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Asymptotic analysis for the Kelvin–Voigt model for a thin laminate

Grigory Panasenko [a,b,*], Ruxandra Stavre [c]

[a] University of Lyon, Institute Camille Jordan UMR CNRS 5208, 23 rue du Docteur-Paul-Michelon, 42023 Saint-Étienne, France
[b] UMI CNRS 2615 Jean-Victor-Poncelet, Moscow, Russia
[c] Simion Stoilow Institute of Mathematics of the Romanian Academy, Research unit nr. 6, P.O. Box 1-764, 014700 Bucharest, Romania

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ABSTRACT

A two dimensional Kelvin–Voigt model of a visco-elastic thin stratified strip with Neumann condition at the lateral boundary is considered. The dimension reduction combined with the homogenization procedure allows us to construct a complete asymptotic expansion of the solution and to justify the limit one dimensional model containing the long-fading memory term while the initial model corresponds to the short memory.

1. Introduction

The dimension reduction of thin heterogeneous plates was considered firstly in [1–3]; in particular, in [2–6] the complete asymptotic expansions are constructed and justified. Such dimension reduction is an important tool that can be applied to the analysis of stresses and strains in thin heterogeneous structures [7,8]. We consider below the Kelvin–Voigt viscoelasticity equations set in a stratified plate. For the sake of simplicity of the presentation, we consider the two-dimensional case, although the three-dimensional plate may be treated using the same approach. Notice that the Kelvin–Voigt model was considered in the homogenization theory for composite materials in [9–11] and the so-called memory effect was discovered. In [12], this effect was obtained for the case of time-depending coefficients. The presence of the fading memory term in the homogenized model means that the homogenized model becomes nonlocal in time while the initial Kelvin–Voigt model does not contain any integral term in time. So, the Kelvin–Voigt model being the short-memory model, it generates the homogenized model that is the long-memory model. However, for thin domains, the dimension reduction of the Kelvin–Voigt model is not studied and this problem is rather different from that of the homogenization of a massive body because in the dimension reduction, one of the homogenized equations has the fourth order with respect to the space derivative, while in the “massive” case, all the homogenized equations are of the second order. We show that the analogous fading memory effect holds for the plates, in the dimension reduction. Moreover, we construct and justify the complete asymptotic expansion of the solution to this problem. This expansion generalizes N. Bakhvalov’s ansatz [13,14], applied to the elastic composite plates and rods in [4] and it contains the integral terms. A similar ansatz was introduced in [15], where it was applied to the homogenization of the long-memory visco-elasticity equations for heterogeneous media. The dimensional reduction of

* Corresponding author.
E-mail addresses: grigory.panasenko@univ-st-etienne.fr (G. Panasenko), Ruxandra.Stavre@imar.ro (R. Stavre).

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a thin visco-elastic stratified plate and the construction of the complete asymptotic expansion are the main results of the present paper.

2. Quasi-steady visco-elastic plates/rodes

Let

$$G_\varepsilon = \mathbb{R} \times (0, \varepsilon) \tag{1}$$

be a thin layer (strip) in $\mathbb{R}^2$, modeling a plate.

Consider the Kelvin–Voigt viscoelasticity equations set in this layer with the 1-periodicity condition in the variable x_1 and with Neumann condition on the boundary of the layer:

$$\begin{cases} P_\varepsilon u_\varepsilon \equiv -\displaystyle\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(B_{ij}(\frac{x_2}{\varepsilon})\frac{\partial \dot{u}_\varepsilon}{\partial x_j}\right) - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(A_{ij}(\frac{x_2}{\varepsilon})\frac{\partial u_\varepsilon}{\partial x_j}\right) = f_\varepsilon(x_1, t) \\ \displaystyle\sum_{j=1}^{2}\left(B_{2j}\frac{\partial \dot{u}_\varepsilon}{\partial x_j} + A_{2j}\frac{\partial u_\varepsilon}{\partial x_j}\right)\Bigg|_{x_2=0,\varepsilon} = 0 \\ u_\varepsilon|_{t=0} = 0 \end{cases} \tag{2}$$

where $\dot{f}$ represents the time derivative of a function f.

