for nanobeams can be obtained by substituting (21) into (19) and (20) to get

$$-c_1^2 GJ\vartheta^{(4)} + GJ\vartheta^{(2)} = -m_\mathrm{t} + c^2 m_\mathrm{t}^{(2)} - c^2 c_1^2 m_\mathrm{t}^{(4)} \tag{22}$$

and the related boundary conditions are

$$\begin{aligned}&\text{specify } \vartheta \quad \text{or} \quad -c_1^2 GJ\vartheta^{(3)} + GJ\vartheta^{(1)} = M_\mathrm{t} + c^2 m_\mathrm{t}^{(1)} - c^2 c_1^2 m_\mathrm{t}^{(3)} \\ &\text{specify } \vartheta^{(1)} \quad \text{or} \quad c_1^2 GJ\vartheta^{(2)} = c^2 c_1^2 m_\mathrm{t}^{(2)}\end{aligned} \tag{23}$$

The expression of the torsional moment follows from an integration by parts of the nonlocal elastic equilibrium condition (18). Enforcement of the boundary condition $(20)_2$ and of Eq. (21) provide the following equalities:

$$\begin{aligned}M_\mathrm{t} &= M_{0\mathrm{t}} - c_1^2 M_{0\mathrm{t}}^{(2)} - c^2 m_\mathrm{t}^{(1)} + c^2 c_1^2 m_\mathrm{t}^{(3)} \\ &= GJ\vartheta^{(1)} - c_1^2 GJ\vartheta^{(3)} - c^2 m_\mathrm{t}^{(1)} + c^2 c_1^2 m_\mathrm{t}^{(3)}\end{aligned} \tag{24}$$

Remark 3. The FTGEM can be specialized to recover the following special cases.

- $c_1 = 0$ – TEM. The variational formulation (18) pertaining to the FTGEM degenerates to the corresponding variational formulation of the nonlocal torsional nanobeam theory based on the Eringen model (TEM) given by Eq. (11). Hence the governing differential equation (22) becomes

$$GJ\vartheta^{(2)} = -m_\mathrm{t} + c^2 m_\mathrm{t}^{(2)} \tag{25}$$

and the boundary conditions (23) are given by

$$\text{specify } \vartheta \quad \text{or } GJ\vartheta^{(1)} = M_\mathrm{t} + c^2 m_\mathrm{t}^{(1)} \tag{26}$$

The TEM is derived in Section 3 following a classical scheme based on the Eringen expression of the shear stress.

- $c = 0$ – TGM. The variational formulation (18) pertaining to the FTGEM collapses to the corresponding variational formulation of the nonlocal torsional nanobeam theory based on the gradient elasticity model (TGM) given by

$$\int_0^L M_\mathrm{t}\dot{\vartheta}^{(1)}\mathrm{d}x = \int_0^L M_{0\mathrm{t}}\dot{\vartheta}^{(1)}\mathrm{d}x + \int_0^L c_1^2 M_{0\mathrm{t}}^{(1)}\dot{\vartheta}^{(2)}\mathrm{d}x \tag{27}$$

Accordingly, the governing differential equation (22) becomes

$$-c_1^2 GJ\vartheta^{(4)} + GJ\vartheta^{(2)} = -m_\mathrm{t} \tag{28}$$

and the boundary conditions (23) are given by

$$\begin{aligned}&\text{specify } \vartheta \quad \text{or} \quad -c_1^2 GJ\vartheta^{(3)} + GJ\vartheta^{(1)} = M_\mathrm{t} \\ &\text{specify } \vartheta^{(1)} \quad \text{or} \quad c_1^2 GJ\vartheta^{(2)} = 0\end{aligned} \tag{29}$$

- $c = c_1 = 0$ – Local model. If both the parameters c and c_1 are equal to zero, the governing equation (23) and the boundary conditions (24) reduce to those of the classical (local) beam model subjected to a torsional load.

Remark 4. It is well known that the Eringen nonlocal model is free of small-scale effects under special cases, see [17,37,38] for a theoretical analysis. On the contrary, the proposed FTGEM does not suffer from these drawbacks.

