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Interfacial instabilities of two-layer plane Poiseuille flows of viscoelastic fluids

Instabilités interfaciales de l'écoulement de Poiseuille plan de deux couches de fluides viscoélastiques

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ABSTRACT

The linear stability of a two-layer flow of viscoelastic fluids on an inclined channel is examined. A simplified model is derived by using a weighted residual method combined with a long-wavelength expansion. A set of two coupled evolution equations for two fields, the film thickness $h(x,t)$ and the flow rate $q(x,t)$, is obtained. Then, the correct threshold of instability of the interface is characterized analytically. The effect of the elasticity stratification on the stability of the interface is considered within the long-wavelength assumption.

RÉSUMÉ

Nous présentons une étude de stabilité linéaire de deux couches de fluides viscoélastiques s'écoulant par gravité dans un canal incliné. Un modèle simplifié basé sur une méthode aux résidus pondérés et une approximation polynomiale du champ des vitesses est développé. Il s'agit d'un système de deux équations couplées, décrivant l'évolution de l'interface $h(x,t)$ et du débit local $q(x,t)$. Le modèle permet de prédire le seuil de l'instabilité de façon précise. L'effet de la différence d'élasticité sur la stabilité de l'interface est examiné dans le cadre de l'approximation des grandes longueurs d'onde.

1. Introduction

The interfacial instability between two fluid layers has been considered by numerous authors, for both Newtonian and viscoelastic liquids [1,2]. These studies are motivated by their various applications, in the field of engineering such as coating processes or natural phenomena and by their significant fundamental interest. Most of these studies in the literature on this topic examined the effects of viscosity, depth and elasticity ratio in the case of non-Newtonian fluids on the interfacial instability by using asymptotic techniques for long-wave [3] and numerical analysis of the Orr–Sommerfeld equations. Such

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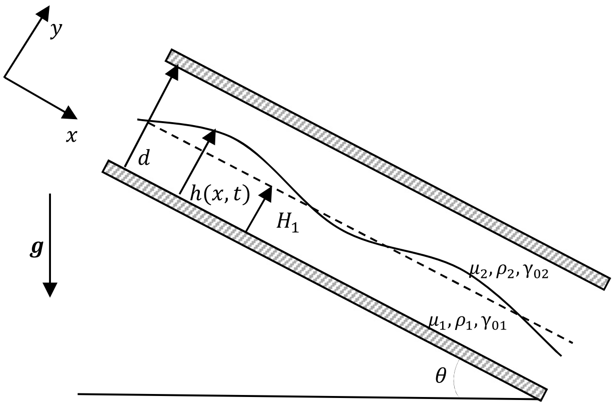

Fig. 1. Configuration of the studied flow.

studies may lead to an unstable viscoelastic flow in the condition for which the Newtonian fluids with the same viscosity ratio would be stable. Note that less theoretical work is devoted to non-Newtonian multilayered films in the framework of thin-film approximation. The aim of this paper is to formulate a low-dimensional model that is consistent up to first order in the long-wave parameter ε. We focus on the long-wavelength instability in a two non-Newtonian viscoelastic fluids of "Walters' liquid B" bounded by an upper and a lower rigid plates. The procedure followed here is an extension of the methodology applied in the case of Newtonian flows by Ruyer-Quil and Manneville [4] for falling thin film and Amaouche et al. [5]. It is based on a weighted residual integral method with polynomial expansions for velocity fields. The proposed model corrects the deficiencies of the Shkadov IBL model [6] and hence is appropriate to accurately predict the behavior of the flow close to criticality and to describe the physical interfacial motion far from the linear stability threshold. These features can be found in the case of one layer of viscoelastic fluid flowing down an inclined plane obtained in our previous work [7].

2. Governing equations

We consider the superposed flow of two viscoelastic fluids through a parallel plate channel as shown in Fig. 1. The two walls of the channel are inclined, making an angle θ with the horizontal. Each layer $k = 1, 2$ has viscosity μ_k, density ρ_k, and relaxation time γ_{0k}. The interface between the two fluids is defined by the surface $y = h(x, t)$ and has a surface tension σ. The flow is assumed to be immiscible and incompressible.