Here the coefficients A_{ij}, B_{ij} are 2×2-matrix-valued functions depending on the transversal variable only and having the following form:

$$A_{11} = \begin{pmatrix} \lambda + 2\mu & 0 \\ 0 & \mu \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} 0 & \mu \\ \lambda & 0 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} \mu & 0 \\ 0 & \lambda + 2\mu \end{pmatrix}$$

$$B_{11} = \begin{pmatrix} \hat{\lambda} + 2\hat{\mu} & 0 \\ 0 & \hat{\mu} \end{pmatrix}, \quad B_{12} = \begin{pmatrix} 0 & \hat{\lambda} \\ \hat{\mu} & 0 \end{pmatrix}$$

$$B_{21} = \begin{pmatrix} 0 & \hat{\mu} \\ \hat{\lambda} & 0 \end{pmatrix}, \quad B_{22} = \begin{pmatrix} \hat{\mu} & 0 \\ 0 & \hat{\lambda} + 2\hat{\mu} \end{pmatrix}$$

where λ, μ, $\hat{\lambda}$, $\hat{\mu}$, are piece-wise smooth positive functions of $\xi_2 = \frac{x_2}{\varepsilon}$; namely, there exist positive numbers $\xi^1 < \ldots < \xi^N < 1$, such that $\lambda, \mu, \hat{\lambda}, \hat{\mu} \in C^1([\xi^i, \xi^{i+1}])$ for all $i = 0, \ldots, N$ ($\xi^0 = 0, \xi^{N+1} = 1$) and there exists a positive constant κ such that $\lambda, \mu, \hat{\lambda}, \hat{\mu} \geq \kappa$. The unknown u_ε is a two-dimensional vector-valued function, as well as the right-hand side f_ε. The right-hand side depends on the longitudinal space variable x_1 and on the time variable t and is scaled as follows:

$$f_\varepsilon(x_1, t) = \begin{pmatrix} \varepsilon f_1(x_1, t) \\ \varepsilon^2 f_2(x_1, t) \end{pmatrix}$$

where f_1 and f_2 are independent of ε $C^\infty(\mathbb{R} \times [0, T])$-smooth functions, 1-periodic in x_1, and such that there exists a positive number t^*, such that $f_j(\cdot, t) = 0$ for $t < t^*$. Assume that for all $t \in [0, T]$, $< f_j(\cdot, t) >= 0$ where $< \cdot >= \int_0^1 \cdot dx$ is the average over the period.

Problem (2) simulates the viscoelastic deformation of a thin stratified plate under a periodic in x_1 mass force; ε is the ratio of the thickness of the plate to the longitudinal period of the applied force and is a small parameter.

Denote by G_ε^1 the rectangle $(0, 1) \times (0, \varepsilon)$. Denote by $H^1_{\mathrm{per}}(G_\varepsilon^1)$ the space of functions defined on G_ε, 1-periodic in x_1 and such that its restriction to any rectangle $(a, b) \times (0, \varepsilon)$ belongs to $H^1((a, b) \times (0, \varepsilon))$. It is supplied by the inner product of the space $H^1(G_\varepsilon^1)$.

The weak solution to problem (2) is defined as a two-dimensional vector-valued function u_ε with $u_\varepsilon \in H^1(0, T; (H^1_{\mathrm{per}}(G_\varepsilon^1)^2)$, satisfying initial condition $(2)_3$, such that for any test function $v \in (H^1_{\mathrm{per}}(G_\varepsilon^1))^2$, for all $t \in (0, T)$ the integral identity holds:

$$\int\limits_{G_\varepsilon^1} \sum_{i,j=1}^{2} \left(B_{ij}\left(\frac{x_2}{\varepsilon}\right)\frac{\partial \dot{u}_\varepsilon}{\partial x_j} + A_{ij}\left(\frac{x_2}{\varepsilon}\right)\frac{\partial u_\varepsilon}{\partial x_j}\right)\frac{\partial v}{\partial x_i}\mathrm{d}x = \int\limits_{G_\varepsilon^1} f_\varepsilon v\,\mathrm{d}x \tag{3}$$

Theorem 1. *Let the right-hand side satisfy the above conditions. There exists a unique solution to (3) satisfying condition*

$$< u_\varepsilon(\cdot,t) >= 0 \tag{4}$$

The proof is given by the Galerkin method.