5. Closed form solutions

Eq. (22) is a fourth-order non-homogeneous linear ordinary differential equation with constant coefficients that can be solved analytically. Assuming a quadratic distribution of torque per-unit-length $m_\mathrm{t}(x) = \frac{m}{L^2}x^2$, the analytical solution of Eq. (22) for the FTGEM is:

$$\vartheta(x) = -\frac{mx^4}{12GJL^2} + \frac{mx^2\left(c^2 - c_1^2\right)}{GJL^2} + \mathrm{e}^{-\frac{x}{c_1}}A_1 + \mathrm{e}^{\frac{x}{c_1}}A_2 + A_3 + xA_4 \tag{30}$$

In this equation, $A_1, \ldots, A_4$ are four constants that will be obtained by applying appropriate four boundary conditions.

In order to delineate the nonlocal torsion of a nanotube, numerical examples for nanocantilevers and fully clamped nanobeams are presented.

5.1. Nanocantilever

The fixed–free boundary conditions for the nanocantilever are provided by the four conditions:

$$\begin{cases} \vartheta(0) = 0 \\ GJ\vartheta^{(2)}(0) = c^2 m_t^{(2)}(0) \\ -c_1^2 GJ\vartheta^{(3)}(L) + GJ\vartheta^{(1)}(L) = c^2 m_t^{(1)}(L) - c^2 c_1^2 m_t^{(3)}(L) \\ GJ\vartheta^{(2)}(L) = c^2 m_t^{(2)}(L) \end{cases} \tag{31}$$

Hence the twist angle (30) becomes:

$$\begin{aligned} \vartheta_{GE}(x) = \vartheta_l(x) &+ \frac{c^2 m x^2}{GJL^2} - \frac{e^{\frac{L}{c_1}-\frac{x}{c_1}} m c_1^2}{\left(-1+e^{\frac{2L}{c_1}}\right) GJ} + \frac{e^{\frac{L}{c_1}+\frac{x}{c_1}} m c_1^2}{\left(-1+e^{\frac{2L}{c_1}}\right) GJ} \\ &- \frac{m x^2 c_1^2}{GJL^2} - \frac{2 m c_1^4}{GJL^2} - \frac{2e^{\frac{L}{c_1}-\frac{x}{c_1}} m c_1^4}{\left(-1+e^{\frac{2L}{c_1}}\right) GJL^2} + \frac{2e^{\frac{2L}{c_1}-\frac{x}{c_1}} m c_1^4}{\left(-1+e^{\frac{2L}{c_1}}\right) GJL^2} \\ &+ \frac{2e^{\frac{L}{c_1}+\frac{x}{c_1}} m c_1^4}{\left(-1+e^{\frac{2L}{c_1}}\right) GJL^2} - \frac{2e^{\frac{x}{c_1}} m c_1^4}{\left(-1+e^{\frac{2L}{c_1}}\right) GJL^2} \end{aligned} \tag{32}$$

where ϑ_l is the twist angle of the local model

$$\vartheta_l(x) = \frac{mLx}{3GJ} - \frac{mx^4}{12GJL^2} \tag{33}$$

Note that the upper bound of the nanocantilever twist angle ϑ_{GE} for $c_1 \to 0$ is provided by the twist angle ϑ_E of the TEM (see Eq. (35) below). The lower bound $\vartheta_{GE\infty}$ can be evaluated by taking the limit of ϑ_{GE} for $c_1 \to \infty$ and is given by

$$\vartheta_{GE\infty}(x) = \frac{mx\left(L^3 + 4c^2 x\right)}{4GJL^2} \tag{34}$$

Hence the twist angle ϑ_{GE} of the FTGEM belongs to the strip bounded by the functions ϑ_E and $\vartheta_{GE\infty}$. Note that the upper and lower bounds ϑ_E and $\vartheta_{GE\infty}$ are prescribed once the length-scale parameter c has been fixed.

