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A simple and effective axisymmetric convected Helmholtz integral equation

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ABSTRACT

In this paper, we develop an axisymmetric boundary integral equation that derives from a reformulation of the 3D Helmholtz integral formula for the acoustic radiation problems in a subsonic uniform flow. Through the use of a new non-standard derivative operator, the axisymmetric convected Helmholtz integral equation substantially reduces the effects of flow incorporated in the classical convected boundary integral formulations, and involved in the normal derivative and the derivative in the flow direction of the axisymmetric convected Green's function. As for the free term derived from the singular integrals, it is given by a new expression independent of complete elliptic integrals and evaluated analytically as a convected angle in the meridian plane. The numerical treatment of singular integrals requires only the use of standard Gauss quadrature rules. Different test cases are presented.

1. Introduction

The Boundary Integral Equation (BIE) method is a technique appropriate to predict the three-dimensional acoustic radiation in a subsonic uniform flow. The integral representations can be formulated in the transformed acoustic medium [1,2] or in the original acoustic medium [3,4]. However, the advantages of using boundary integral formulations in an untransformed acoustic medium are that the convection effects are explicit and that complex boundary conditions can be handled easily.

Nonetheless, when considering the convected acoustic radiation in axisymmetric domains, it is necessary to develop new integral representations. Although the main problems posed by the 2D axisymmetric integral formulations in a uniform flow are the same as in the no-flow case [5–9], the presence of flow substantially increases these difficulties and makes significantly more complicated the convected integral representations [10,11] derived from the three-dimensional boundary integral formulation developed in [3].

The aim of this paper is to contribute to reduce the complexity of convected boundary integral formulas for axisymmetric domains and boundary conditions. Thus, we develop a new boundary integral formula resulting from a simplified BIE for the convected Helmholtz problem in a 3D exterior domain.

A study of the axisymmetric Convected Helmholtz Integral Equation (CHIE) is presented in details. It is expressed only in terms of the axisymmetric convected Green's function and its non-standard normal derivative. This non-standard normal derivative kernel is very similar to the normal derivative of the axisymmetric Green function in the classical case without flow. The singular parts of the kernels involved in this integral formula are evaluated analytically in terms of complete

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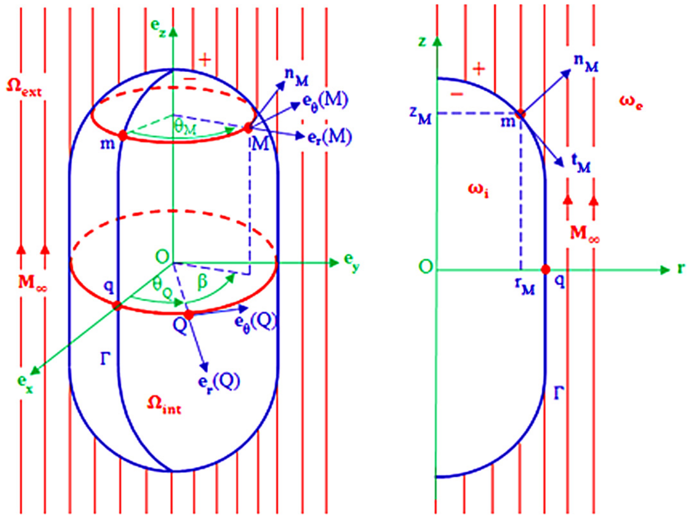

Fig. 1. (Color online.) Geometrical model of the surface of revolution (S).

elliptic integrals. Analytic expression is given for the free term resulting from the singular integrals and may be interpreted as a convected angle in the meridian plane. The numerical implementation of the present convected boundary integral equation is developed with quadratic unidimensional isoparametric elements. An efficient method is proposed to evaluate the elementary integrals containing singular complete elliptic integrals.

Finally, the axisymmetric convected boundary integral equation is validated by comparison with the exact solution to the acoustic radiation problem generated from a monopole source in a uniform flow.

2. A reformulation of the 3D convected Helmholtz integral equation

The boundary integral formulation for acoustic radiation in a subsonic uniform flow of Wu and Lee [3] has its origin in the formulation of sound radiation problems by bodies in the presence of a subsonic nonuniform flow. Thus, the unbounded propagation medium was split into two areas: a domain bounded by a fictitious closed surface S where the flow is non-uniform around the radiating body, called interior domain, noted Ω_{int}, and an unbounded domain outside of S where the flow is uniform, called exterior domain, noted Ω_{ext}. The streamlines are not perturbed by the presence of this separation interface S. The acoustic velocity potential in the exterior domain Ω_{ext} is then represented by a 3D convected boundary integral equation over the fictitious surface S [3]. We observe that the obtained formula is established on a fixed fictitious surface S and without the need to impose boundary conditions.

In this section, we consider the acoustic radiation problems in the exterior domain Ω_{ext}. The acoustic medium is occupied by a compressible perfect fluid characterized by the density ρ_∞ and the speed sound c_∞, isentropic, and in subsonic uniform motion of velocity $\mathbf{v}_\infty = v_\infty \mathbf{e_z}$ in the z-direction of a Cartesian coordinate system (O, x, y, z) associated with the physical space $\mathbb{R}^3$, with the standard basis $(\mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z})$. According to the linear acoustic analysis in harmonic-time $e^{(+j\omega t)}$, the acoustic pressure perturbation p_a in Ω_{ext} is given by $p_a = p e^{(+j\omega t)}$, where p is the complex amplitude, ω is the angular frequency, and $j = \sqrt{-1}$.

Fig. 1 shows the exterior domain Ω_{ext} and its fictitious boundary S submerged in a uniform flow. However, this scheme does not describe a physical test-case in $\mathbb{R}^3$, but is motivated by the obtaining the boundary integral equation. Using radiation conditions at infinity, the acoustic radiation problem in the exterior domain Ω_{ext} is governed by the classical boundary integral representation of the acoustic pressure p given for all point $M \in \Omega_{\mathrm{ext}}$ by [4]:

$$\begin{aligned} p(M) = \int_S \bigg\{ & p(Q)\frac{\partial G_c^k}{\partial n_Q}(M,Q) - G_c^k(M,Q)\frac{\partial p}{\partial n_Q}(Q) \\ & + 2jk(\mathbf{M}_\infty\cdot\mathbf{n_Q})\,p(Q)\,G_c^k(M,Q) + (\mathbf{M}_\infty\cdot\mathbf{n_Q})\Big[G_c^k(M,Q)(\mathbf{M}_\infty\cdot\mathbf{grad_Q}\,p(Q)) \\ & - p(Q)(\mathbf{M}_\infty\cdot\mathbf{grad_Q}\,G_c^k(M,Q))\Big]\bigg\}\,\mathrm{d}S_Q,\ M \notin S \end{aligned} \tag{1}$$

where $G_c^k(M,Q)$ is the convected Green's function, $k = \omega/c_\infty$ is the wave number, $\mathbf{M}_\infty = M_\infty \mathbf{e_z}$ is the uniform flow Mach vector, the constant $M_\infty = v_\infty/c_\infty$ is the Mach number ($0 \le M_\infty < 1$) and $\mathbf{grad_Q}$ is the gradient operator at point Q.

Eq. (1) is very similar to conventional integral formulas in Ref. [3]. It requires the evaluation of the kernel $G_c^k(\mathrm{M},\mathrm{Q})$ and its derivatives, the normal derivative $\partial G_c^k(\mathrm{M},\mathrm{Q})/\partial n_\mathrm{Q}$ and the derivative in the flow direction $\mathbf{M}_\infty \cdot \mathbf{grad}_\mathbf{Q} G_c^k(\mathrm{M},\mathrm{Q})$, which are given for $\mathrm{M} \neq \mathrm{Q}$ by [4]:

$$G_c^k(\mathrm{M},\mathrm{Q}) = \frac{\mathrm{e}^{-\mathrm{j}k^*(\mathbf{MQ}\cdot\mathbf{M}_\infty^* + R^*)}}{4\pi(1-M_\infty^2)^{\frac{1}{2}}\, R^*} \tag{2}$$

$$\begin{aligned}\frac{\partial G_c^k}{\partial n_\mathrm{Q}}(\mathrm{M},\mathrm{Q}) = {} & -G_c^k(\mathrm{M},\mathrm{Q})(1+\mathrm{j}k^*R^*)\frac{\mathbf{MQ}\cdot\mathbf{n_Q}}{R^{*2}} \\ & -(\mathbf{M}_\infty\cdot\mathbf{n_Q})G_c^k(\mathrm{M},\mathrm{Q})\Big[\mathrm{j}\frac{k}{1-M_\infty^2} + (1+\mathrm{j}k^*R^*)\frac{\mathbf{MQ}\cdot\mathbf{M}_\infty}{(1-M_\infty^2)R^{*2}}\Big]\end{aligned} \tag{3}$$

$$\mathbf{M}_\infty\cdot\mathbf{grad_Q}G_c^k(\mathrm{M},\mathrm{Q}) = -\mathrm{j}k\frac{M_\infty^2}{1-M_\infty^2}G_c^k(\mathrm{M},\mathrm{Q}) - G_c^k(\mathrm{M},\mathrm{Q})(1+\mathrm{j}k^*R^*)\frac{\mathbf{MQ}\cdot\mathbf{M}_\infty}{(1-M_\infty^2)R^{*2}} \tag{4}$$

with $\mathbf{n_Q}$ is the unit normal on S pointing in Ω_{ext},

$$k^* = k/(1-M_\infty^2)^{\frac{1}{2}}, \mathbf{M}_\infty^* = \mathbf{M}_\infty/(1-M_\infty^2)^{\frac{1}{2}} \tag{5}$$

$$R = \|\mathbf{MQ}\|, R^* = \Big(\|\mathbf{MQ}\|^2 + (\mathbf{MQ}\cdot\mathbf{M}_\infty^*)^2\Big)^{\frac{1}{2}} \tag{6}$$

The function R is the physical distance between the source and the observer, while R^* is the convected distance.

