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Solving the vibration problem of inhomogeneous orthotropic cylindrical shells with hoop-corrugated oval cross section

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A R T I C L E I N F O

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A B S T R A C T

Based on the framework of Flügge's shell theory, transfer matrix approach and Romberg integration method, this paper investigates how corrugation parameters and material homogeneity affect the vibration behavior of isotropic and orthotropic oval cylindrical shells with sine-shaped hoop. Assume that the Young's moduli, shear moduli and density of the orthotropic material are continuous functions of the coordinate in the circumferential direction. The governing equations of non-homogeneous, orthotropic oval cylindrical shells with variable homogeneity along its circumference are derived and put in a matrix differential equation as a boundary-value problem. The trigonometric functions are used with Fourier's approach to approximate the solution in the longitudinal direction, and also to reduce the two-dimensional problem to a one-dimensional one. Using the transfer matrix approach, the equations can be written in a matrix differential equation of the first order and solved numerically as an initial-value problem. The proposed model is applied to get the vibration frequencies and mode shapes of the symmetric and antisymmetric vibration modes. The sensitivity of the vibration behavior to the corrugation parameters, homogeneity variation, ovality, and orthotropy of the shell is studied for different type modes of vibration.

1. Introduction

Orthotropic shells with circular and noncircular profiles are more efficient than isotropic ones, not only because of their higher wall stiffness, but also because they are less sensitive to initial imperfections. A special type of orthotropic shells, discretely reinforced longitudinal or circumferential corrugation, is very cost effective and widely used in various areas of modern engineering applications, such as air engineering, marine structures, agricultural silos, and piping. Circumferentially corrugated shells are a type of sensitive elastic elements widely used in precision instruments and meters, and they are used in structures with low flexural stiffness in the longitudinal direction and high flexural stiffness in the circumferential direction. The vibration, bending and stability of corrugated shells should be studied systematically to obtain the entire mechanical properties to help designers to achieve a reduction in weight and an increase in stiffness. Mathematically, the consideration of non-homogeneity, orthotropy, corrugations and aspect ratios leads to a very complex problem involving several parameters. So, numerical or approximate techniques are necessary for their analysis. Since more attention is paid to the analysis of the behavior of shells of a non-uniform cross-section with inhomogeneous materials, researches on the vibration of isotropic and orthotropic circular cylindrical shells have been undertaken by many researchers, since the basic equations for this had been established by Love [1], Rayleigh [2], and Flügge [3]. The best collection of documents can be

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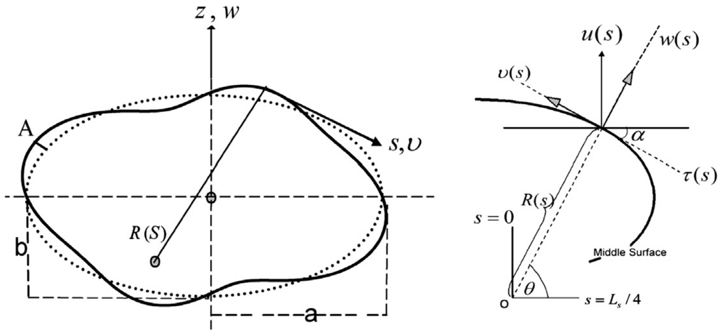

Fig. 1. Coordinate system and geometry of an inhomogeneous elastic cylindrical shell with a hoop-corrugated oval cross section.

found in Leissa [4], in which more than 500 publications are analyzed and discussed in both linear and non-linear vibration cases. Other related literatures for the vibration of homogeneous oval cylindrical shells can be found in [5–11]. On the other hand, there are a few publications that studied the vibration characteristics of the isotropic and orthotropic corrugated cylindrical shells, e.g., [12–17], whereas Tsiolkovsky [18] was the first one to see that the corrugated shells can be used in lighter-than-air engineering. As it has been found recently, a few studies have been directed toward the vibration behavior of smooth, inhomogeneous, orthotropic cylindrical shells [19–24]. Despite extensive works have been carried out, it is likely that the combined effects of corrugations, orthotropy, and non-homogeneity of materials on the vibration behavior of oval and circular cylindrical shells have not been analyzed. This paper aims at studying the combined effects of non-homogeneity of materials and corrugations of surface on the vibration behavior of elastic orthotropic shells. The basic equations of the vibration of a non-homogeneous, corrugated orthotropic oval cylindrical shell are derived from the framework of Flügge's shell theory. The transfer matrix is employed to get the vibration frequencies and the corresponding mode shapes for the symmetric- and antisymmetric-type modes. The results reveal that the variations of the oval cylindrical shell parameters, homogeneity variation, orthotropy of material and corrugation parameters have significant effects on the values of the natural frequencies and the mode shapes. The results are cited in tabular and graphical forms.