The complete Navier–Stokes and continuity equations for each fluid layer k are:

$$\rho_k\left(\frac{\partial \boldsymbol{u}_k}{\partial t} + \boldsymbol{u}_k.\nabla \boldsymbol{u}_k\right) = \mathbf{div}\bar{\bar{\boldsymbol{\tau}}}_{\boldsymbol{k}} + \rho_k \boldsymbol{g} \tag{1}$$

$$\nabla \cdot \boldsymbol{u}_k = 0 \tag{2}$$

The constitutive equation governing the "Walters' liquid B" fluid model are given by

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij} - 2\mu\gamma_0\left(\frac{\partial e_{ij}}{\partial t} + u_k\frac{\partial e_{ij}}{\partial x_k} - \frac{\partial u_j}{\partial x_k}e_{ik} - \frac{\partial u_i}{\partial x_k}e_{kj}\right) \tag{3}$$

where $\bar{\bar{\tau}}$ is the shear stress tensor and $\bar{\bar{e}}$ is the rate of strain tensor.

The immiscibility condition can be written at interface $h(x, t)$

$$v_k = h_t + u_k h_x \tag{4}$$

Boundary and interfacial conditions (i.e., no slip at the solid walls, continuity of velocity shear, and normal stresses across the interface):

$$u_1(0) = v_1(0) = 0 \tag{5}$$

$$u_2(d) = v_2(d) = 0 \tag{6}$$

$$[\mathbf{u}] = 0 \tag{7}$$

$$\left[\boldsymbol{t}(\bar{\bar{\tau}}\mathbf{n})\right] = 0 \tag{8}$$

$$\left[\mathbf{n}(\bar{\bar{\tau}}\mathbf{n})\right] = \sigma(\nabla.\mathbf{n}) \tag{9}$$

where the symbol $[\mathrm{f}] = \mathrm{f}_2 - \mathrm{f}_1$ is the jump of the function f at the interface described by the equation $y - h(x, t) = 0$ and σ denote the surface tension coefficient.

$\boldsymbol{t}$ and $\mathbf{n}$ represent the unit tangent and the unit outward normal vectors respectively, at the interface.

In the preceding equations, $\boldsymbol{u_k} = (u_k, v_k)$, $\boldsymbol{g}$ and p refer to the velocity, the gravitational acceleration, and the pressure fields.

For plane Poiseuille flow, the velocity scale is the depth-averaged velocity $U_m = Q_0/d$ and the corresponding velocity fields are given by solving the differential system (1)–(9):

$$U_1(y) = \alpha\left[(1 + r\beta)\left(\frac{y}{H_1}\right) - \left(\frac{y}{H_1}\right)^2\right], \qquad U_2(y) = \alpha r^2 \frac{\rho}{\mu}\left[\left(1 + \frac{\mu}{\rho r}\beta\right)\left(\frac{d-y}{d-H_1}\right) - \left(\frac{d-y}{d-H_1}\right)^2\right] \tag{10}$$

where d is the total height, $H_2 = d - H_1$ is the upper fluid thickness, $r = H_2/H_1$, $\mu = \mu_2/\mu_1$, $\rho = \rho_2/\rho_1$, $\alpha = \frac{g\rho_1 \sin\theta}{2\mu_1} H_1^2$ and $\beta = (1 + \rho r)/(r + \mu)$. We note that the total flow rate is $Q_0 = \alpha H_1/2\{r\beta(1 + r) + \frac{1}{3}(1 + (r)^3\rho/\mu)\}$ and the basic Poiseuille solution corresponds to flat interface ($h(x,t) = H_1$) and to a parabolic velocity profile whose coefficients depend on viscosity, density, and thickness ratio.

The stability of the Poiseuille flow will be considered here within the long-wave assumption.

In order to non-dimensionalize the governing equations (1)–(9), the following transformations are used:

$$u^* = u/U_m, \qquad x^* = x/\lambda, \qquad y^* = y/d, \qquad v^* = v\lambda/dU_m, \qquad t^* = tU_m/\lambda, \qquad p^* = pd^2/U_m\lambda\mu_1$$

where λ is the characteristic wavelength of the perturbation. The channel height is normalized to 1.