The complete asymptotic expansion of the solution to problem (2) is constructed below. The error of the asymptotic approximations in the form of partial sums of the expansion is evaluated. As in the case of the homogenization of the composite materials, in the case of the dimension reduction for a stratified plate, the long-memory terms in the homogenized equation will be obtained, i.e. the homogenized model contains the terms of the form

$$-\frac{\partial}{\partial x_i}\int\limits_0^t K_{ij}^V(t-t')\frac{\partial \dot{u}}{\partial x_j}(x,t')\,\mathrm{d}t', \quad -\frac{\partial}{\partial x_i}\int\limits_0^t K_{ij}^E(t-t')\frac{\partial u}{\partial x_j}(x,t')\,\mathrm{d}t'$$

3. Complete asymptotic expansion of the solution

The asymptotic approximation of order J is sought in the form

$$u_\varepsilon^{(J)}(x,t) = v_\varepsilon^{(J)}(x_1,t) + \sum_{l=1}^{J}\varepsilon^l\int\limits_0^t \Big(N_l^V(\frac{x_2}{\varepsilon},t-t')D^l\dot{v}_\varepsilon^{(J)}(x_1,t')$$
$$+ N_l^E(\frac{x_2}{\varepsilon},t-t')D^l v_\varepsilon^{(J)}(x_1,t')\Big)\,\mathrm{d}t' \tag{5}$$

where $D^l = \partial^l/\partial x_1^l$, 2×2 matrix-valued functions N_l^V, N_l^E are described below, $v_\varepsilon^{(J)}$ is a two-dimensional vector function, such that its two components have the following expansion with respect to ε:

$$v_{\varepsilon,1}^{(J)} = \sum_{j=0}^{J}\varepsilon^{j+1}v_{j,1}(x_1,t)\ , \quad v_{\varepsilon,2}^{(J)} = \sum_{j=0}^{J}\varepsilon^{j}v_{j,2}(x_1,t)$$

and functions $v_{j,1}, v_{j,2}$ do not depend on the small parameter and will be defined below.

This ansatz generalizes N. Bakhvalov's ansatz [13,14], applied to the elastic composite plates and rods in [4]. Ansatz (5) contains the integral terms.

Substituting the ansatz (5) into the equation and the boundary condition (2), taking together the terms of the same order with respect to ε, and letting constant the coefficients of the derivatives $D^l\dot{v}_\varepsilon^{(J)}$ and $D^l v_\varepsilon^{(J)}$, we get as in [4] equations for the matrix-valued coefficients N_l^V, N_l^E. Define first the right hand side functions of these equations and the boundary conditions. Denote

$$\tilde{F}_l^V(\xi_2,s) = B_{12}\frac{\partial^2 N_{l-1}^V}{\partial s\partial\xi_2} + A_{12}\frac{\partial N_{l-1}^V}{\partial\xi_2} + B_{11}\frac{\partial N_{l-2}^V}{\partial s} + A_{11}N_{l-2}^V$$
$$F_l^V(\xi_2,s) = \tilde{F}_l^V(\xi_2,s) + \frac{\partial}{\partial\xi_2}\left(B_{21}\frac{\partial N_{l-1}^V}{\partial s} + A_{21}N_{l-1}^V\right)$$
$$\tilde{F}_l^E(\xi_2,s) = B_{12}\frac{\partial^2 N_{l-1}^E}{\partial s\partial\xi_2} + A_{12}\frac{\partial N_{l-1}^E}{\partial\xi_2} + B_{11}\frac{\partial N_{l-2}^E}{\partial s} + A_{11}N_{l-2}^E$$
$$F_l^E(\xi_2,s) = \tilde{F}_l^E(\xi_2,s) + \frac{\partial}{\partial\xi_2}\left(B_{21}\frac{\partial N_{l-1}^E}{\partial s} + A_{21}N_{l-1}^E\right)$$

and

$$\tilde{G}_l^V(\xi_2) = B_{12}\frac{\partial N_{l-1}^V}{\partial\xi_2}(\xi_2,0) + B_{11}N_{l-2}^V(\xi_2,0) + B_{11}\delta_{l2}$$
$$G_l^V(\xi_2) = \tilde{G}_l^V(\xi_2) + \frac{\partial}{\partial\xi_2}\Big(B_{21}(N_{l-1}^V(\xi_2,0) + I_2\delta_{l1})\Big)$$

$$\tilde{G}_l^E(\xi_2) = B_{12}\frac{\partial N_{l-1}^E}{\partial \xi_2}(\xi_2,0) + B_{11}N_{l-2}^E(\xi_2,0) + A_{11}\delta_{l2}$$