Finally, the torsional moment (24) of the nanocantilever under the quadratic torque distribution reduces to its classical (local) counterpart $M_t(x) = \dfrac{m(L^3 - x^3)}{3L^2}$.

According to the results reported in Remark 3, the FTGEM can be specialized to the following nonlocal models.

- $c_1 = 0$ – Torsional Eringen model (TEM). The twist angle ϑ_E can be obtained from (32) by setting $c_1 = 0$

$$\vartheta_E(x) = \vartheta_l(x) + \frac{c^2 m x^2}{GJL^2} \tag{35}$$

The twist angle ϑ_E of the TEM is bounded below, i.e. for $c \to 0$, by the local twist angle ϑ_l but it is not bounded above, i.e. for $c \to \infty$.

- $c = 0$ – Torsional gradient elasticity model (TGM). The twist angle ϑ_G can be obtained from (32) by setting $c = 0$

$$\begin{aligned} \vartheta_G(x) = \vartheta_l(x) &- \frac{e^{\frac{L}{c_1}-\frac{x}{c_1}} m c_1^2}{\left(-1+e^{\frac{2L}{c_1}}\right) GJ} + \frac{e^{\frac{L}{c_1}+\frac{x}{c_1}} m c_1^2}{\left(-1+e^{\frac{2L}{c_1}}\right) GJ} \\ &- \frac{m x^2 c_1^2}{GJL^2} - \frac{2 m c_1^4}{GJL^2} - \frac{2e^{\frac{L}{c_1}-\frac{x}{c_1}} m c_1^4}{\left(-1+e^{\frac{2L}{c_1}}\right) GJL^2} + \frac{2e^{\frac{2L}{c_1}-\frac{x}{c_1}} m c_1^4}{\left(-1+e^{\frac{2L}{c_1}}\right) GJL^2} \\ &+ \frac{2e^{\frac{L}{c_1}+\frac{x}{c_1}} m c_1^4}{\left(-1+e^{\frac{2L}{c_1}}\right) GJL^2} - \frac{2e^{\frac{x}{c_1}} m c_1^4}{\left(-1+e^{\frac{2L}{c_1}}\right) GJL^2} \end{aligned} \tag{36}$$

The upper bound of ϑ_{G} is obtained by $c \to 0$ and is provided by the local twist angle. The lower bound $\vartheta_{G\infty}$ can be evaluated by taking the limit of ϑ_{G} for $c_1 \to \infty$ or, equivalently, by setting $c = 0$ in Eq. (34), and is given by

$$\vartheta_{G\infty}(x) = \frac{mLx}{4GJ} \tag{37}$$

Hence the twist angle ϑ_{G} of the TGM belong to the strip bounded by the functions ϑ_l and $\vartheta_{\mathrm{G}\infty}$. Note that the upper and lower bounds ϑ_l and $\vartheta_{G\infty}$ are independent of the length-scale parameters.

- $c = c_1 = 0$ – Classical (local) torsional model. In this case, the nonlocal model collapses to the local model and the twist angle ϑ_{GE} coincides to ϑ_l given by Eq. (33).

5.2. Fully clamped nanobeam

The fixed–fixed boundary conditions for the fully clamped nanobeam are provided by the four conditions:

$$\begin{cases} \vartheta(0) = 0 \\ GJ\vartheta^{(2)}(0) = c^2 m_{\mathrm{t}}^{(2)}(0) \\ \vartheta(L) = 0 \\ GJ\vartheta^{(2)}(L) = c^2 m_{\mathrm{t}}^{(2)}(L) \end{cases} \tag{38}$$