The presence of the flow in Eqs. (1)–(4) shows that the integral formula of the acoustic pressure $p(\mathrm{M})$ becomes significantly more complicated than in the no-flow case. To obtain a formulation that is less complicated and suitable for theoretical and numerical developments, the normal derivative and the derivative in the flow direction of the convected Green's function G_c^k can be converted into a non-standard normal derivative operator $\mathrm{D}G_c^k/\mathrm{D}n_\mathrm{Q}$ defined by

$$\frac{\mathrm{D}G_c^k}{\mathrm{D}n_\mathrm{Q}} = \frac{\partial G_c^k}{\partial n_\mathrm{Q}} - (\mathbf{M}_\infty\cdot\mathbf{n_Q})(\mathbf{M}_\infty\cdot\mathbf{grad_Q}G_c^k) + \mathrm{j}k(\mathbf{M}_\infty\cdot\mathbf{n_Q})G_c^k \tag{7}$$

We note that the idea of introducing the operator $\mathrm{D}G_c^k/\mathrm{D}n_\mathrm{Q}$ is due to the formulation of a radiation condition at infinity in [4]. Substituting Eqs. (3) and (4) in Eq. (7), one obtains:

$$\frac{\mathrm{D}G_c^k}{\mathrm{D}n_\mathrm{Q}} \equiv H_c^k(\mathrm{M},\mathrm{Q}) = -G_c^k(\mathrm{M},\mathrm{Q})(1+\mathrm{j}k^*R^*)\frac{\mathbf{MQ}\cdot\mathbf{n_Q}}{R^{*2}} \tag{8}$$

Thereafter, the convected integral formula of the acoustic pressure $p(\mathrm{M})$ given by Eq. (1) can be rewritten as

$$\begin{aligned}p(\mathrm{M}) = {} & \int_S p(\mathrm{Q})H_c^k(\mathrm{M},\mathrm{Q})\,\mathrm{d}S - \int_S \Big[\Big(1-(\mathbf{MQ}\cdot\mathbf{n_Q})^2\Big)\frac{\partial p}{\partial n_\mathrm{Q}}(\mathrm{Q}) \\ & -(\mathbf{M}_\infty\cdot\mathbf{n_Q})(\mathbf{M}_\infty.\mathbf{grad}_{\mathbf{s_Q}}p(\mathrm{Q})) - \mathrm{j}k(\mathbf{M}_\infty\cdot\mathbf{n_Q})p(\mathrm{Q})\Big]G_c^k(\mathrm{M},\mathrm{Q})\,\mathrm{d}S_\mathrm{Q}, \mathrm{M}\notin S\end{aligned} \tag{9}$$

where $\mathbf{grad}_{\mathbf{s_Q}} = \mathbf{grad_Q} - \mathbf{n_Q}\partial/\partial n_\mathrm{Q}$ is the surface gradient operator.

The 3D convected Helmholtz integral equation given by Eq. (9) presents a major advantage over the conventional convected integral equation (1). The first integral containing the integrand H_c^k in Eq. (9) only requires the evaluation of two convective terms instead of several terms in Eq. (1) due to the normal derivative and the derivative in the flow direction of the convected Green's function $G_c^k(\mathrm{M},\mathrm{Q})$. This significant reduction of convection effects can be interpreted by the fact that the non-standard normal derivative of the Green's function in a uniform motion free space has a very similar form to the standard normal derivative of the Green's function in a free space at rest. Thus, the use of non-standard normal derivative operator H_c^k has significantly reduced the complexity due to flow in the conventional integral formula (1). In addition, using the boundary element method, the numerical approximation of Eq. (1) requires the evaluation of a large number of additional elementary integrals compared to the numerical approximation of Eq. (9) due to the derivatives of $G_c^k(\mathrm{M},\mathrm{Q})$. However, the computation of these additional elementary integrals involving the integrands, the kernel $\mathbf{M}_\infty\cdot\mathbf{grad_Q}G_c^k(\mathrm{M},\mathrm{Q})$ given by Eq. (4) and the terms in $(\mathbf{M}_\infty\cdot\mathbf{n_Q})$ of $\partial G_c^k(\mathrm{M},\mathrm{Q})/\partial n_\mathrm{Q}$ in Eq. (3) leads to a significant increase in computation and implementation expenses. In contrast, the numerical approximation of Eq. (9) does not contain these elementary integrals, which significantly reduces the computational burden of elementary integrals and makes easy numerical implementation.

3. Axisymmetric convected Helmholtz integral equation formulation

Consider the revolution boundary S around the axis $(\mathrm{O}z)$ of cylindrical coordinate system (r,θ,z) where θ is the angle of revolution. The cylindrical coordinates of the points M and Q are respectively denoted by $(r_\mathrm{M},\theta_\mathrm{M},z_\mathrm{M})$ and $(r_\mathrm{Q},\theta_\mathrm{Q},z_\mathrm{Q})$.

The generator of S is a piecewise regular curve Γ, contained in the r–z meridian plane in which the points M and Q are identified by lowercase letters m and q having respective coordinates $(r_{\mathrm{M}}, z_{\mathrm{M}})$ and $(r_{\mathrm{Q}}, z_{\mathrm{Q}})$. Thus, the unit tangent vector $\mathbf{t_M}$ and unit normal $n_{\mathbf{M}}$ to the generator Γ at point M on S are defined by (Fig. 1)

$$\mathbf{t_M} = t_{r_{\mathrm{M}}}\mathbf{e_r}(\mathrm{M}) + t_{z_{\mathrm{M}}}\mathbf{e_z} \tag{10}$$

$$\mathbf{n_M} = \mathbf{t_M} \wedge \mathbf{e}_\theta(\mathrm{M}) = n_{r_{\mathrm{M}}}\mathbf{e_r}(\mathrm{M}) + n_{z_{\mathrm{M}}}\mathbf{e_z} \tag{11}$$

with $(\mathbf{e_r}(\mathrm{M}), \mathbf{e}_\theta(\mathrm{M}), \mathbf{e_z})$ is the cylindrical natural basis at point M. Then, the geometric quantities can be expressed in terms of the difference between azimuthal angles $\beta = \theta_{\mathrm{M}} - \theta_{\mathrm{Q}}$, as follows:

$$\mathbf{MQ}\cdot\mathbf{n_Q} = \Big(r_{\mathrm{Q}}n_{r_{\mathrm{Q}}} + (z_{\mathrm{Q}} - z_{\mathrm{M}})n_{z_{\mathrm{Q}}}\Big) - r_{\mathrm{M}}n_{r_{\mathrm{Q}}}\cos(\beta) \tag{12}$$

$$R = \|\mathbf{MQ}\| = \Big(r_{\mathrm{M}}^2 + r_{\mathrm{Q}}^2 + (z_{\mathrm{Q}} - z_{\mathrm{M}})^2 - 2r_{\mathrm{M}}r_{\mathrm{Q}}\cos(\beta)\Big)^{\frac{1}{2}} \tag{13}$$

$$R^* = \Big(r_{\mathrm{M}}^2 + r_{\mathrm{Q}}^2 + (z_{\mathrm{Q}} - z_{\mathrm{M}})^2 + (z_{\mathrm{Q}} - z_{\mathrm{M}})^2 M_\infty^{*2} - 2r_{\mathrm{M}}r_{\mathrm{Q}}\cos(\beta)\Big)^{\frac{1}{2}} \tag{14}$$

By introducing the convected distance $\rho^*(\mathrm{m}, \mathrm{q})$ and the physical distance $\rho(\mathrm{m}, \mathrm{q})$ between the two points m and q in the meridian plane:

$$\rho^*(\mathrm{m}, \mathrm{q}) = \Big(\|\mathbf{mq}\|^2 + (\mathbf{mq}\cdot\mathbf{M}_\infty^*)^2\Big)^{\frac{1}{2}} \tag{15}$$

$$\rho(\mathrm{m}, \mathrm{q}) = \|\mathbf{mq}\| = \Big((r_{\mathrm{Q}} - r_{\mathrm{M}})^2 + (z_{\mathrm{Q}} - z_{\mathrm{M}})^2\Big)^{\frac{1}{2}} \tag{16}$$

R^* can be rewritten as

$$R^* = d^*\Big(1 - \kappa^{*^2}\cos^2(\beta/2)\Big)^{\frac{1}{2}} \tag{17}$$

where d^* and κ^* are defined as

$$d^{*^2} = \rho^{*^2} + 4r_{\mathrm{Q}}r_{\mathrm{M}}, \kappa^{*^2} = 1 - \frac{\rho^{*^2}}{d^{*^2}} \tag{18}$$

The parameter d^* can be interpreted as a convected characteristic distance and corresponds to $\beta = \pi$. Using Eqs. (15) and (16), d^* can be rewritten in the form

$$d^{*^2} = d^2 + (\mathbf{mq}\cdot\mathbf{M}_\infty^*)^2, d^2 = \rho^2 + 4r_{\mathrm{Q}}r_{\mathrm{M}} \tag{19}$$