The dimensionless groups that appear are the Reynolds number $R = Q_0\rho_1/\mu_1$, the capillary number $W = \sigma/\mu_1 U_m$, the viscoelastic parameters $\Gamma_1 = \gamma_{01}\mu_1/\rho_1 d$, $\Gamma_2 = \gamma_{02}\mu_2/\rho_2 d$ and the long-wave parameter $\varepsilon = d/\lambda$.

Assuming that $\varepsilon = d/\lambda \ll 1$ allows us to simplify the dimensionless governing equations and the corresponding boundary conditions. We have dropped all the terms of order higher than ε^2 and we have kept the main contribution of surface tension, which is also formally of $\mathcal{O}(\varepsilon^3)$. It is assumed that the surface tension is large; a convenient order of magnitude assignment is $W = \mathcal{O}(\varepsilon^{-2})$. This term is related to capillary forces that prevents the waves from breaking and hence ensures a slow time and space modulation of the basic flat solution (10). We consider in the following the case of the weak elastic properties of the viscoelastic fluids, which ensures that the viscoelastic parameters Γ_1, Γ_2 are retained in the $\mathcal{O}(\varepsilon)$ approximation.

3. Method of solution

The procedure followed here is an extension of the methodology applied in the case of Newtonian flows (see Amaouche et al. [5]). The basic idea is to separate the variables and to expand the velocity field in each layer on a set of test functions which satisfies the boundary conditions (5)–(6) defining the velocity at the walls, we write:

$$u_1(x,y,t) = a_0(x,t)\eta + \sum_{k=1}^{K} a_k(x,t) f_{k-1}(\eta) \tag{11}$$

$$u_2(x,y,t) = b_0(x,t)\xi + \sum_{k=1}^{K} b_k(x,t) f_{k-1}(\xi) \tag{12}$$

where $\eta = \frac{y}{h(x,t)}$, $\xi = \frac{1-y}{1-h(x,t)}$ and $f_{k-1}(\eta) = \eta^k - \frac{k}{k+1}\eta^{k+1}$

In order to obtain a reduced model, the streamwise velocity components are expanded in power of ε as

$$u_i(x,y,t) = u_i^{(0)} + \varepsilon u_i^{(1)} + O\left(\varepsilon^2\right) \quad i = 1, 2 \tag{13}$$

where leading-order solutions $u_1^{(0)}$ and $u_2^{(0)}$ are of degree 2 in y and $u_i^{(1)}$ are of degree 6. Our analysis is based on the assumption of slow time and space modulations of the basic-state solution that corresponds to a parabolic velocity profile (10). Note that Eqs. (11) and (12) satisfy identically the continuity equation and the no-slip conditions (5) and (6). Integration of y-momentum equations provide the pressure field in the upper and the lower fluids and using the dynamic condition at the interface, this gives $p_1(x,y,t)$ and $p_2(x,y,t)$. Substituting the pressure field in the x-momentum equations yields the boundary-layer equation in each layer.

Now, we apply the weighted residual approach developed by Amaouche et al. [5] for Newtonian fluids. A simplified equation is obtained by vanishing residual that corresponds to suitable weight functions $g_1(x,y,t)$ and $g_2(x,y,t)$:

$$\int_0^h \Big[-R\varepsilon\frac{Du_1}{Dt} + u_{1yy} - \mu u_{2yy}|_{y=1} + (1-\rho)G\{1 - \varepsilon\cot\theta h_x\} + W\varepsilon^3 h_{xxx} + \varepsilon^2\big\{2u_{1xx} + (u_{1x}|_h - \mu u_{2x}|_h)_x\big\}$$
$$- \Gamma_1 R\varepsilon(u_{1yyt} + u_1 u_{1yyx} + v_1 u_{1yyy} - u_{1y}u_{1xy} - v_{1y}u_{1yy}) + \Gamma_2 R\varepsilon\big\{(u_{2yyt} - u_{2y}u_{2xy})|_{y=1}\big\}\Big] g_1 \mathrm{d}y$$

$$+\int_h^1 \left[-R\rho\varepsilon\frac{Du_2}{Dt} + \mu\left(u_{2yy} - u_{2yy}|_{y=1} + 2\varepsilon^2 u_{2xx}\right) + \Gamma_2 R\varepsilon\left\{(u_{2yyt} - u_{2y}u_{2xy})|_{y=1}\right\}\right.$$
$$\left. -\Gamma_2 R\varepsilon(u_{2yyt} + u_2 u_{2yyx} + v_2 u_{2yyy} - u_{2y}u_{2xy} - v_{2y}u_{2yy})\right] g_2 \mathrm{d}y = 0 \qquad (14)$$

where $G = 2\alpha H_1(1+r)^3/Q_0$.