The twist angle (30) becomes

$$\begin{aligned} \vartheta_{GE}(x) = \vartheta_l(x) + \frac{\mathrm{e}^{-\frac{x}{c_1}} m}{\left(\mathrm{e}^{\frac{2L}{c_1}} - 1\right) GJL^2} \Bigg(& 2c_1^4 \left(\mathrm{e}^{\frac{L}{c_1}} - 1\right)\left(\mathrm{e}^{\frac{x}{c_1}} - 1\right)\left(\mathrm{e}^{\frac{x}{c_1}} - \mathrm{e}^{\frac{L}{c_1}}\right) \\ & + c^2 \mathrm{e}^{\frac{x}{c_1}} \left(\mathrm{e}^{\frac{2L}{c_1}} - 1\right) x(x-L) + c_1^2 \left(-\mathrm{e}^{\frac{L}{c_1}} L^2 + \mathrm{e}^{\frac{L}{c_1} - \frac{2x}{c_1}} L^2 + \mathrm{e}^{\frac{x}{c_1}} x^2 - \mathrm{e}^{\frac{L}{c_1} - \frac{2x}{c_1}} x^2\right)\Bigg) \end{aligned} \tag{39}$$

where the twist angle of the local model ϑ_l is

$$\vartheta_l(x) = \frac{mLx}{12GJ} - \frac{mx^4}{12GJL^2} \tag{40}$$

The upper bound of the nanocantilever twist angle ϑ_{GE} for $c_1 \to 0$ is provided by the twist angle ϑ_{E} of the TEM (see Eq. (42) below). The lower bound $\vartheta_{\mathrm{GE}\infty}$ can be evaluated by taking the limit of ϑ_{GE} for $c_1 \to \infty$ and is given by

$$\vartheta_{\mathrm{GE}\infty}(x) = -\frac{c^2 mx(L-x)}{GJL^2} \tag{41}$$

Hence the twist angle ϑ_{GE} of the FTGEM belongs to the strip bounded by the functions ϑ_{E} and $\vartheta_{\mathrm{GE}\infty}$. Note that the upper and lower bounds ϑ_E and $\vartheta_{GE\infty}$ are prescribed once the length-scale parameter c has been fixed.

According to the results reported in Remark 3, the FTGEM can be specialized to the following nonlocal models.

- $c_1 = 0$ – Torsional Eringen model (TEM). The twist angle ϑ_{E} can be obtained from (39) by setting $c_1 = 0$:

$$\vartheta_{\mathrm{E}}(x) = \vartheta_l(x) + \frac{c^2 mx^2}{GJL^2} - \frac{c^2 mx}{GJL} \tag{42}$$

The twist angle ϑ_{E} of the TEM is bounded below, i.e. for $c \to 0$, by the local twist angle ϑ_l, but it is not bounded above, i.e. for $c \to \infty$.

- $c = 0$ – Torsional gradient elasticity model (TGM). The twist angle ϑ_{G} can be obtained from (39) by setting $c = 0$:

$$\begin{aligned} \vartheta_G(x) = \vartheta_l(x) + \frac{\mathrm{e}^{-\frac{x}{c_1}} c_1^2 m}{\left(\mathrm{e}^{\frac{2L}{c_1}} - 1\right) GJL^2} \Bigg(& 2c_1^2 \left(\mathrm{e}^{\frac{L}{c_1}} - 1\right)\left(\mathrm{e}^{\frac{x}{c_1}} - 1\right)\left(\mathrm{e}^{\frac{x}{c_1}} - \mathrm{e}^{\frac{L}{c_1}}\right) \\ & - \mathrm{e}^{\frac{L}{c_1}} L^2 + \mathrm{e}^{\frac{L}{c_1} - \frac{2x}{c_1}} L^2 + \mathrm{e}^{\frac{x}{c_1}} x^2 - \mathrm{e}^{\frac{L}{c_1} - \frac{2x}{c_1}} x^2\Bigg) \end{aligned} \tag{43}$$

The upper bound of ϑ_{G} is obtained by $c_1 \to 0$ and is provided by the local twist angle. The lower bound $\vartheta_{G\infty}$ can be evaluated by taking the limit of ϑ_{G} for $c_1 \to \infty$ and is given by $\vartheta_{G\infty}(x) = 0$.

Hence the twist angle ϑ_{G} of the TGM belongs to the strip bounded by the functions ϑ_l and the vanishing function $\vartheta_{G\infty}$. Note that the upper and lower bounds ϑ_l and $\vartheta_{G\infty}$ are independent of the length-scale parameters.