Thus, Eq. (2) rewritten as

$$G_{\mathrm{c}}^k(\mathrm{M}, \mathrm{Q}) = \frac{\mathrm{e}^{-\mathrm{j}k^*\left(\mathbf{mq}\cdot\mathbf{M}_\infty^* + d^*\left(1 - \kappa^{*^2}\cos^2(\beta/2)\right)^{\frac{1}{2}}\right)}}{4\pi d^*(1 - M_\infty^2)^{\frac{1}{2}}\Big(1 - \kappa^{*^2}\cos^2(\beta/2)\Big)^{\frac{1}{2}}} \tag{20}$$

The convected Green function $G_{\mathrm{c}}^k(\mathrm{M}, \mathrm{Q})$ and its non-standard normal derivative $H_{\mathrm{c}}^k(\mathrm{M}, \mathrm{Q})$ are then 2π periodic in the variable β. In Eq. (20), the parameter κ^* is dimensionless $(0 \le \kappa^* \le 1)$ and its limit value is $\kappa_{\mathrm{L}}^* = 1$, which can be obtained when $\rho^*(\mathrm{m}, \mathrm{q}) = 0$ in the case when $\mathrm{m} = \mathrm{q}$. Thus, the limit value κ_{L}^* describes the singularity of the functions $G_{\mathrm{c}}^k(\mathrm{M}, \mathrm{Q})$ and $H_{\mathrm{c}}^k(\mathrm{M}, \mathrm{Q})$ when $\mathrm{M} = \mathrm{Q}$. The parameter κ^* is an indicator to measure "the reduced distance" ρ^{*^2}/d^{*^2} to the singularity $\kappa_{\mathrm{L}}^* = 1$ when $\beta = 0$.

On the other hand, for a surface of revolution S that is generated by a curve Γ. The area element $\mathrm{d}S(\mathrm{M})$ at a point M on S and the surface gradient $\mathbf{grad}_{\mathbf{s_M}}\phi(\mathrm{M})$ of a scalar fields $\phi(\mathrm{M})$ are conventionally given by

$$\mathrm{d}S(M) = r_{\mathrm{M}}\mathrm{d}\theta_{\mathrm{M}}\Gamma(\mathrm{m}) \tag{21}$$

$$\mathbf{grad}_{\mathbf{s_M}}\phi(\mathrm{M}) = \frac{\partial\phi(\mathrm{M})}{\partial t_{\mathrm{M}}}\mathbf{t_M} + \frac{1}{r_{\mathrm{M}}}\frac{\partial\phi(\mathrm{M})}{\partial\theta_{\mathrm{M}}}\mathbf{e}_\theta(\mathrm{M}) \tag{22}$$

where $\mathrm{d}\Gamma(\mathrm{m})$ and $\partial/\partial t_{\mathrm{M}}$ are respectively the elementary length and the tangential differential operator at point m on the generator Γ.

The procedure for deriving the 3D convected boundary integral formulation is based on the transformation of Eq. (9) into the cylindrical coordinated system. For the fully axisymmetric problems, the sound pressure $p(\mathrm{M})$ and its normal derivative

$\sigma(\mathrm{M}) = \partial p(\mathrm{M})/\partial n_{\mathrm{M}}$ are independent of the revolution angle θ_{M}. Taking into account that $S = \Gamma \times [0, 2\pi]$ and $\mathbf{M}_\infty = M_\infty \mathbf{e_z}$, then using Eqs. (10), (11), (21), and (22), we have

$$\begin{aligned} p(\mathrm{m}) = &\int_\Gamma p(\mathrm{q}) r_{\mathrm{Q}} \left\{ \int_0^{2\pi} H_{\mathrm{c}}^k(\mathrm{M},\mathrm{Q}) \mathrm{d}\theta_{\mathrm{Q}} \right\} \mathrm{d}\Gamma(\mathrm{q}) \\ &- \int_\Gamma \left(1 - M_\infty^2 n_{z_{\mathrm{Q}}}^2\right) \sigma(\mathrm{q}) r_{\mathrm{Q}} \left\{ \int_0^{2\pi} G_{\mathrm{c}}^k(\mathrm{M},\mathrm{Q})\, \mathrm{d}\theta_{\mathrm{Q}} \right\} \mathrm{d}\Gamma(\mathrm{q}) \\ &- M_\infty^2 \int_\Gamma n_{z_{\mathrm{Q}}} n_{r_{\mathrm{Q}}} \frac{\partial p(\mathrm{q})}{\partial t_{\mathrm{Q}}} r_{\mathrm{Q}} \left\{ \int_0^{2\pi} G_{\mathrm{c}}^k(\mathrm{M},\mathrm{Q})\, \mathrm{d}\theta_{\mathrm{Q}} \right\} \mathrm{d}\Gamma(\mathrm{q}) \\ &+ \mathrm{j}kM_\infty \int_\Gamma n_{z_{\mathrm{Q}}} p(\mathrm{q}) r_{\mathrm{Q}} \left\{ \int_0^{2\pi} G_{\mathrm{c}}^k(\mathrm{M},\mathrm{Q})\, \mathrm{d}\theta_{\mathrm{Q}} \right\} \mathrm{d}\Gamma(\mathrm{q}) \end{aligned} \tag{23}$$

As the functions $G_{\mathrm{c}}^k(\mathrm{M},\mathrm{Q})$ and $H_{\mathrm{c}}^k(\mathrm{M},\mathrm{Q})$ are 2π periodic in the variable β, we get

$$\int_0^{2\pi} G_{\mathrm{c}}^k(\mathrm{M},\mathrm{Q})\, \mathrm{d}\theta_{\mathrm{Q}} = \int_0^{2\pi} G_{\mathrm{c}}^k(\mathrm{M},\mathrm{Q})\, \mathrm{d}\beta = 2\pi G_{\mathrm{c0}}^k(\mathrm{m},\mathrm{q}) \tag{24}$$

$$\int_0^{2\pi} H_{\mathrm{c}}^k(\mathrm{M},\mathrm{Q})\, \mathrm{d}\theta_{\mathrm{Q}} = \int_0^{2\pi} H_{\mathrm{c}}^k(\mathrm{M},\mathrm{Q})\, \mathrm{d}\beta = 2\pi H_{\mathrm{c0}}^k(\mathrm{m},\mathrm{q}) \tag{25}$$

where G_{c0}^k and H_{c0}^k are the axisymmetric convected Green's function and its non-standard normal derivate, respectively. Thus, using Eqs. (24) and (25), Eq. (23) can be re-written for all point m located in the exterior medium ω^{e} (see Fig. 1) as follows:

$$\begin{aligned} p(\mathrm{m}) = 2\pi &\int_\Gamma p(\mathrm{q}) r_{\mathrm{Q}} H_{\mathrm{c0}}^k(\mathrm{m},\mathrm{q})\, \mathrm{d}\Gamma(\mathrm{q}) \\ &- 2\pi \int_\Gamma \left(1 - M_\infty^2 n_{z_{\mathrm{Q}}}^2\right) \sigma(\mathrm{q}) r_{\mathrm{Q}} G_{\mathrm{c0}}^k(\mathrm{m},\mathrm{q})\, \mathrm{d}\Gamma(\mathrm{q}) \\ &- 2\pi M_\infty^2 \int_\Gamma n_{z_{\mathrm{Q}}} n_{r_{\mathrm{Q}}} \frac{\partial p(\mathrm{q})}{\partial t_{\mathrm{Q}}} r_{\mathrm{Q}} G_{\mathrm{c0}}^k(\mathrm{m},\mathrm{q})\, \mathrm{d}\Gamma(\mathrm{q}) \\ &+ \mathrm{j}2\pi kM_\infty \int_\Gamma n_{z_{\mathrm{Q}}} p(\mathrm{q}) r_{\mathrm{Q}} G_{\mathrm{c0}}^k(\mathrm{m},\mathrm{q})\, \mathrm{d}\Gamma(\mathrm{q}),\ \mathrm{m} \notin \Gamma \end{aligned} \tag{26}$$

When the source point m is taken on the generator, the kernels $G_{\mathrm{c0}}^k(\mathrm{m},\mathrm{q})$ and $H_{\mathrm{c0}}^k(\mathrm{m},\mathrm{q})$ are singular at $\mathrm{m} = \mathrm{q}$.

Following the usual practice [12,13], we isolate the singular point m located on the outside face Γ^+ of Γ, by adding to the exterior medium ω^{e} an exclusion neighborhood D_ε^- of a disk $\mathrm{D}_\varepsilon = \mathrm{D}_\varepsilon^- \cup \mathrm{D}_\varepsilon^+$ with centre m and radius $\varepsilon = \rho(\mathrm{m},\mathrm{q})$, while D_ε^- is the lower half disk and D_ε^+ is the upper half disk. The boundary of this disc D_ε is the circle $\mathrm{C}_\varepsilon = \mathrm{C}_\varepsilon^- \cup \mathrm{C}_\varepsilon^+$ (Fig. 2).