2. The model

2.1. Shell theory

It is known by researchers that the study of vibration problems in shells depends on the shell geometry and on the model of shell theory adopted. The present problem is modeled using Flügge's theory, which is very accurate in the case of thin shells due to the consideration of the shear deformation effect in the equilibrium equations.

2.2. Geometric formulation

The present shell is a corrugated orthotropic oval cylindrical shell with a sine-shaped hoop. The curvature of the cross-section profile of a corrugated oval shell is defined by the equation $r_0/R(s) = f(s)$, where R is the variable radius of curvature along the cross-section mid-line, r_0 is the reference radius of curvature, chosen to be the radius of a circle having the same circumference as the oval profile, and $f(s)$ is a prescribed function of s and can be taken as a simple form by the first two terms of the Fourier series representation of the curvature of a general cylindrical shell, see [25]; the corrugated surface can be taken as:

$$f(s) = 1 + \varepsilon\cos(2s) + A\sin(2Ns), \quad 0 \le s \le 2\pi \tag{1}$$

where A is the corrugation amplitude and N is the number of circumferential corrugations over the shell surface, as ε is the ovality parameter, which measures the eccentricity of the cross section of the shell, and to prevent negative curvature, $|\varepsilon|$ should be equal to or less than unity. Such a function describes a doubly-symmetric oval cross section and can be approximated by an ellipse. The ovality parameter ε is expressed by the semi-major, a, and semi-minor, b, axes of the oval profile given by [9]:

$$\begin{aligned}
&\varepsilon = 3Q - (36/35)Q^3 \\
&Q = (A/B - 1)/(A/B + 1) \\
&B = b/r, \qquad A = a/r_0
\end{aligned} \tag{2}$$

The position of a point on the middle surface of the shell is defined by the cylindrical coordinates (x, s, z), as shown in Fig. 1. The displacements of the middle surface of the shell are denoted by u, υ and w in the axial, hoop and transverse directions, respectively. The shell geometry is described by T_0, the thickness of the shell, L_x, the axial length, and $(L_s = 2\pi r_0)$, the circumferential length of the shell. If we suppose that the axial and circumferential directions are principal axes of the orthotropic material. Then we have:

$$\nu_x E_s = \nu_s E_x \tag{3}$$

where E_x, E_s are Young's moduli and ν_x, ν_s are Poisson's ratios.

2.3. Model of non-homogeneous materials

In general, non-homogeneous materials can be found in nature and man-made structures and its elastic constants are varying continuously in different spatial directions. In most of solutions for elastic non-homogeneity, it is assumed that the material of the shell is isotropic or orthotropic, the Poisson's ratio is constant, the Young's moduli and the density of the shell are either an exponential or a power function of a spatial variable, see [26]. In the present study, the material properties of the inhomogeneous orthotropic oval cylindrical shell is assumed to arise due to the variation of Young's moduli, of the shear modulus and of the density along the circumferential direction of the shell. However, in order to take into account the non-homogeneity of the present shell material, we assumed that:

$$(E_x, E_\theta, G, \rho) = (E_{x0}, E_{\theta 0}, G_0, \rho_0)\phi(s) \tag{4}$$