- $c = c_1 = 0$ – Classical (local) torsional model. In this case, the nonlocal model collapses to the local model and the twist angle ϑ_{GE} coincides to ϑ_l given by Eq. (40).

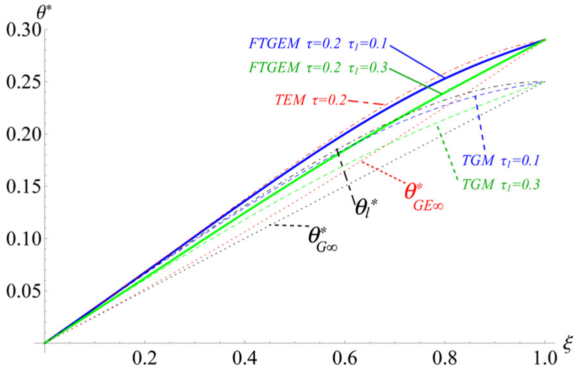

Fig. 1. (Color online.) Dimensionless twist angle $\vartheta^*_{\mathrm{GE}}$ of the FTGEM for a nanocantilever, with $\tau = 0.2$, $\tau_1 = 0.1$ and $\tau = 0.2$, $\tau_1 = 0.3$ compared with the solutions of the TEM ϑ^*_{E} with $\tau = 0.2$, TGM with $\tau_1 = 0.1$ and $\tau_1 = 0.3$ and the local behavior ϑ^*_l. Upper and lower bounds of $\vartheta^*_{\mathrm{GE}}$ are provided by ϑ^*_{E} and $\vartheta^*_{\mathrm{GE}\infty}$ with $\tau = 0.2$. Upper and lower bounds of ϑ^*_{G} are provided by ϑ^*_l and $\vartheta^*_{\mathrm{G}\infty}$.

The torsional moment (24) of the nanocantilever under the quadratic torque distribution differs from its classical (local) counterpart M_{t_l} by the term $-c^2 m/L$, so that we have:

$$M_{\mathrm{t}}(x) = \frac{m\left(L^3 - 4x^3\right)}{12L^2} - \frac{c^2 m}{L} = M_{\mathrm{t}_l}(x) - \frac{c^2 m}{L} \tag{44}$$

It is apparent that M_{t} depends only on the length-scale parameter c, since the torsional moment of the TGM coincides with the classical one M_{t_l}.

6. Examples and discussion

Let us now introduce the following dimensionless quantities

$$\xi = \frac{x}{L}, \qquad \tau = \frac{c}{L}, \qquad \tau_1 = \frac{c_1}{L}, \qquad \vartheta^* = \vartheta \frac{GJ}{mL^2} \tag{45}$$

Nanocantilever under torsional loading. The dimensionless twist angle $\vartheta^*_{\mathrm{GE}}$ obtained by the FTGEM is reported in Fig. 1 considering the following pairs of dimensionless small-scale parameters $\tau = 0.2$, $\tau_1 = 0.1$ and $\tau = 0.2$, $\tau_1 = 0.3$.

The solutions of the FTGEM turn out to be stiffer than the solution obtained by the TEM with the corresponding value of τ, i.e. $\tau = 0.2$, as shown in Fig. 1. For a given τ and for increasing τ_1, the dimensionless twist angle $\vartheta^*_{\mathrm{GE}}$ of the FTGEM decreases. Moreover, both FTGEM and TEM result to be stiffer than the TGM for the corresponding parameters $\tau_1 = 0.1$ and $\tau_1 = 0.3$. The upper and lower bounds of the twist angle $\vartheta^*_{\mathrm{GE}}$ are provided by ϑ^*_{E}, obtained by the TEM, and by $\vartheta^*_{\mathrm{GE}\infty}$ with $\tau = 0.2$.