Let s_ε denote the boundary between D_ε^- and D_ε^+, in the increased external domain $\omega_\varepsilon^{\mathrm{e}}$ of boundary $\Gamma_\varepsilon \cup \mathrm{C}_\varepsilon^-$ with $\Gamma_\varepsilon = \Gamma - s_\varepsilon$, Eq. (26) is valid. Thus, by passage to the limit $\varepsilon \to 0$ ($\omega^{\mathrm{e}} = \lim_{\varepsilon\to 0} \omega_\varepsilon^{\mathrm{e}}$ and $\Gamma = \lim_{\varepsilon\to 0} \Gamma_\varepsilon$), the integral equation for the acoustic pressure p at point $\mathrm{m} \in \Gamma$ is given as follows:

$$\begin{aligned} p(\mathrm{m}) = 2\pi &\lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} p(\mathrm{q}) H_{\mathrm{c0}}^k(\mathrm{m},\mathrm{q}) r_{\mathrm{Q}} \mathrm{d}\Gamma(\mathrm{q}) \\ &- 2\pi \lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} \left((1 - M_\infty^2 n_{z_{\mathrm{Q}}}^2) \sigma(\mathrm{q}) + M_\infty^2 n_{z_{\mathrm{Q}}} n_{r_{\mathrm{Q}}} \frac{\partial p(\mathrm{q})}{\partial t_{\mathrm{Q}}} - \mathrm{j}kM_\infty n_{z_{\mathrm{Q}}} p(\mathrm{q}) \right) G_{\mathrm{c0}}^k(\mathrm{m},\mathrm{q}) r_{\mathrm{Q}} \mathrm{d}\Gamma(\mathrm{q}) \\ &+ 2\pi \lim_{\varepsilon\to 0} \int_{\mathrm{C}_\varepsilon^-} p(\mathrm{q}) H_{\mathrm{c0}}^k(\mathrm{m},\mathrm{q}) r_{\mathrm{Q}} \mathrm{d}\Gamma(\mathrm{q}) \end{aligned}$$

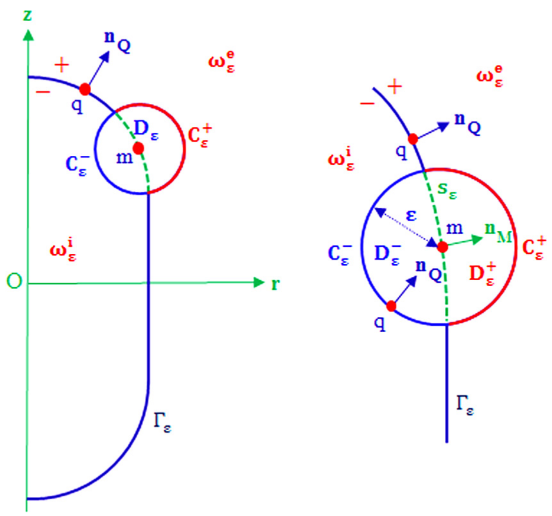

Fig. 2. (Color online.) The singular point m on the boundary Γ: the exclusion procedure.

$$-2\pi \lim_{\varepsilon\to 0} \int_{C_\varepsilon^-} \left((1 - M_\infty^2 n_{z_Q}^2)\,\sigma\,(\mathrm{q}) + M_\infty^2 n_{z_Q} n_{r_Q} \frac{\partial p(\mathrm{q})}{\partial t_Q} - \mathrm{j}kM_\infty n_{z_Q} p(\mathrm{q}) \right) G_{\mathrm{c}0}^k(\mathrm{m},\mathrm{q}) r_Q \,\mathrm{d}\Gamma(\mathrm{q}) \tag{27}$$

It is noted that the passage to the limit in Eq. (27) requires to study of behavior of kernels $G_{\mathrm{c}0}^k(\mathrm{m},\mathrm{q})$ and $H_{\mathrm{c}0}^k(\mathrm{m},\mathrm{q})$ near the singularity.

3.1. Behavior of kernels $G_{\mathrm{c}0}^k$ and $H_{\mathrm{c}0}^k$ near the singularity

The modal Green's functions $G_{\mathrm{c}0}^k(\mathrm{m},\mathrm{q})$ and $H_{\mathrm{c}0}^k(\mathrm{m},\mathrm{q})$ are singular at $\mathrm{m} = \mathrm{q}$, but the singularities can be isolated by decomposing the convected Green function $G_{\mathrm{c}0}^k(\mathrm{M},\mathrm{Q})$ and its non-standard normal derivative $H_{\mathrm{c}0}^k(\mathrm{M},\mathrm{Q})$ into a singular static part ($k = 0$) and non-singular part depending on the wave number k as follows:

$$G_{\mathrm{c}}^k(\mathrm{M},\mathrm{Q}) = G_{\mathrm{c}}^0(\mathrm{M},\mathrm{Q}) + g_{\mathrm{c}}^k(\mathrm{M},\mathrm{Q}) \tag{28}$$

$$H_{\mathrm{c}}^k(\mathrm{M},\mathrm{Q}) = H_{\mathrm{c}}^0(\mathrm{M},\mathrm{Q}) + h_{\mathrm{c}}^k(\mathrm{M},\mathrm{Q}) \tag{29}$$

with

$$G_{\mathrm{c}}^0(\mathrm{M},\mathrm{Q}) = \frac{1}{4\pi\left(1 - M_\infty^2\right)^{\frac{1}{2}} R^*} \tag{30}$$

$$g_{\mathrm{c}}^k(\mathrm{M},\mathrm{Q}) = \frac{\mathrm{e}^{-\mathrm{j}k^*(\mathbf{MQ}\cdot\mathbf{M}_\infty^* + R^*)} - 1}{4\pi(1 - M_\infty^2)^{\frac{1}{2}} R^*} \tag{31}$$

while

$$H_{\mathrm{c}}^0(\mathrm{M},\mathrm{Q}) = -\frac{\mathbf{MQ}\cdot\mathbf{n}_Q}{4\pi(1 - M_\infty^2)^{\frac{1}{2}} R^{*3}} \tag{32}$$

$$h_{\mathrm{c}}^k(\mathrm{M},\mathrm{Q}) = \frac{\left(-\,\mathrm{e}^{-\mathrm{j}k^*(\mathbf{MQ}\cdot\mathbf{M}_\infty^* + R^*)}(1 + \mathrm{j}k^* R^*) + 1\right)}{4\pi(1 - M_\infty^2)^{\frac{1}{2}} R^{*2}} \frac{\mathbf{MQ}\cdot\mathbf{n}_\mathbf{Q}}{R^*} \tag{33}$$

The functions g_{c}^k and h_{c}^k are non-singular near the singularity. This result is easily obtained by a Taylor expansion of function $\mathrm{e}^{(-\mathrm{j}k^*(.))}$ in Eqs. (31) and (33), knowing that $(\mathbf{MQ}\cdot\mathbf{n}_\mathbf{Q})/R^* = \mathrm{O}(R)$ [13] and $(\mathbf{MQ}\cdot\mathbf{M}_\infty^*)/R^* = \mathrm{O}(\|\mathbf{M}_\infty^*\|)$, and by taking into account $R/R^* \leq 1$. Thus, when the source point m approaches the field point q, the Fourier coefficients of zero order $g_{\mathrm{c}0}^k(\mathrm{m},\mathrm{q})$ and $h_{\mathrm{c}0}^k(\mathrm{m},\mathrm{q})$ are regular.

By virtue of parity and 2π-periodicity in the variable β of the function $G_{\mathrm{c}0}^0(\mathrm{M},\mathrm{Q})$, then using a change of variable $\theta = \beta/2$, the axisymmetric static convected Green's function $G_{\mathrm{c}0}^0(\mathrm{m},\mathrm{q})$ can be expressed in the form [5,6,10]

$$G_{c0}^{0}(\mathrm{m},\mathrm{q})=\frac{1}{2\pi}\frac{1}{\pi d^{*}\sqrt{1-M_{\infty}^{2}}}F(\pi/2,\kappa^{*}) \tag{34}$$

where $F(\pi/2,\kappa^*)$ is the elliptic complete integral of first kind.

Then, substituting Eq. (12) in Eq. (32), and using Eq. (25) with $k=0$ and taking into account the properties of the complex Fourier coefficients, the non-standard normal derivative of $G_{c0}^{0}(\mathrm{m},\mathrm{q})$ satisfies the following relation,

$$H_{c0}^{0}(\mathrm{m},\mathrm{q})=\left[n_{r_\mathrm{Q}}r_\mathrm{Q}+n_{z_\mathrm{Q}}\left(z_\mathrm{Q}-z_\mathrm{M}\right)\right]h_{c0}(\mathrm{m},\mathrm{q})-\frac{n_{r_\mathrm{Q}}r_\mathrm{M}}{2}\left(h_{c1}(\mathrm{m},\mathrm{q})+h_{c-1}(\mathrm{m},\mathrm{q})\right) \tag{35}$$

where

$$h_{cn}(\mathrm{m},\mathrm{q})=\frac{1}{2\pi}\int_{0}^{2\pi}h_{c}(\mathrm{M},\mathrm{Q})\,\mathrm{e}^{-\mathrm{j}n\beta}\mathrm{d}\beta \tag{36}$$

with h_{cn} is the nth Fourier coefficient of the function h_c defined by

$$h_{c}(\mathrm{M},\mathrm{Q})=\frac{-1}{4\pi\sqrt{1-M_{\infty}^{2}}R^{*^{3}}} \tag{37}$$

According to [15], when $\mathrm{m}\to\mathrm{q}$ (or $\kappa^*\to 1$), one obtains

$$\lim_{\kappa^{*}\to 1}F(\pi/2,\kappa^{*})=\frac{1}{2}\mathrm{Log}\left(16/(1-\kappa^{*^{2}})\right) \tag{38}$$

Thus, taking into account Eqs. (18), (34) and (38), the static modal Green's function $G_{c0}^{0}(\mathrm{m},\mathrm{q})$ exhibits a logarithmic behavior near the singularity, as follows:

$$G_{c0}^{0}(\mathrm{m},\mathrm{q})\sim-\frac{1}{2\pi\sqrt{1-M_{\infty}^{2}}}\frac{\mathrm{Log}\left(\rho^{*}(\mathrm{m},\mathrm{q})\right)}{2\pi r_\mathrm{Q}} \tag{39}$$

On the other hand, by introducing a regular function $E_\mathrm{c}(\mathrm{M},\mathrm{Q})=R^{*}/4\pi\sqrt{1-M_{\infty}^{2}}$, we infer that h_c and E_c satisfy the following relation:

$$d^{*^{4}}(\kappa^{*^{2}}-1)h_\mathrm{c}=E_\mathrm{c}+4\frac{\mathrm{d}^{2}E_\mathrm{c}}{\mathrm{d}\beta^{2}} \tag{40}$$

where E_c is considered as a function of variable β only.