where E_{x0} and $E_{\theta 0}$ are the Young's moduli in the x- and s-directions, respectively, G_0 is the shear modulus, and ρ_0 is the density of homogeneous material of the shell, and ϕ is a prescribed continuous function of s accounting for the non-homogeneity of materials. For the present study, the non-homogeneous function was chosen to be dependent on the tangential angle α, which is between the tangent at the origin of s, located at the end of the semi-major axis, a, and the one at any point on the centerline, see Fig. 1. The non-homogeneous function is a parabolic, and takes the form:

$$\phi(\alpha) = 1 + \Delta\alpha^2 \tag{5}$$

where Δ is a real number expressing the amplitude of the homogeneity variation along the shell's circumference. Furthermore, we will propose a new ratio, used for the first time, to express comprehensively the non-homogeneity of the shell, since the function $\phi(\alpha)$ takes a minimum value for $\alpha = 0$, and a maximum value for $\alpha = \pi/2$. Hereby, substituting (5) into (4) yields:

$$E_{x\mathrm{Max}}/E_{x\mathrm{Min}} = E_{\theta\mathrm{Max}}/E_{\theta\mathrm{Min}} = G_{\mathrm{Max}}/G_{\mathrm{Min}} = \rho_{\mathrm{Max}}/\rho_{\mathrm{Min}} = 1 + 2.47\Delta \tag{6}$$

Then, the dependence of the non-homogeneity ratio ($H = Max/Min$) on Δ takes the form:

$$H = 1 + 2.47\Delta, \quad \Delta \geq 0 \tag{7}$$

2.4. Governing equations

When the above assumptions are taken into consideration, and for studying the free vibrations of general circular cylindrical shells, the equilibrium equations of forces, based on Flügge's theory [3], take the forms:

$$\begin{aligned}
&N'_x + N^\bullet_{sx} + \rho T_0\omega^2 u = 0\\
&N'_{xs} + N^\bullet_s + Q_s/R + \rho T_0\omega^2 \upsilon = 0\\
&Q'_x + Q^\bullet_s - N_s/R + \rho T_0\omega^2 w = 0\\
&M_x + M^\bullet_{sx} - Q_x = 0\\
&M'_{xs} + M^\bullet_s - Q_s = 0\\
&S_s - Q_s - M'_{sx} = 0\\
&N_{xs} - N_{sx} - M_{sx}/R = 0
\end{aligned} \tag{8}$$

where N_x, N_s and Q_x, Q_s are the normal and transverse shearing forces in the x- and s-directions, respectively, N_{sx} and N_{xs} are the in-plane shearing forces, M_x, M_s and M_{xs}, M_{sx} are the bending moment and the twisting moment, respectively, S_s is the equivalent shearing force, ρ is the mass density, and ω is the angular frequency of vibration, $' \equiv \partial/\partial x$, and $^\bullet \equiv \partial/\partial s$. The relations between strains and deflections for the cylindrical shells used here are taken from [27,28] as follows:

$$\begin{aligned}
&\varepsilon_x = u', \qquad \varepsilon_s = \upsilon^\bullet + w/R, \qquad \gamma_{xs} = \upsilon' + u^\bullet\\
&\gamma_{xz} = w' + \psi_x = 0\\
&\gamma_{sz} = w^\bullet + \psi_s - \upsilon/R = 0\\
&K_x = \psi'_x, \qquad K_s = \psi^\bullet_s + \left(\upsilon^\bullet + w/R\right)/R\\
&K_{sx} = \psi'_s, \qquad K_{xs} = \psi^\bullet_x + u^\bullet/R
\end{aligned} \tag{9}$$

where ε_x and ε_s are the normal strains of the middle surface of the shell, γ_{xs}, γ_{xz} and γ_{sz} are the shear strains, and the quantities K_x, K_s, K_{sx} and K_{xs} represent the change of curvature and the twist of the middle surface, ψ_x is the bending slope, and ψ_s is the angular rotation. The components of force and moment resultants in terms of (9) are given based on [3]:

$$\begin{aligned}
&N_x = D_x(\varepsilon_x + \nu_s\varepsilon_s) + (K_x/R)K_x \\
&N_{xs} = D_{xs}\gamma_{xs} + (K_{sx}/R)K_{sx} \\
&N_s = D_s(\varepsilon_s + \nu_x\varepsilon_x) - (K_s/R)(K_s - \varepsilon_s/R) \\
&N_{sx} = D_{sx}\gamma_{sx} + (K_{xs}/R)K_{xs} \\
&M_{sx} = 2K_{xs}(K_{xs} - \gamma_{xs}/2R) \\
&M_x = K_x(K_x + \nu_s K_s + \varepsilon_x/R) \\
&M_s = K_s(K_s + \nu_x K_x - \varepsilon_s/R)
\end{aligned} \tag{10}$$

where the quantities D_x, D_s and D_{xs} are the extensional rigidities, and on considering the non-homogeneity of the shell, using (5), they become:

$$\begin{aligned}
&D_x = E_x T_0/(1 - \nu_x\nu_s) = \big(E_{x0}T_0/(1 - \nu_x\nu_s)\big)\phi = D_{x0}\phi(\alpha) \\
&D_s = E_s T_0(1 - \nu_s\nu_x) = \big(E_{s0}T_0(1 - \nu_s\nu_x)\big)\phi = D_{s0}\phi(\alpha) \\
&D_{xs} = T_0 G = D_{xs0}\phi(\alpha)
\end{aligned} \tag{11}$$

and K_x, K_s, and K_{xs} are the flexural rigidities defined as:

$$\begin{aligned}
&K_x = E_x T_0^3/12(1 - \nu_x\nu_s) = \big(E_{x0}T_0^3/12(1 - \nu_x\nu_s)\big)\phi = K_{x0}\phi(\alpha) \\
&K_s = E_s T_0^3/(1 - \nu_s\nu_x) = K_{s0}\phi \\
&K_{xs} = GT_0^3/12 = K_{xs0}\phi(\alpha)
\end{aligned} \tag{12}$$

and the components with subscript zero denote the homogeneous ones.

3. Solution methodology

For complex geometries of non-circular shells, exact solutions are scarce. In this case, one has to make many approximations and simplifications in order to reduce the governing differential equations that reduce the original equations to a suitable form and to deal with the problem numerically. The effect of the non-homogeneity and orthotropy of material as well as shell corrugation complicates considerably the solution to the vibration problem.

3.1. Matrix representation

From Eqs. (8)–(12), by eliminating the variables Q_x, Q_s, N_x, N_{xs}, M_x, M_{xs} and M_{sx}, which are not differentiated with respect to s, the vibration system of the partial differential equations for the state variables u, υ, w, ψ_s, M_s, S_s, N_s and N_{sx} of the shell are obtained as follows:

$$\begin{aligned}
&u^\bullet = (1/BD_{xs})N_{sx} + (\Gamma/RB)\psi_s' - \upsilon', \qquad w^\bullet = \upsilon/r - \psi_s \\
&\upsilon^\bullet = N_s/D_s - w/R - \nu_x u' - \nu_x\Gamma\psi_x' - (1/RD_s)M_s \\
&\psi_s^\bullet = (B/K_s)M_s - \nu_x B\psi_x' - (1/RD_s)N_s + (\nu_x/R)u' \\
&M_s^\bullet = S_s - K_{xs}(\Gamma/B - 4)\psi_s'' - (R\Gamma/B)N_{sx}' \\
&N_s^\bullet = -S_s/R - N_{sx}' - \rho_0\omega^2 T_0\phi(\alpha)\upsilon \\
&N_{sx}^\bullet = D_x(1 - \nu_x\nu_s)u'' - (D_x\nu_s/D_s)N_s' + (K_x\nu_s/RK_s)M_s' \\
&\qquad - K_x\big((1 - \nu_x\nu_s)/R\big)\psi_x'' - \rho_0\omega^2 T_0\phi(\alpha)u \\
&S_s^\bullet = N_s/R + (\nu_s BD_x/D_s)M_s'' - K_x(1 - \nu_x\nu_s B)w''' - (K_x/R)(1 - \nu_x\nu_s)u'' \\
&\qquad - (\nu_s D_x\Gamma/D_s)N_s'' - \rho_0\omega^2 T_0\phi(\alpha)w, \quad B = 1 + \Gamma, \quad \Gamma = T_0^2/12R^2
\end{aligned} \tag{13}$$