The maximum dimensionless twist angle $\vartheta^*_l(1)$ of the local model coincides with $\vartheta^*_{\mathrm{G}\infty}(1)$ and with $\vartheta^*_{\mathrm{G}}(1)$ obtained by the FTGEM. The maximum dimensionless twist angle $\vartheta^*_{\mathrm{E}}(1)$ of the TEM coincides with $\vartheta^*_{\mathrm{GE}\infty}(1)$ and with $\vartheta^*_{\mathrm{GE}}(1)$. Moreover, $\vartheta^*_{\mathrm{E}}(1)$ of the TEM is related to the dimensionless twist angle of the local problem by the relation $\vartheta^*_{\mathrm{E}}(1) = \vartheta^*_{\mathrm{l}}(1) + \tau^2$.

The two-dimensional plot of the maximum dimensionless twist angle $\vartheta^*_{\mathrm{GE}}$, i.e. $\vartheta^*_{\mathrm{GE}}(1)$, of the FTGEM versus the length-scale parameters τ and τ_1 is reported in Fig. 2.

For a given τ, the maximum dimensionless twist angle $\vartheta^*_{\mathrm{GE}}(1)$ of the FTGEM is independent of the length-scale parameter τ_1, as shown in Fig. 3. On the contrary, if τ_1 is fixed, the maximum dimensionless twist angle $\vartheta^*_{\mathrm{GE}}(1)$ depends on the length-scale parameter τ, but it is independent of the assumed value for τ_1, see Fig. 3.

The dimensionless maximum twist angles $\vartheta^*_{\mathrm{E\,max}}$, $\vartheta^*_{\mathrm{G\,max}}$, $\vartheta^*_{\mathrm{GE\,max}}$ of the TEM, TGM and of the proposed FTGEM are reported in Table 1 for several values of the length-scale parameters τ and τ_1. The results reported in Table 1 show that the dimensionless maximum twist angle increases for increasing values of the length-scale parameters. The dimensionless twist angle field of the TEM differs from the local one, see, e.g., Fig. 1, but the dimensionless maximum twist angle $\vartheta^*_{\mathrm{G\,max}}$ coincides with the local one $\vartheta^*_{\mathrm{l\,max}}$ for any value of τ_1. As a consequence, the dimensionless twist angle fields of the TEM and FTGEM are different from each other, and, for a given τ, the dimensionless maximum twist angles $\vartheta^*_{\mathrm{E\,max}}$ and $\vartheta^*_{\mathrm{GE\,max}}$ coincide for any value of τ_1. The dimensionless maximum twist angles $\vartheta^*_{\mathrm{E\,max}} = \vartheta^*_{\mathrm{GE\,max}}$ increases for increasing τ and the TEM and FTGEM are less stiff than in the local model, as shown in Table 1.

Fully clamped nanobeam under torsional loading. The dimensionless twist angle $\vartheta^*_{\mathrm{GE}}$ obtained by the FTGEM is reported in Fig. 4 considering the following pairs of dimensionless small-scale parameters $\tau = 0.2$, $\tau_1 = 0.1$ and $\tau = 0.2$, $\tau_1 = 0.3$.

The dimensionless maximum twist angles $\vartheta^*_{\mathrm{E\,max}}$, $\vartheta^*_{\mathrm{G\,max}}$, $\vartheta^*_{\mathrm{GE\,max}}$ of the TEM, TGM and of the proposed FTGEM are different from each other and, moreover, they are attained at different values of the dimensionless nanobeam abscissa $\xi_{\max}$ depending on the values of the length-scale parameters τ and τ_1, as shown in the next Table 2.

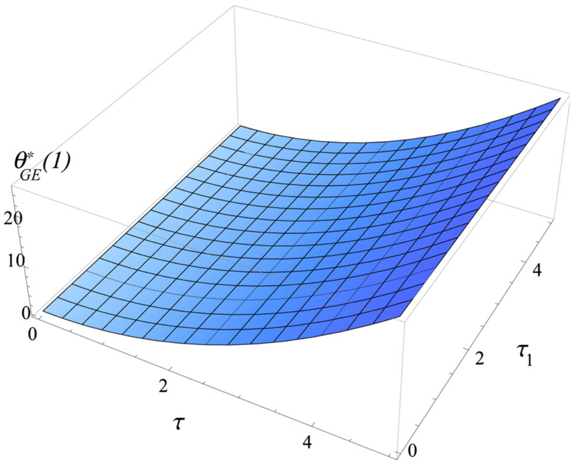

Fig. 2. (Color online.) Two-dimensional plot of the maximum dimensionless twist angle $\vartheta^*_{\mathrm{GE}}$ for the nanocantilever.