$$G_l^E(\xi_2) = \tilde{G}_l^E(\xi_2) + \frac{\partial}{\partial \xi_2}\left(B_{21}N_{l-1}^E(\xi_2,0) + A_{21}\delta_{l1}\right)$$

and consider the following boundary value problems for matrices $N_l^V,\ N_l^E$:

$$\begin{cases} -\dfrac{\partial}{\partial \xi_2}\left(B_{22}\dfrac{\partial^2 N_l^V}{\partial \xi_2 \partial s}\right) - \dfrac{\partial}{\partial \xi_2}\left(A_{22}\dfrac{\partial N_l^V}{\partial \xi_2}\right) = F_l^V - \left\langle \tilde{F}_l^V \right\rangle, \xi_2 \in (0,1) \\ B_{22}\dfrac{\partial^2 N_l^V}{\partial \xi_2 ds} + A_{22}\dfrac{\partial N_l^V}{\partial \xi_2} = -B_{21}\dfrac{\partial N_{l-1}^V}{\partial s} - A_{21}N_{l-1}^V\,, \ \ \xi_2 = 0,1 \\ -\dfrac{\partial}{\partial \xi_2}\left(B_{22}\dfrac{\partial N_l^V}{\partial \xi_2}(\xi_2,0)\right) = G_l^V - \left\langle \tilde{G}_l^V \right\rangle, \ \ \xi_2 \in (0,1) \\ B_{22}\dfrac{\partial N_l^V}{\partial \xi_2}(\xi_2,0) = -B_{21}(N_{l-1}^V(\xi_2,0) + I_2\delta_{l1})\,, \ \ \xi_2 = 0,1 \\ \left\langle N_l^V(\cdot,s)\right\rangle = 0 \end{cases} \tag{6}$$

and

$$\begin{cases} -\dfrac{\partial}{\partial \xi_2}\left(B_{22}\dfrac{\partial^2 N_l^E}{\partial \xi_2 \partial s}\right) - \dfrac{\partial}{\partial \xi_2}\left(A_{22}\dfrac{\partial N_l^E}{\partial \xi_2}\right) = F_l^E - \left\langle \tilde{F}_l^E \right\rangle, \xi_2 \in (0,1) \\ B_{22}\dfrac{\partial^2 N_l^E}{\partial \xi_2 ds} + A_{22}\dfrac{\partial N_l^E}{\partial \xi_2} = -B_{21}\dfrac{\partial N_{l-1}^E}{\partial s} - A_{21}N_{l-1}^E\,, \ \ \xi_2 = 0,1 \\ -\dfrac{\partial}{\partial \xi_2}\left(B_{22}\dfrac{\partial N_l^E}{\partial \xi_2}(\xi_2,0)\right) = G_l^E - \left\langle \tilde{G}_l^E \right\rangle, \ \ \xi_2 \in (0,1) \\ B_{22}\dfrac{\partial N_l^E}{\partial \xi_2}(\xi_2,0) = -B_{21}N_{l-1}^E(\xi_2,0) - A_{21}\delta_{l1}\,, \ \ \xi_2 = 0,1 \\ \left\langle N_l^E(\cdot,s)\right\rangle = 0 \end{cases} \tag{7}$$

Note that these problems are non-steady and non-local with respect to the variable s, and that the initial conditions are given by boundary value problems for the ordinary differential equations $(6)_{3,4}$ and $(7)_{3,4}$. These problems can be solved analytically and so there exist the unique solutions N_l^V and N_l^E. Then the result of substitution of the ansatz into the equation has the form

$$\begin{aligned} P_\varepsilon u_\varepsilon^{(J)} = &-\sum_{l=1}^{J}\varepsilon^{l-2}\int_0^t \Big(\left\langle \tilde{F}_l^V\right\rangle(t-t')D^l\dot{v}_\varepsilon^{(J)}(x_1,t') \\ &+ \left\langle \tilde{F}_l^E\right\rangle(t-t')D^l v_\varepsilon^{(J)}(x_1,t')\Big)\,\mathrm{d}t' - \sum_{l=1}^{J}\varepsilon^{l-2}\Big(\left\langle \tilde{G}_l^V\right\rangle D^l\dot{v}_\varepsilon^{(J)}(x_1,t) \\ &+ \left\langle \tilde{G}_l^E\right\rangle D^l v_\varepsilon^{(J)}(x_1,t)\Big) + r_\varepsilon^{(J)} \end{aligned} \tag{8}$$