The solution to the two-dimensional boundary-value problem for the system of partial differential equations (13) that satisfies the boundary conditions for a simply supported shell takes the forms:

$$\begin{aligned}
&u(x,s)=\overline{U}(s)\cos\beta x, \qquad \big(\upsilon(x,s),w(x,s)\big)=\big(\overline{V}(s),\overline{W}(s)\big)\sin\beta x, \qquad \psi_s(x,s)=\bar{\psi}_s(s)\sin\beta x\\
&\big(N_x(x,s),N_s(x,s),Q_s(x,s),S_s(x,s)\big)=\big(\bar{N}_x(s),\bar{N}_s(s),\bar{Q}_s(s),\bar{S}_s(s)\big)\sin\beta x\\
&\big(N_{xs}(x,s),N_{sx}(x,s),Q_x(x,s)\big)=\big(\bar{N}_{xs}(s),\bar{N}_{sx}(s),\bar{Q}_x(s)\big)\cos\beta x\\
&\big(M_x(x,s),M_s(x,s)\big)=\big(\bar{M}_x(s),\bar{M}_s(s)\big)\sin\beta x\\
&\big(M_{xs}(x,s),M_{sx}(x,s)\big)=\big(\bar{M}_{xs}(s),\bar{M}_{sx}(s)\big)\cos\beta x, \quad \beta=m\pi/L_x,\ m=1,2,\ldots
\end{aligned} \tag{14}$$

m is the axial half wave number. The nature of the axial mode β must be chosen so as to satisfy the required boundary conditions at the two ends of the cylindrical shell, where for the simply-supported ends boundary conditions, $\beta=m\pi/L_x$, for the clamped–clamped ends, $\beta=(2m+1)\pi/2L_x$, and for the clamped-simply supported ones, $\beta=(4m+1)\pi/4L_x$. The quantities $\overline{U}(s),\overline{V}(s),\ldots$ are state variables, which are undetermined functions of s. To reduce the two-dimensional boundary-value problem to one-dimensional problem, we introduce Eqs. (14) into system (13), and after appropriate algebraic operations, the system of vibration equations can be written in the form of linear ordinary differential equations referred to the variable s only are obtained, in the following matrix form:

$$r_0\frac{\mathrm{d}}{\mathrm{d}s}\begin{Bmatrix}\tilde{U}\\ \tilde{V}\\ \tilde{W}\\ \tilde{\psi}_s\\ \tilde{M}_s\\ \tilde{S}_s\\ \tilde{N}_s\\ \tilde{N}_{sx}\end{Bmatrix}=\begin{bmatrix}0&V_{12}&0&V_{14}&0&0&0&V_{18}\\ V_{21}&0&V_{23}&0&V_{25}&0&V_{27}&0\\ 0&V_{32}&0&V_{34}&0&0&0&0\\ V_{41}&0&V_{43}&0&V_{45}&0&V_{47}&0\\ 0&0&0&V_{54}&0&V_{56}&0&V_{58}\\ V_{61}&0&V_{63}&0&V_{65}&0&V_{67}&0\\ 0&V_{72}&0&0&0&V_{76}&0&V_{78}\\ V_{81}&0&V_{83}&0&V_{85}&0&V_{87}&0\end{bmatrix}\begin{Bmatrix}\tilde{U}\\ \tilde{V}\\ \tilde{W}\\ \tilde{\psi}_s\\ \tilde{M}_s\\ \tilde{S}_s\\ \tilde{N}_s\\ \tilde{N}_{sx}\end{Bmatrix} \tag{15}$$