A model consistent at first order in ε can be formulated in terms of two coupled spatiotemporal evolution equations for the thickness h and the local flow rate q. One can eliminate the explicit effects of the corrections $u_i^{(1)}$ with a suitable choice of the weight functions g_i. Thus (14) is simplified and written formally in the form

$$F_0(q,h) + \varepsilon R\left\{F_1(q,h)q_t + \tilde{F}(\partial_x, q, h)\right\} + \varepsilon^2 F_2(\partial_{xx}, q, h) = 0 \qquad (15)$$

Integrating the continuity Eq. (2) in layer 1 and using the kinematic condition (4), one obtains

$$q_x + h_t = 0 \qquad (16)$$

where $q(x,t) = \int_0^{h(x,t)} u_1(x,y,t)\mathrm{d}y$ is the flow rate in layer 1.

Eqs. (15) and (16) are two nonlinear partial differential equations describing the temporal and spatial evolution of thickness h and local flow rate q. It is worth mentioning that setting $\Gamma_1 = \Gamma_2 = 0$ in Eq. (15) now corresponds exactly to Eq. (36) truncated at first order in ε in Amaouche et al. [5] for Newtonian fluids.

4. Linear stability analysis

In order to investigate the stability characteristics of the flow, we have linearized the Eqs. (15) and (16) about the base state. Owing to the normal mode assumption, the perturbations can be expressed in this form:

$$h(x,t) = A\mathrm{e}^{\mathrm{i}(kx-\omega t)} \quad \text{and} \quad q(x,t) = B\mathrm{e}^{\mathrm{i}(kx-\omega t)} \qquad (17)$$

where k is the wavenumber and $\omega = \omega_r + \mathrm{i}\omega_i$ is the wave frequency. In temporal stability analysis, the disturbance is applied in space and is observed as it evolves in time. For this, k is assumed real and ω complex. The imaginary part of ω represents the temporal growth rate of disturbances. Modes with positive growth rate are unstable and those with negative growth rate are stable.

The condition for a non-trivial solution of Eqs. (15)–(16) to exist is that ω and k must satisfy a dispersion relation that can be put in the form:

$$D(k, \omega, h_1, \rho, \mu, R, \Gamma_1, \Gamma_2, \cot\theta) = 0 \qquad (18)$$

The celerity and the Reynolds number near the instability threshold were obtained from (18) by expanding the pulsation ω in power of small wavenumbers. We have found

$$c_0 = g(h_1, \rho, \mu) \qquad (19)$$

$$R_c = \frac{(h_1-1)(\rho-1)f(h_1,\rho,\mu)\cot(\theta)}{f_0(h_1,\rho,\mu) + f_1(h_1,\rho,\mu)\Gamma_1 + f_2(h_1,\rho,\mu)\Gamma_2} \qquad (20)$$

These critical properties are also obtained by linearizing equations (1)–(9) about the base flow and this leads to the Orr–Sommerfeld equations. Whatever the game of parameters we consider, the results are identical to those given by Eqs. (19) and (20), which validates our model. Furthermore, by setting $\Gamma_1 = \Gamma_2 = 0$ (lack of viscoelasticity) in the Eq. (18), we find the critical Reynolds number obtained in [5,8] for Newtonian fluids (see Fig. 2a).