Then, when $\mathrm{m}\to\mathrm{q}$, the function h_{cn} exhibits a singular behavior of the form

$$h_{cn}(\mathrm{m},\mathrm{q})\sim-\frac{1}{2\pi}\frac{1}{2\pi r_\mathrm{Q}}\frac{1}{\sqrt{1-M_{\infty}^{2}}\rho^{*^{2}}(\mathrm{m},\mathrm{q})} \tag{41}$$

It follows from Eqs. (35) and (41) that the behavior of the static modal Green's function H_{c0}^{0} near the singularity is characterized by

$$H_{c0}^{0}(\mathrm{m},\mathrm{q})\sim-\frac{1}{2\pi}\frac{1}{2\pi r_\mathrm{Q}}\frac{\mathbf{mq}\cdot\mathbf{n_Q}}{\sqrt{1-M_{\infty}^{2}}\rho^{*^{2}}(\mathrm{m},\mathrm{q})},\ \rho(\mathrm{m},\mathrm{q})\to 0 \tag{42}$$

Since $\rho/\rho^*\le 1$ and $(\mathbf{mq}\cdot\mathbf{n_Q})/\rho=\mathrm{O}(\rho)$ as $\rho(\mathrm{m},\mathrm{q})\to 0$, then the Fourier coefficient $H_{c0}^{0}(\mathrm{m},\mathrm{q})$ is bounded at the neighborhood of the singular point. The singularity of the static kernel $H_{c0}^{0}(\mathrm{m},\mathrm{q})$ is thus only apparent.

3.2. Analytical evaluation of the free terms

We are now able to evaluate the integrals along C_ε^- when $\varepsilon\to 0$ in integral Eq. (27). In the polar coordinate system with origin at source point m located on the outer face Γ^+, the elementary length at point $\mathrm{q}\in\mathrm{C}_\varepsilon^-$ is given by $\mathrm{d}\Gamma(q)=\rho(\mathrm{m},\mathrm{q})\mathrm{d}\varphi(\mathrm{q})$ and, taking into account the regularity of functions p and σ, then the integral on C_ε^-, when $\rho(\mathrm{m},\mathrm{q})\to 0$ and due to Eqs. (28), (29), (39) and (42), is given by

$$\lim_{\varepsilon\to 0}2\pi\int_{\mathrm{C}_\varepsilon^-}\left(p(\mathrm{q})H_{c0}^{k}(\mathrm{m},\mathrm{q})-(\ldots)G_{c0}^{k}(\mathrm{m},\mathrm{q})\right)r_\mathrm{Q}\,\mathrm{d}\Gamma(\mathrm{q})=\lim_{\varepsilon\to 0}2\pi p(\mathrm{m})\int_{\mathrm{C}_\varepsilon^-}H_{c0}^{0}(\mathrm{m},\mathrm{q})r_\mathrm{Q}\,\mathrm{d}\Gamma(\mathrm{q}) \tag{43}$$

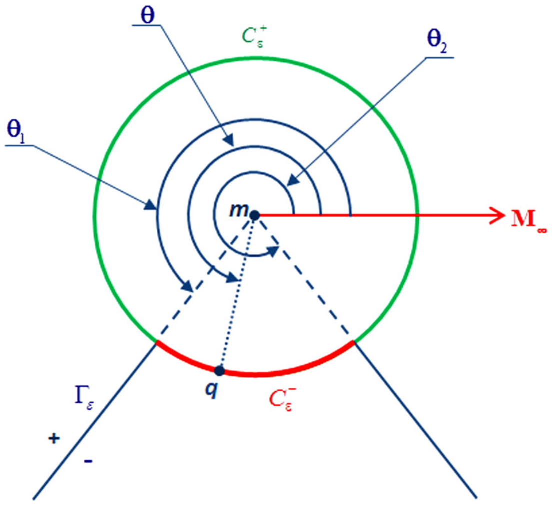

Fig. 3. (Color online.) Geometric model for the evaluation of free terms at an angular point m of boundary Γ.

with

$$\lim_{\varepsilon\to 0} 2\pi p\,(\mathrm{m}) \int_{C_\varepsilon^-} H_{c0}^0\,(\mathrm{m},\mathrm{q})\, r_{\mathrm{Q}}\,\mathrm{d}\Gamma\,(\mathrm{q}) = p\,(\mathrm{m})\,\frac{\theta_c^-\,(\mathrm{m})}{2\pi} \tag{44}$$

where

$$\theta_c^-\,(\mathrm{m}) = -\lim_{\varepsilon\to 0}\int_{C_\varepsilon^-} \frac{\mathbf{mq}\cdot\mathbf{n_q}}{\sqrt{1-M_\infty^2\rho^{*2}}}\mathrm{d}\Gamma\,(\mathrm{q})\,,\ \mathrm{m}\in\Gamma^+ \tag{45}$$

The function $\theta_c^-(\mathrm{m})$ is the free term related to singular integrals derived from the exterior acoustic problem in ω^{e}. We set:

$$\mathrm{d}\theta_c^-(\mathrm{q}) = \frac{\rho^2}{\sqrt{1-M_\infty^2\rho^{*2}}}\mathrm{d}\theta(\mathrm{q}) \tag{46}$$

where

$$\mathrm{d}\theta\,(\mathrm{q}) = -\frac{\mathbf{mq}\cdot\mathbf{n_q}}{\rho^2}\mathrm{d}\Gamma\,(\mathrm{q}) \tag{47}$$

We observe that the differential element $\mathrm{d}\theta(\mathrm{q})$ given by Eq. (47) represents in the meridian plane the elementary angle under which the arc element at point q of elementary length $\mathrm{d}\Gamma(\mathrm{q})$ is seen from point m. In this no-flow case, it is easy to evaluate analytically the free term $\theta(\mathrm{m})$ that takes the value π when m is a regular point on Γ where a unique tangent exists, and the value of the angle between the two tangent vectors at an angular point m of Γ where the normal $\mathbf{n_M}$ is discontinuous. Consequently, in the presence of uniform flow, the differential element $\mathrm{d}\theta_c^-(\mathrm{q})$ can be regarded as a convected elementary angle. Substituting Eqs. (15) and (16) into Eq. (45), the convected angle $\theta_c^-(\mathrm{m})$ is given by

$$\theta_c^-(\mathrm{m}) = \int_{\theta_1}^{\theta_2} \mathrm{d}(\arctan(\frac{\tan\theta(\mathrm{q})}{\sqrt{1+M_\infty^{*2}}})) = \arctan(\frac{\tan\theta_2}{\sqrt{1+M_\infty^{*2}}}) - \arctan(\frac{\tan\theta_1}{\sqrt{1+M_\infty^{*2}}}) \tag{48}$$

where for $\mathrm{q}\in C_\varepsilon^-$, $\theta(\mathrm{q}) = (\widehat{\mathbf{M}_\infty,\mathbf{mq}})$ is the angle between the Mach vector $\mathbf{M}_\infty$ and the position vector $\mathbf{mq}$, with θ_1 and θ_2 are the angles of the half-tangents to the generator Γ at point m (Fig. 3).

To complete this interpretation, it is necessary to introduce the free term $\theta_c^+(\mathrm{m})$ associated with the interior acoustic problem in ω^{i}. In this case, for a point m located on the inner face Γ^-, we obtain an equation similar to Eq. (26) where the normal $\mathbf{n_Q}$ is substituted by $-\mathbf{n_Q}$, the exclusion neighborhood is D_ε^+, and the increased interior domain is given by $\omega_\varepsilon^{\mathrm{i}}$ with the boundary $\partial\omega_\varepsilon^{\mathrm{i}} = \Gamma_\varepsilon \cup C_\varepsilon^-$ (see Fig. 2). Then, the free term $\theta_c^+(\mathrm{m})$ is defined as follows

$$\theta_c^+\,(\mathrm{m}) = \lim_{\varepsilon\to 0}\int_{C_\varepsilon^+} \frac{\mathbf{mq}\cdot\mathbf{n_q}}{\sqrt{1-M_\infty^2\rho^{*2}}}\mathrm{d}\Gamma\,(\mathrm{q})\,,\ \mathrm{m}\in\Gamma^- \tag{49}$$

and satisfies the following properties:

$$\theta_c^+(\mathrm{m}) + \theta_c^-(\mathrm{m}) = 2\pi, \mathrm{m} \in \Gamma \tag{50}$$

$$\theta_c^+(\mathrm{m}) = \theta_c^-(\mathrm{m}) = \pi, \text{ if m is a regular point on } \Gamma \tag{51}$$

Note that these results are not dependent on the shape of the curves $\mathrm{C}_\varepsilon^\pm$. For a regular point m on the generator Γ, where $\theta_2 = \theta_1 + \pi$, the convected angles $\theta_c^\pm(\mathrm{m})$ are independent of the Mach number M_∞ and take the classical values of the angle subtended by the half space 2D at point m. When m is an angular point of the generator Γ, $\theta_c^-(m)$ and $\theta_c^+(m)$ are complementary angles that may be interpreted as the convected angles subtended by the arc C_ε^- and the arc C_ε^+ at the centre m, respectively. The value of $\theta_c^-(\mathrm{m})$ is given by Eq. (48).