where the residual $r_\varepsilon^{(J)}$ can be evaluated and its order is $O(\varepsilon^{J-1}\sqrt{\varepsilon})$ in the norm $L^2(G_\varepsilon^1)$. Calculating the first coefficients, we get the leading term of Eq. $(2)_1$ after the substitution of the ansatz (the higher order terms are replaced by ...):

$$
\begin{cases}
-\varepsilon\Bigg(\hat{E}^V D^2\dot{v}_{01}+\hat{E}^E D^2 v_{01}+\hat{\hat{E}}^V D^3\dot{v}_{02}+\hat{\hat{E}}^E D^3 v_{02} \\
+\displaystyle\int_0^t \Big(\Big\langle(\tilde{F}_2^V)_{11}\Big\rangle(t-t')D^2\dot{v}_{01}(t')+\Big\langle(\tilde{F}_2^E)_{11}\Big\rangle D^2 v_{01} \\
+\Big\langle(\tilde{F}_3^V)_{12}\Big\rangle D^3\dot{v}_{02}+\Big\langle(\tilde{F}_3^E)_{12}\Big\rangle D^3 v_{02}\Big)\,\mathrm{d}t'\Bigg) \\
+\ldots=\varepsilon f_1 \\
-\varepsilon^2\Bigg(\hat{\hat{\hat{E}}}^V D^3\dot{v}_{01}+\hat{\hat{\hat{E}}}^E D^3 v_{01}+\hat{J}^V D^4\dot{v}_{02}+\hat{J}^E D^4 v_{02} \\
+\displaystyle\int_0^t \Big(\Big\langle(\tilde{F}_3^V)_{21}\Big\rangle D^3\dot{v}_{01}+\Big\langle(\tilde{F}_3^E)_{21}\Big\rangle D^3 v_{01} \\
+\Big\langle(\tilde{F}_4^V)_{22}\Big\rangle D^4\dot{v}_{02}+\Big\langle(\tilde{F}_4^E)_{22}\Big\rangle D^4 v_{02}\Big)\,\mathrm{d}t'\Bigg) \\
+\ldots=\varepsilon^2 f_2
\end{cases}
\tag{9}
$$

The coefficients $\hat{E}^V, \hat{E}^E, \widehat{E^V}, \widehat{E^E}, \widehat{\widehat{E^V}}, \widehat{\widehat{E^E}}, \hat{J}^V, \hat{J}^E$ have the following expressions

$$
\begin{cases}
\hat{E}^V=\left\langle\dfrac{4\hat{\mu}(\hat{\lambda}+\hat{\mu})}{\hat{\lambda}+2\hat{\mu}}\right\rangle,\ \hat{E}^E=\left\langle\dfrac{(\lambda+2\mu)(\hat{\lambda}+2\hat{\mu})-\lambda\hat{\lambda}}{\hat{\lambda}+2\hat{\mu}}\right\rangle \\
\widehat{E^V}=\left\langle\dfrac{4\hat{\mu}(\hat{\lambda}+\hat{\mu})}{\hat{\lambda}+2\hat{\mu}}\left(\dfrac{1}{2}-\xi_2\right)\right\rangle,\ \widehat{E^E}=\left\langle\dfrac{4\hat{\mu}(\hat{\lambda}+\hat{\mu})}{\hat{\lambda}+2\hat{\mu}}\underline{\left(\dfrac{\mu}{\hat{\mu}}\right)}\right\rangle \\
\widehat{\widehat{E^V}}=\left\langle\overline{\dfrac{4\hat{\mu}(\hat{\lambda}+\hat{\mu})}{\hat{\lambda}+2\hat{\mu}}}\right\rangle,\ \widehat{\widehat{E^E}}=\left\langle\overline{\dfrac{(\lambda+2\mu)(\hat{\lambda}+2\hat{\mu})-\lambda\hat{\lambda}}{\hat{\lambda}+2\hat{\mu}}}\right\rangle \\
\hat{J}^V=\left\langle\overline{\dfrac{4\hat{\mu}(\hat{\lambda}+\hat{\mu})}{\hat{\lambda}+2\hat{\mu}}\left(\dfrac{1}{2}-\xi_2\right)}\right\rangle,\ \hat{J}^E=\left\langle\overline{\dfrac{4\hat{\mu}(\hat{\lambda}+\hat{\mu})}{\hat{\lambda}+2\hat{\mu}}\underline{\left(\dfrac{\mu}{\hat{\mu}}\right)}}\right\rangle
\end{cases}
\tag{10}
$$