Figs. 2a–c represent the stability maps in the plane (h_1, μ) for two fluids of same density ($R_c = 0$). They show the influence of the viscoelastic property at the criticality. For two Newtonian fluids (see Fig. 2a), we find the so-called thin-layer effect [8]; the flow is always unstable when the largest layer fluid is also the more viscous one. The curve in Fig. 2(a) has as equation $(\mu-1)(\mu-\mu_c) = 0$ with $\mu_c = (1/h_1 - 1)^2$. Fig. 2b corresponds to the case where the upper fluid is viscoelastic and the lower is Newtonian. We show that the viscoelasticity may stabilize or destabilize the interface in the presence of viscosity stratification. For values of $\Gamma_2 \geq 0.03$, a zone of instability appears when the less viscous fluid is in a minority. This situation is illustrated in Fig. 2b for $\Gamma_2 = 0.05$. The flow becomes unstable when $\mu > 1.57$. Furthermore, the viscoelasticity reduces significantly the instability zone when the more viscous fluid is the thinner one. The viscoelastic two-layer configuration is illustrated in Fig. 2c, the upper layer is the less elastic; in this case $\Gamma_2 = 0.01$. Increasing the elasticity Γ_1 of the lower fluid stabilizes the flow. This stabilizing effect is observed when the upper fluid is the most viscous and only in the regions of large h_1.

We have also examined the influence of the elasticity far from criticality. We have first considered a Newtonian lower layer ($\Gamma_1 = 0$) and we have varied the viscoelastic parameter Γ_2 of the upper layer. Fig. 3a represents the growth rate ω_i

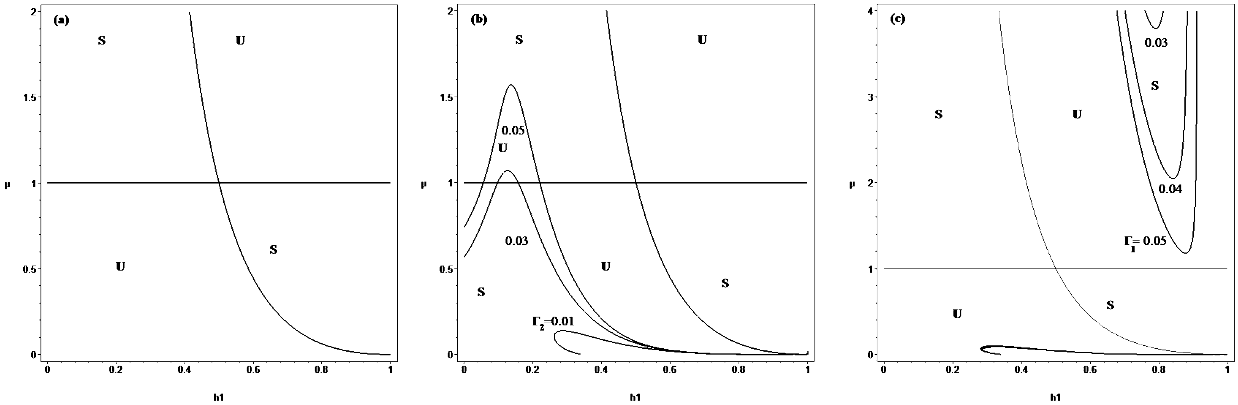

Fig. 2. Neutral stability diagram in the (h_1, μ) plane in the long wavelength limit for $\rho = 1$: (a) two Newtonian layers, (b) upper viscoelastic layer and lower Newtonian layer, (c) upper and lower viscoelastic layers. S denotes stable and U denotes unstable regions.