However, we show that the function $\theta_c^-(\mathrm{m})$ can be extended to a function defined in the half-plane $\bar{\omega} = \mathbb{R}^+ \times \mathbb{R}$, namely:

$$\frac{\theta_c^-(\mathrm{m})}{2\pi} = \begin{cases} 0, & \mathrm{m} \in \omega^{\mathrm{e}} \\ \int_\Gamma -2\pi H_{c0}^0(\mathrm{m},\mathrm{q})\, r_{\mathrm{Q}}\, \mathrm{d}\Gamma(\mathrm{q}), & \mathrm{m} \in \Gamma \\ 1, & \mathrm{m} \in \omega^{\mathrm{i}} \end{cases} \tag{52}$$

By virtue of Eqs. (35) and (40), the static axisymmetric nonstandard kernel H_{c0}^0 can be rewritten as

$$\begin{aligned} H_{c0}^0(\mathrm{m},\mathrm{q}) = -\frac{1}{2\pi}\frac{1}{\pi\sqrt{1-M_\infty^2 d^*\rho^{*2}}}\Big((\mathbf{mq}\cdot\mathbf{n_Q})E(\pi/2,\kappa^*) \\ + \rho^{*2}\frac{n_{r_\mathrm{Q}}}{2r_\mathrm{Q}}\big(F(\pi/2,\kappa^*) - E(\pi/2,\kappa^*)\big)\Big) \end{aligned} \tag{53}$$

where $F(\pi/2,\kappa^*)$ and $E(\pi/2,\kappa^*)$ are the complete elliptic integrals of the first and the second kind, respectively, defined by

$$\begin{aligned} F(\pi/2,\kappa^*) &= \int_0^{\frac{\pi}{2}} \left(1-\kappa^{*^2}\cos^2(\beta)\right)^{\frac{-1}{2}} \mathrm{d}\beta \\ E(\pi/2,\kappa^*) &= \int_0^{\frac{\pi}{2}} \left(1-\kappa^{*^2}\cos^2(\beta)\right)^{\frac{1}{2}} \mathrm{d}\beta \end{aligned} \tag{54}$$

We observe that the integrand $H_{c0}^0(\mathrm{m},\mathrm{q})$ is expressed by an extremely simple formula as that used in the literature [9,10], and that in addition it has a similar form to the classical formula in the no-flow case [14].

A numerical integration of integral in Eq. (52) can be used to evaluate the free term $\theta_c^-(\mathrm{m})$, but in this work we use the analytical expression given by Eq. (48), which has the advantage of not depending on complete elliptic integrals. This analytical method can be used as an alternative method for the evaluation of free terms derived from the boundary integral formulations for acoustic problems in 2D axisymmetric domain with an acoustic medium at rest [5–9] or in uniform flow [10,11].

3.3. The axisymmetric convected Helmholtz integral equation

The study of the behavior of the kernels G_{c0}^k and H_{c0}^k near the singularity show that G_{c0}^k present a logarithmic singularity (weak), whereas H_{c0}^k present an apparent singularity. Hence, the integrals along Γ_ε in Eq. (27) are convergent in the usual sense when $\varepsilon \to 0$, and it is not necessary to consider them in Cauchy's principal value sense. In addition, taking into account the properties of the free terms $\theta_c^\pm(\mathrm{m})$, the acoustic pressure field p may be written for $\mathrm{m} \in \mathbb{R}^+ \times \mathbb{R}$, thus yielding the following convected integral representation

$$\begin{aligned} c^+(\mathrm{m})p(\mathrm{m}) = {} & 2\pi\int_\Gamma p(\mathrm{q})\, H_{c0}^k(\mathrm{m},\mathrm{q})\, r_\mathrm{Q}\, \mathrm{d}\Gamma(\mathrm{q}) \\ & - 2\pi\int_\Gamma \Big((1-M_\infty^2 n_{z_\mathrm{Q}}^2)\sigma(\mathrm{q}) + M_\infty^2 n_{z_\mathrm{Q}} n_{r_\mathrm{Q}}\frac{\partial p(\mathrm{q})}{\partial t_\mathrm{Q}} \\ & - \mathrm{j}kM_\infty n_{z_\mathrm{Q}} p(\mathrm{q})\Big) G_{c0}^k(\mathrm{m},\mathrm{q}) r_\mathrm{Q}\, \mathrm{d}\Gamma(q) \end{aligned} \tag{55}$$

with

$$c^+(\mathrm{m}) = 1 - \frac{\theta_c^-(m)}{2\pi} \tag{56}$$

Eq. (55) is the axisymmetric convected Helmholtz integral equation for exterior acoustic problems. As in the classical no-flow case [9], Eq. (55) is not valid for certain values of the wave number k, called irregular frequencies. This irregular frequencies problem is not discussed in this study.

Compared to conventional axisymmetric convected boundary integral formulas [10,11], the integral formula (55) does not contain the derivative in the flow direction of the axisymmetric kernel G^k_{c0} and significantly reduces the convected effects incorporated in the normal derivative $\partial G^k_{c0}/\partial n_Q$. An additional advantage is that the dimensionless outer convected angle $c^+(\mathrm{m})$ at point $\mathrm{m} \in \Gamma$ is expressed independently of complete elliptic integrals and evaluated analytically. Thus, the convected boundary integral formulation represented by Eqs. (55)–(56) and (48)–(53) simplifies the numerical treatments allowing one to predict the axisymmetric radiated acoustic field in a subsonic uniform flow.

However, integral equation (55) requires an accurate evaluation of the axisymmetric kernels $G^k_{c0}(\mathrm{m},\mathrm{q})$ and $H^k_{c0}(\mathrm{m},\mathrm{q})$. But the main difficulty in computing these kernels is the treatment of the singularity of the axisymmetric convected Green's function $G^k_{c0}(\mathrm{m},\mathrm{q})$ at $\mathrm{m}=\mathrm{q}$.

This problem also arises in solving the standard Helmholtz integral equation in a 2D axisymmetric space. The singularity extraction method is used extensively to compute the Fourier coefficients of the Green's function in free space at rest and its normal derivative by decomposing them into a static singular part and a dynamic regular part depending on the wave number k. The regular parts are computed by using standard Gauss quadrature rules while the static singular parts are evaluated analytically [5–9].

When the acoustic medium is in uniform motion, an extension of this singularity extraction method has been employed in [10,11] for evaluation of the axisymmetric convected kernel $G^k_{c0}(\mathrm{m},\mathrm{q})$, its normal derivative $\partial G^k_{c0}(\mathrm{m},\mathrm{q})/\partial n_Q$ and the derivative in the flow direction $\partial G^k_{c0}(\mathrm{m},\mathrm{q})/\partial z_Q$. This technique is based on the splitting of the symmetric part of the convected Green function into a static singular part and a dynamic regular part. Therefore, the singular and regular parts of $\partial G^k_{c0}(\mathrm{m},\mathrm{q})/\partial n_Q$ and $\partial G^k_{c0}(\mathrm{m},\mathrm{q})/\partial z_Q$ contain several additional terms compared to the singular part and the regular part of the normal derivative of classical axisymmetric Green's function in free space at rest. The non-singular parts are evaluated by a standard Gaussian quadrature, whereas the non-static singular parts are evaluated analytically. They are expressed in terms of the non-symmetric part of the convected Green function $G^k_c(\mathrm{M},\mathrm{Q})$ and several terms involving complete elliptic integrals. However, we observe that the computational burden of the evaluation of these singulars kernels is significantly higher in comparison with the treatment of singular kernels of the standard axisymmetric Helmholtz integral equation in a fluid at rest [5].

As regards the evaluation of singular axisymmetric kernels in formula (55), we use the singularity extraction method by applying the Fourier transform to Eqs. (28) and (29). The axisymmetric convected Green's function $G^k_{c0}(\mathrm{m},\mathrm{q})$ and its non-standard normal derivative $H^k_{c0}(\mathrm{m},\mathrm{q})$ can be split into a static singular part and a non-singular part depending on the wave number k as follows

$$G^k_{c0}(\mathrm{m},\mathrm{q}) = G^0_{c0}(m,q) + g^k_{c0}(\mathrm{m},\mathrm{q}) \tag{57}$$

$$H^k_{c0}(\mathrm{m},\mathrm{q}) = H^0_{c0}(\mathrm{m},\mathrm{q}) + h^k_{c0}(\mathrm{m},\mathrm{q}) \tag{58}$$

The regular functions $\mathrm{g}^k_{c0}(\mathrm{m},\mathrm{q})$ and $h^k_{c0}(\mathrm{m},\mathrm{q})$ can be evaluated numerically using a standard Gauss quadrature according to the azimuth angle. In contrast, the static kernels $G^0_{c0}(\mathrm{m},\mathrm{q})$ and $H^0_{c0}(\mathrm{m},\mathrm{q})$ given by Eqs. (34) and (53) are evaluated analytically, because they are expressed in terms of the complete elliptic integrals $F(\pi/2,\kappa^*)$ and $E(\pi/2,\kappa^*)$, which can be computed with high precision even when κ^* is very close to 1 using a polynomial approximation [15] or by a recent method [16] with an error of the order 10^{-8} and 10^{-16}, respectively.