where we used the notation $\overline{F}(\xi_2)=\xi_2\langle F\rangle-\int_0^{\xi_2}F(\theta)\mathrm{d}\theta$, $\underline{F}(\xi_2)=\left\langle\int_0^{\alpha}F(\theta)\mathrm{d}\theta\right\rangle-\int_0^{\xi_2}F(\theta)\mathrm{d}\theta$.

Divide the first equation by ε, the second by ε^2. Denote $\hat{P}$ the operator in the left-hand side:

$$
\begin{cases}
(\hat{P}v_0)_1=-\Bigg(\hat{E}^V D^2\dot{v}_{01}+\hat{E}^E D^2 v_{01}+\hat{\hat{E}}^V D^3\dot{v}_{02}+\hat{\hat{E}}^E D^3 v_{02} \\
+\displaystyle\int_0^t \Big(\Big\langle(\tilde{F}_2^V)_{11}\Big\rangle(t-t')D^2\dot{v}_{01}(t')+\Big\langle(\tilde{F}_2^E)_{11}\Big\rangle D^2 v_{01} \\
+\Big\langle(\tilde{F}_3^V)_{12}\Big\rangle D^3\dot{v}_{02}+\Big\langle(\tilde{F}_3^E)_{12}\Big\rangle D^3 v_{02}\Big)\,\mathrm{d}t'\Bigg) \\
(\hat{P}v_0)_2=-\Bigg(\hat{\hat{\hat{E}}}^V D^3\dot{v}_{01}+\hat{\hat{\hat{E}}}^E D^3 v_{01}+\hat{J}^V D^4\dot{v}_{02}+\hat{J}^E D^4 v_{02} \\
+\displaystyle\int_0^t \Big(\Big\langle(\tilde{F}_3^V)_{21}\Big\rangle D^3\dot{v}_{01}+\Big\langle(\tilde{F}_3^E)_{21}\Big\rangle D^3 v_{01} \\
+\Big\langle(\tilde{F}_4^V)_{22}\Big\rangle D^4\dot{v}_{02}+\Big\langle(\tilde{F}_4^E)_{22}\Big\rangle D^4 v_{02}\Big)\,\mathrm{d}t'\Bigg)
\end{cases}
\tag{11}
$$

Inserting next the ansatz into (9), we get a recurrent chain of 1-periodic in x_1 problems determining all vectors v_j with the components v_{j1} and v_{j2}. Namely, for $j=0$ we get the leading homogenized equation:

$$
\begin{cases}
\hat{P}v_0=\begin{pmatrix} f_1 \\ f_2 \end{pmatrix},\ x_1\in\mathbb{R},\ t>0 \\
v_0|_{t=0}=0, \qquad <v_0>=0
\end{cases}
\tag{12}
$$

For v_j we get

$$\begin{cases} \hat{P} v_j = \mathcal{F}_j(x_1, t)\,, \ x_1 \in \mathbb{R}\,, \ \ t > 0 \\ v_j|_{t=0} = 0\,, \qquad\quad < v_j >= 0 \end{cases} \tag{13}$$

where the right-hand side $\mathcal{F}_j$ depends on the values of functions $v_0, \ldots, v_{j-1}$ and their derivatives. The last two conditions in problems (13), $j = 0, \ldots, J$, are generated by conditions $(2)_3$, (4).

By induction, we prove that

$$< \mathcal{F}_j >= 0$$

so that every Eq. (13), $j = 0, \ldots, J$ has a unique solution. The existence and uniqueness are proved using a version of the fixed-point theorem on the reiterate integral operator (see [16]). This idea was used as well to the long-memory viscoelastic equations in a bounded domain in [17].

Then using an a priori estimate for problem (2) we get the theorem below.

Theorem 2. *The following estimate holds*

$$\|u_\varepsilon - u_\varepsilon^{(J)}\|_{H^1((0,T)\times G_\varepsilon^1)} \le C\varepsilon^J$$

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