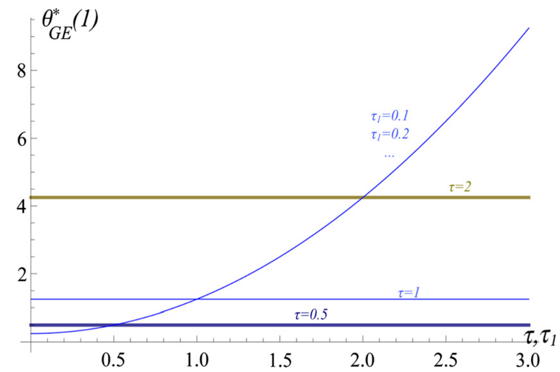

Fig. 3. (Color online.) Cutting of the two-dimensional plot of Fig. 2 for different values of $\tau \in \{0.5, 1, 2\}$ and for any value of τ_1.

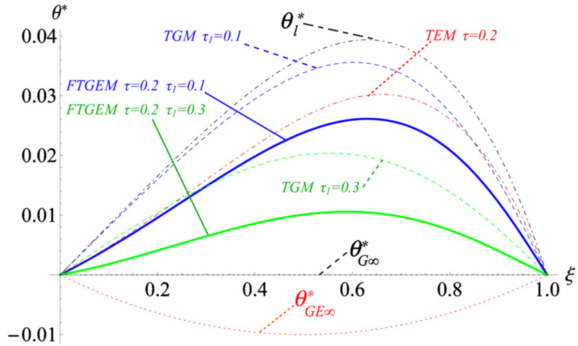

Fig. 4. Dimensionless twist angle $\vartheta^*_{\mathrm{GE}}$ of the FTGEM for a fully clamped nanobeam, with $\tau = 0.2$, $\tau_1 = 0.1$ and $\tau = 0.2$, $\tau_1 = 0.3$ compared with the solutions of the TEM ϑ^*_{E} with $\tau = 0.2$, TGM with $\tau_1 = 0.1$ and $\tau_1 = 0.3$ and the local behavior ϑ^*_{l}. Upper and lower bounds of $\vartheta^*_{\mathrm{GE}}$ are provided by ϑ^*_{E} and $\vartheta^*_{\mathrm{GE}\infty}$ with $\tau = 0.2$.

The results reported in Table 2 show that the dimensionless maximum twist angle of the considered models decreases for increasing values of the length-scale parameters. The FTGEM is the stiffest model and the TEM is the most deformable one so that, for a given values of the dimensionless length-scale parameter τ, the FTGEM is stiffer that the TEM for any value of the dimensionless length-scale parameter τ_1.

The dimensionless maximum twist angle $\vartheta^*_{\mathrm{E\,max}}$ for the TEM decreases for increasing values of the dimensionless length-scale parameter τ and the related dimensionless abscissa $\xi_{\max}$ increases. On the contrary, both the dimensionless maximum twist angle $\vartheta^*_{\mathrm{G\,max}}$ and the related dimensionless abscissa $\xi_{\max}$ for the TGM decreases for increasing values of the dimensionless length-scale parameter τ_1. For a given τ, both the dimensionless maximum twist angle $\vartheta^*_{\mathrm{GE\,max}}$ and the related dimensionless abscissa $\xi_{\max}$ for the FTGEM decreases for increasing values of the dimensionless length-scale parameter τ_1,

Table 1
Dimensionless maximum twist angles of TEM, TGM and FTGEM in terms of the length-scale parameters τ and τ_1.