We observe that the evaluation of convected singular kernels $G^k_{c0}(\mathrm{m},\mathrm{q})$ and $H^k_{c0}(\mathrm{m},\mathrm{q})$ in the axisymmetric integral formula (55) presents a major advantage compared to the evaluation of convected singular kernels $G^k_{c0}(\mathrm{m},\mathrm{q})$, $\partial G^k_{c0}(\mathrm{m},\mathrm{q})/\partial n_Q$ and $\partial G^k_{c0}(\mathrm{m},\mathrm{q})/\partial z_Q$ involved in the conventional axisymmetric integral formulas [10,11]. The static singular parts and the regular parts of the axisymmetric convected Green function $G^k_{c0}(\mathrm{m},\mathrm{q})$ and of its non-standard normal derivative $H^k_{c0}(\mathrm{m},\mathrm{q})$ have very similar forms compared to the static singular part and the regular part of the conventional axisymmetric Green function in a free space at rest, and its normal derivative, which makes the computational burden of the evaluation of these convected singular kernels comparable to that of singular kernels involved in the standard axisymmetric Helmholtz integral formula.

Furthermore, we note that the modal non-standard kernel $H^k_{c0}(\mathrm{m},\mathrm{q})$ can be evaluated by standard Gauss quadrature rules, because it has an apparent singularity at $\mathrm{m}=\mathrm{q}$. However, a combination of the analytical integration of the static part $H^0_{c0}(\mathrm{m},\mathrm{q})$ coupled with a numerical integration of the dynamic part $h^k_{c0}(\mathrm{m},\mathrm{q})$ leads to a greater accuracy than the use of a purely numerical integration procedure to compute $H^k_{c0}(\mathrm{m},\mathrm{q})$.

4. Numerical implementation

The Boundary Element Method is used for solving Eq. (55) by discretizing the generator Γ with N one-dimensional quadratic isoparametric elements, denoted by T^i, $i=1,\ldots,N$. Each of these curvilinear elements is mapped onto the reference element $t^i=[-1,1]$. This transformation is used for describing both the geometric quantities (r_Q, z_Q) and the physical

Table 1
Relative error for the evaluation of the weakly singular integral J by the standard Gauss quadrature.

Number of Gaussian points	3	5	7	10
J_{num}	3.8892	3.9547	3.9755	3.9875
E_1 (%)	2.7	1.1	0.6	0.3

quantities (p, σ) on the reference element using the same quadratic shape functions [5]. In the local coordinate system, the tangential derivative operator $\partial/\partial t_{\mathrm{q}}$ along the generator Γ at point q of T^i is given by

$$\frac{\partial}{\partial t_{\mathrm{q}}} = \frac{1}{J^i(\xi_{\mathrm{q}})}\frac{\partial}{\partial \xi_{\mathrm{q}}} \tag{59}$$

where ξ_{q} is the local coordinate, $\xi_{\mathrm{q}} \in t^i = [-1, 1]$, and $J^i(\xi_{\mathrm{q}})$ is the Jacobian of the transformation. Thus, the tangential derivative $\partial p/\partial t_{\mathrm{q}}$ is expressed in terms of nodal values of the sound pressure p on the element T^i.

Thereafter, the discretization of Eq. (55) is obtained by the collocation method so that the point m coincides with the nodes of the discretized generator Γ. This leads to a complex linear system

$$[\mathbf{A}]\mathbf{p} = [\mathbf{B}]\boldsymbol{\sigma} \tag{60}$$

where $[\mathbf{A}]$ and $[\mathbf{B}]$ are the axisymmetric acoustic matrices, whereas $\mathbf{p}$ and $\boldsymbol{\sigma}$ are the nodal vectors containing the values of the acoustic pressure p and its normal derivative σ at each node, respectively. The elements of matrix $[\mathbf{A}]$ contain the convected angles and elementary integrals related to the axisymmetric convected Green's function $G^k_{\mathrm{c}0}$ and its non-standard normal derivative $H^k_{\mathrm{c}0}$, while the elements of matrix $[\mathbf{B}]$ are composed only by elementary integrals containing the integrand $G^k_{\mathrm{c}0}$.

When the collocation point m is located on the element T^i and the field point q is on the element T^j with $(i \neq j)$, these elementary integrals are regular when T^i and T^j are disjoined and singular in the contrary case. If the elements T^i and T^j coincide $(i = j)$ or have a common border (one node in this case), the singular elementary integrals are of the form

$$\int_{-1}^{1} f(\xi_{\mathrm{q}}) G^0_{\mathrm{c}0}\Big(\mathrm{m}(\xi_{\mathrm{m}}), \mathrm{q}(\xi_{\mathrm{q}})\Big) \mathrm{d}\xi_{\mathrm{q}} \tag{61}$$

where f is a regular function, whereas the static kernel $G^0_{\mathrm{c}0}\Big(m(\xi_{\mathrm{m}}), \mathrm{q}(\xi_{\mathrm{q}})\Big)$ is singular at $\xi_{\mathrm{m}} = \xi_{\mathrm{q}}$ $(\kappa^* = 1)$.

In this work, the collocation points m are located at nodes of the element T^i, which leads to $\xi_{\mathrm{m}} = \xi_{\mathrm{q}} = -1, 0$ or 1. It must be noted that $G^0_{\mathrm{c}0}\Big(\mathrm{m}(\xi_{\mathrm{m}}), \mathrm{q}(\xi_{\mathrm{q}})\Big)$ is evaluated analytically with high accuracy, even if the point ξ_{m} is very close to the point ξ_{q}. As a result, it is not necessary to isolate the logarithmic singularity of $G^0_{\mathrm{c}0}$ for the evaluation of the weakly singular integral (61).

Therefore, when the singular point ξ_{m} is localized at the boundaries of integration ± 1, an accurate evaluation of Eq. (61) can be performed numerically using a standard Gaussian quadrature. We therefore illustrate the use of this numerical scheme when compared to an analytical method. Taking into account the expression $G^0_{\mathrm{c}0}$ given by Eq. (34), we will consider the evaluation of the weakly singular integral

$$\mathrm{I} = \int_{-1}^{1} F(\pi/2, \kappa)\, \mathrm{d}\kappa = 4 \times \mathrm{cat} \tag{62}$$

where cat is Catalan's constant. In Table 1, we compare numerical values of the integral $J = \mathrm{I}/\mathrm{cat}$ obtained using various numbers of Gaussian points with the exact one, and the corresponding relative error (in percent) defined by

$$\mathrm{E}_1(\%) = 100 \times \left| \frac{J_{\mathrm{exa}} - J_{\mathrm{num}}}{J_{\mathrm{exa}}} \right| \tag{63}$$

where $J_{\mathrm{exa}} = 4$ is the exact value and J_{num} is the computed value.

The numerical results show the effectiveness and accuracy of this standard numerical scheme where, with only seven Gaussian points, the relative error is 0.6%. Note that the principal advantages of the numerical procedure proposed are that it requires no special treatment of the integrands containing complete elliptic integral of the first kind with singular values at boundaries of integration, and that only a standard Gaussian quadrature is required.

An extension of this numerical scheme to the case when $\xi_{\mathrm{m}} = 0$ involves the subdivision of the reference element $t^i = [-1, 1]$ into two elements $t^i_1 = [-1, \xi_{\mathrm{m}}]$ and $t^i_2 = [\xi_{\mathrm{m}}, 1]$. Thus, the singular point ξ_{m} is moved from the inside of the element t^i at the common border of the subelements t^i_1 and t^i_2 (Fig. 4).

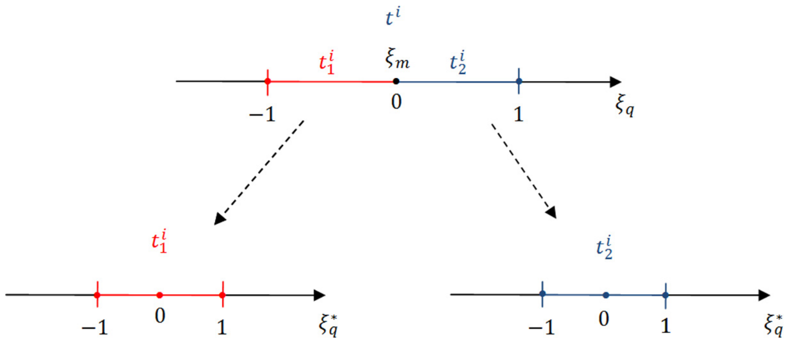

Fig. 4. (Color online.) Subdivision of the reference element.