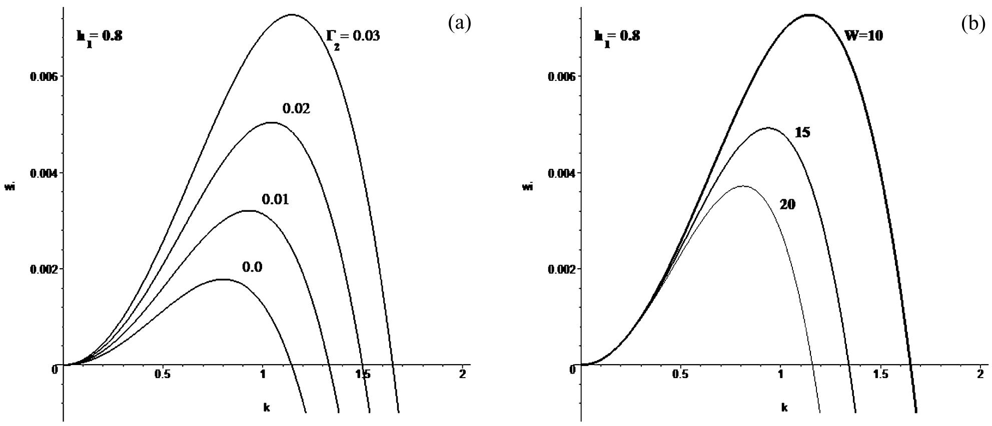

Fig. 3. Growth rate ω_i as function of the wavenumber k: upper viscoelastic layer and lower Newtonian layer (a) effect of the viscoelastic parameter $\Gamma_2 (W = 10)$; (b) effect of the capillary number $W (\Gamma_2 = 0.03)$. The other fixed parameters are $\rho = 1, \mu = 5$ and $R = 10$.

of perturbations as a function of the wavelength k for different values of Γ_2. This figure shows that as the value of Γ_2 increases, the growth rate increases. The maximum growth rate is obtained for k of the order of unity. These results show a good qualitative agreement with those of Theofanous et al. [9] obtained for the same configuration with an Oldroyd-B model.

Capillarity plays also an important role in the interfacial stability problem. As shown in Fig. 3b, capillarity does not affect the long wave instability; however the surface tension affects considerably the region of large wavenumber. Indeed, it prevents a destabilization induced by short-wave perturbations.

We have also analyzed the effect of elasticity stratification on the evolution of the perturbation growth rate in the case of two viscoelastic fluids (see Figs. 4a–b). The special case of two Newtonian fluids was also represented. It can be seen that the elasticity stratification increases significantly the growth rate of long-wave instabilities whatever the value of h_1. Moreover, the flow is predominantly unstable if the less elastic fluid occupies a largest section of the flow than the more elastic one. Here also the results are in good agreement with those of Wilson et al. [1] obtained with polypropylene and high-density polyethylene.

5. Conclusion

A simplified model is derived to describe the interfacial instability between two viscoelastic fluid layers flowing down an inclined channel. To solve this problem, the weighted residual approach was preferred. The model predicts the accurate linear stability conditions. The linear stability study showed that many factors influence the stability of the interface; the most influential of them being the layers thickness, the viscosity ratio and the difference of elasticity. The elastic stratification can be stabilizing when the more elastic fluid occupies a largest section of the flow than the less elastic one, but it is not the only influence on the stability of the flow. In this case, viscosity stratification is also stabilizing if the less viscous fluid occupies a smallest section of the flow. Note that the results are in good agreement with those of the literature [2,3,10].

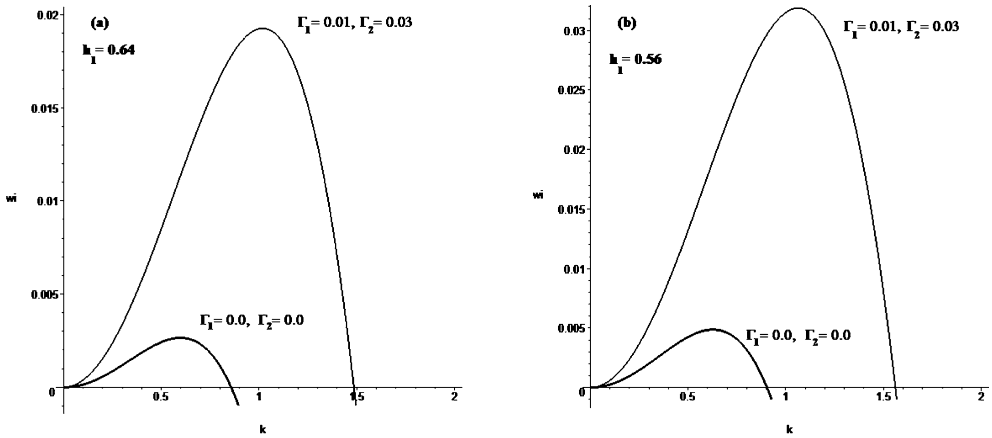

Fig. 4. Growth rate ω_i vs. the wavenumber k. Flow parameters are: $\rho = 1$, $\mu = 5$, $W = 10$ and $R = 10$.

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