Nonlocal model	τ	τ_1	$\vartheta^*_{\max}$
TEM	0.0		0.25
	0.1		0.26
	0.2		0.29
	0.3		0.34
TGM		0.0	0.25
		0.1	0.25
		0.2	0.25
		0.3	0.25
FTGEM	0.0	0.0	0.25
	0.1	0.1	0.26
		0.2	0.26
		0.3	0.26
	0.2	0.1	0.29
		0.2	0.29
		0.3	0.29
	0.3	0.1	0.34
		0.2	0.34
		0.3	0.34
	0.4	0.1	0.41
		0.2	0.41
		0.3	0.41
	0.5	0.1	0.50
		0.2	0.50
		0.3	0.50

Table 2
Dimensionless maximum twist angles of TEM, TGM and FTGEM versus the related dimensionless abscissa $\xi_{\max}$.

Nonlocal model	τ	τ_1	$\xi_{\max}$	$\vartheta^*_{\max}$
TEM	0.0		0.629956	0.0393725
	0.1		0.63678	0.0370503
	0.2		0.660843	0.0302118
TGM		0.0	0.629956	0.0393725
		0.1	0.603297	0.0355893
		0.2	0.571123	0.0277056
FTGEM	0.0	0.0	0.629956	0.0393725
	0.1	0.1	0.60924	0.0332021
		0.2	0.576656	0.0252601
	0.2	0.1	0.630967	0.0261322
		0.2	0.599173	0.0179887
	0.3	0.1	0.685607	0.014861
		0.2	0.677354	0.0063844

but, for a given τ_1, the dimensionless maximum twist angle $\vartheta^*_{\mathrm{GE}\,\max}$ for the FTGEM decreases for increasing values of the dimensionless length-scale parameter τ, and the related dimensionless abscissa $\xi_{\max}$ increases.

7. Conclusion

The elastostatic problem of nanobeams under torsion, with nonlocal constitutive behavior, proposed by Eringen (TEM) has been preliminarily investigated. It has been proven that small-scale effects can be effectively assessed by resorting to an analogy with a local nanobeam subjected to a torsional curvature distortion equivalent to the nonlocality effect. The shear strain field in an Eringen nonlocal nanobeam is thus interpreted as the sum of elastic and inelastic contributions. This observation leads naturally to the definition of the new first-gradient nonlocal model (FTGEM) formulated in Section 4, in terms of the elastic shear strain field and of its derivative along the nanobeam axis. The treatment involves the shear modulus of a linearly elastic isotropic material and two length scale parameters τ and τ_1. The ensuing equations of elastic equilibrium for the FTGEM have been consistently derived by a variational thermodynamic approach. The associated strong form has been analytically solved for nanocantilevers and fully clamped nanobeams under quadratic distributions of torques.

Further comments are briefly listed as follows.

- By virtue of the analogy enlightened between Eringen and local nanobeams under torsion, the solutions of nanobeams formulated according to the FTGEM can be detected by solving the elastic equilibrium problem of classical gradient nanobeams (TGM) under prescribed torsional distortions describing the Eringen effect.
- Variational formulations based on Eringen and gradient constitutive models (TEM and TGM) are obtained by the FTGEM for special values of the length scale parameters.
- TEM coincides with the local linearly elastic model for point or uniformly distributed torques. This pathological behavior is overcome by the new gradient model FTGEM.
- Exact solutions contributed for nanocantilevers and fully clamped nanobeams, according to the FTGEM, provide new benchmarks for computational analyses.
- For nanocantilevers, formulated according to the FTGEM, the maximum dimensionless twist angle increases by increasing τ and is independent of τ_1. The structural response is stiffer than the one associated with local model. On the contrary, the structural response of fully clamped nanobeams formulated according to the FTGEM is soften with respect to the one corresponding to the local model.

Acknowledgement

The present study has been developed within the activities of the research program FARO 2012 – Compagnia San Paolo, Polo delle Scienze e delle Tecnologie, University of Naples Federico II.

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