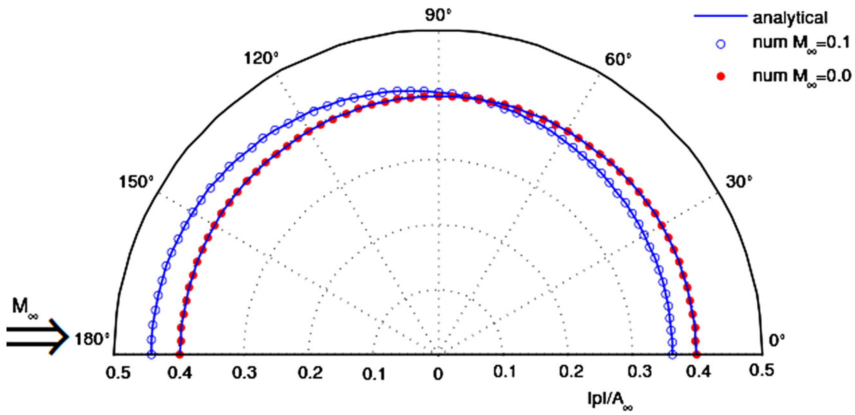

Fig. 5. (Color online.) The directivity pattern of the amplitude of the monopole source along the boundary for $ka = 5$, with $M_\infty = 0$ and $M_\infty = 0.1$.

5. Test cases

In this section, several numerical tests are proposed to validate the accuracy of the boundary integral formulation developed in this paper. Numerical results are compared with analytical solutions to sound radiation problems generated from a monopole source, of strength S located at the origin O of the coordinate system.

However, the sound pressure p due to the monopole source is accessible only by the acoustic velocity potential ϕ via $p = -\rho_\infty \mathrm{D}_\infty \phi/\mathrm{D}t$, where ρ_∞ is the static density of the medium at $\mathrm{D}_\infty/\mathrm{D}t = \partial/\partial t + v_\infty \partial/\partial z$. Then, ϕ/S is obtained by substituting $\mathbf{M}_\infty$ by $-\mathbf{M}_\infty$ into Eq. (2), and is given by [3]:

$$\varphi\,(\mathrm{Q}) = S\frac{\mathrm{e}^{-\mathrm{j}k^*\left[-\mathbf{OQ}\cdot\mathbf{M}_\infty^* + R^*\right]}}{4\pi\left(1 - M_\infty^2\right)^{\frac{1}{2}} R^*},\ \mathrm{Q} \neq \mathrm{O} \tag{64}$$

Thereafter, the acoustic pressure p is related to the acoustic velocity potential through

$$p(\mathrm{Q}) = -\rho_\infty c_\infty \left[\mathrm{j}k\varphi(\mathrm{Q}) + \mathbf{M}_\infty\cdot\mathbf{grad}\varphi(\mathrm{Q})\right] \tag{65}$$

Substituting Eqs. (4) and (5) into Eq. (65), then the sound pressure p is obtained in the form:

$$p\,(\mathrm{Q}) = -\frac{\rho_\infty c_\infty S}{\sqrt{1 - M_\infty^2}}\left[\mathrm{j}k^* - \left(1 + \mathrm{j}k^* R^*\right)\frac{\mathbf{OQ}\cdot\mathbf{M}_\infty^*}{R^{*2}}\right] G_\mathrm{c}^k\,(\mathrm{Q}, \mathrm{O}) \tag{66}$$

We note that compared to the case without flow, two additional terms appear in Eq. (66). Now consider the monopole source inside a virtual sphere S_a of center O and radius a, where the poles axis coincides with the flow direction. Thus, when $\partial p/n_\mathrm{Q}$ is a given Neumann boundary condition on the generator Γ_a of S_a for the normal derivative $\sigma = \partial p/\partial n_\mathrm{Q}$, then p is the analytic solution to the acoustic radiation problem in a 2D axisymmetry space outside the generator Γ_a.

In all numerical tests, the dimensionless wave number ka is fixed to a value equal to 5, while the Mach number takes values $M_\infty = 0, 0.1, 0.3$, and 0.5. The boundary Γ_a is discretized by 30 isoparametric quadratic elements with a total of 61 nodes. We put $A_\infty = \rho_\infty c_\infty S/a^2$. Figs. 5, 6 and 7 show the directivity patterns of the dimensionless amplitude $|p|/A_\infty$ of the numerical and analytical sound pressure along the generator Γ_a, which are in very good agreement. We denote by p_exa

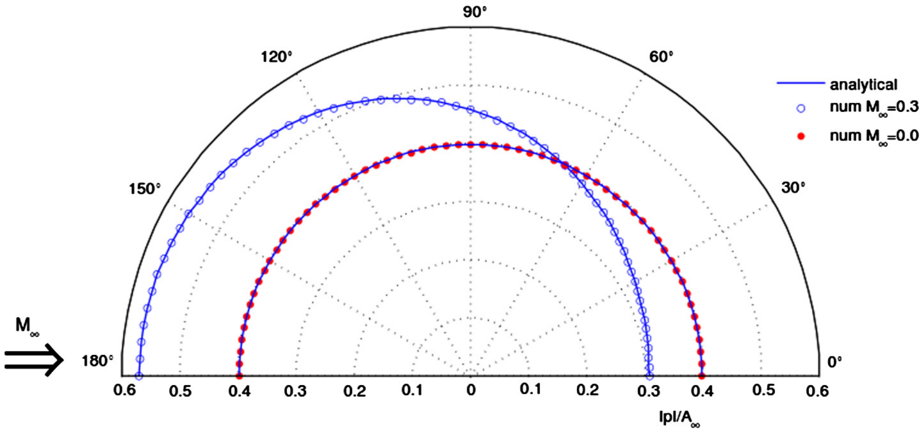

Fig. 6. (Color online.) The directivity pattern of the amplitude of the monopole source along the boundary for $ka = 5$, with $M_\infty = 0$ and $M_\infty = 0.3$.

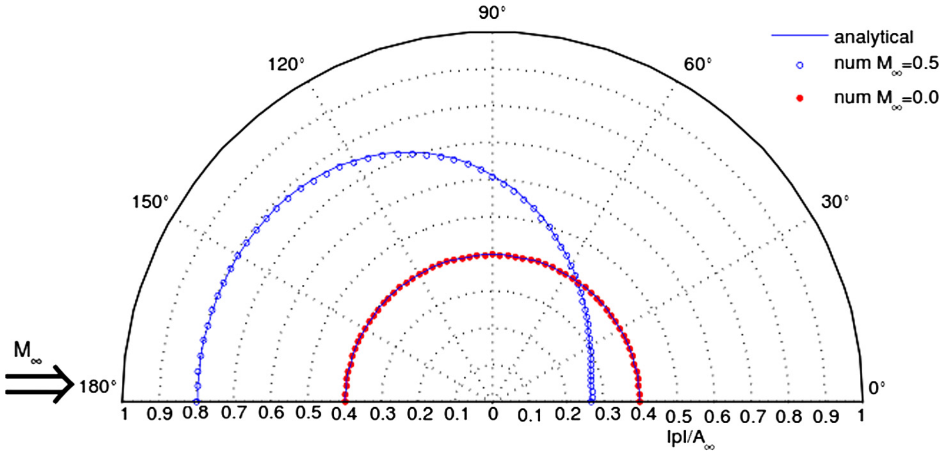

Fig. 7. (Color online.) The directivity pattern of the amplitude of monopole source along the boundary for $ka = 5$, with $M_\infty = 0$ and $M_\infty = 0.5$.

Table 2
The relative L_2 error of the numerical solution on the boundary for $ka = 5$.

Mach number M_∞	0	0.1	0.3	0.5
E_2 (%)	0.024	0.029	0.11	0.4

the exact solution computed using Eq. (66) and by p_{num} the numerical solution to Eq. (60). The relative error in the L_2 norm is defined by (in percent)

$$E_2(\%) = 100 \times \frac{\|p_{\text{exa}} - p_{\text{num}}\|_2}{\|p_{\text{exa}}\|_2} \tag{67}$$

where $\|p_{\text{exa}} - p_{\text{num}}\|_2 = [\frac{1}{N}\sum_{j=1}^{N}(p_{\text{exa}}^j - p_{\text{num}}^j)^2]^{1/2}$ and p_{exa}^j and p_{num}^j are the exact and computed values at the nodes j, with N is the total number of mesh node. Table 2 gives the relative L_2 error for different values of the Mach number.

We observe that, in the absence of flow, the distribution of this amplitude is uniform and symmetrical. But it becomes asymmetric non-uniform in the presence of a flow. In addition, the effect of the flow led to the subdivision of the generator Γ_a into two regions. In the upstream region, the amplitude of the sound pressure field is amplified to be greater than that obtained in a medium without flow. On the contrary, in the downstream region, the convected effect caused a decrease in the acoustic amplitude.

6. Conclusion

In this work, we have developed an axisymmetric Helmholtz integral formula in a subsonic uniform flow. The use of a non-standard kernel has proved its effectiveness to substantially reduce the complexity due to the presence of flow in the conventional axisymmetric boundary integral formulations. A new property of free term resulting from the singular integrals shows that it can be expressed independently of complete elliptic integrals and evaluated analytically as a convected angle in the meridian plane. An additional advantage is that the numerical implementation avoids special treatments for the evaluation of integrals containing a logarithmic singularity of the complete elliptic integral of the first kind, and requires only the use of standard Gaussian quadrature formulae. Numerical tests confirm the accuracy and efficiency of the numerical methods developed in this paper.

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