By using the state vector of fundamental unknowns $Z(s)$, system (15) can be written as:

$$\begin{aligned}
&r_0\frac{\mathrm{d}Z(s)}{\mathrm{d}s}=[V]Z(s)\\
&Z(s)=(\tilde{U},\tilde{V},\tilde{W},\tilde{\psi}_s,\tilde{M}_s,\tilde{S}_s,\tilde{N}_s,\tilde{N}_{sx})^T\\
&(\tilde{U},\tilde{V},\tilde{W})=k_{x0}(\bar{U},\bar{V},\bar{W})\\
&\tilde{\psi}_s=(k_{x0}/\beta)\bar{\psi}_s, \qquad \tilde{M}_s=\big(1/\beta^2\big)\bar{M}_s\\
&(\tilde{S}_s,\tilde{N}_s,\tilde{N}_{sx})=\big(1/\beta^3\big)(\bar{S}_s,\bar{N}_s,\bar{N}_{sx})
\end{aligned} \tag{16}$$

and the coefficients of matrix $[V]$ are given here:

$$\begin{aligned}
&V_{12}=-\beta, \qquad V_{14}=\Gamma_1\beta^2\phi^2/CB_1, \qquad V_{18}=\Gamma_1\beta^3/\phi\mu B_1, \qquad V_{21}=\nu_x\beta\\
&V_{23}=\big(1+\nu_x\Gamma_1\beta^2\big)/C, \qquad V_{25}=\Gamma_1\beta^2/\phi\eta, \qquad V_{27}=\Gamma_1\beta^3/\phi\eta\\
&V_{32}=1/C, \qquad V_{34}=-\beta, \qquad V_{41}=-\nu_x/C, \qquad V_{43}=-\nu_x\beta^2B_1\\
&V_{45}=\beta B_1/\phi\eta, \qquad V_{47}=\beta^2\Gamma_1/C\phi\eta, \qquad V_{54}=\mu\beta\phi\big(4-\Gamma_1/C^2B_1\big)\\
&V_{56}=-\beta, \qquad V_{58}=\beta^2\Gamma_1/C^2B_1, \qquad V_{61}=\phi(1-\nu_s\nu_x)/C\\
&V_{63}=\beta\phi(1-\nu_x\nu_s)/2-\lambda^2\phi/\Gamma_1\beta^3, \qquad V_{65}=B_1\beta/\eta\\
&V_{67}=\big(1+\nu_s\Gamma_1\beta^2/\eta\big)/C, \qquad V_{72}=-\lambda^2\phi/\Gamma_1\beta^3\\
&V_{76}=-1/C, \qquad V_{78}=\beta, \qquad V_{81}=\phi(1-\nu_x\nu_s)/\Gamma_1\beta-\lambda^2\phi/\Gamma_1\beta^3\\
&V_{83}=\phi(1-\nu_x\nu_s)/C, \qquad V_{85}=\nu_s/C\eta, \qquad V_{87}=-\nu_s\beta/\eta
\end{aligned} \tag{17}$$

in terms of the following dimensionless parameters: natural frequency parameter $\lambda=r_0\omega\sqrt{\rho_0(1-\nu_x\nu_s)/E_{x0}}$, thickness ratio $h_0=T_0/r_0$, axial wave parameter $X=\beta/r_0$, circumferential and axial orthotropic parameters $\eta=D_s/D_x=K_s/K_x=\nu_s/\nu_x$, $\mu=D_{xs}/D_x=K_{xs}/K_x=G(1-\nu_x\nu_s)/E_x$, while $B_1=1+\Gamma_1/C^2$ and $\Gamma_1=h_0^2/12$.

By using the transfer matrix $T(s)$ of the shell, the state vector $Z(s)$ of the fundamental unknowns can be expressed as:

$$Z(s)=\big[T(s)\big]Z(0) \tag{18}$$

The substitution of $Z(s)$ into Eq. (16), assuming $(\bar{s}=s/L_s)$, yields:

$$\begin{aligned}
&(\mathrm{d}/\mathrm{d}\bar{s})\big[T(\bar{s})\big]=2\pi\big[V\big(\lambda,\phi(\alpha)\big)\big]\big[T(\bar{s})\big]\\
&\big[T(0)\big]=[I], \quad 0\le\bar{s}\le 1
\end{aligned} \tag{19}$$

The governing system of vibration (19) is too complicated to obtain any closed-form solution, and this problem should be solved by a numerical method. Hence, the matrix $[T(\bar{s})]$ is obtained by using the Romberg integration method, with the starting value $[T(0)] = [I]$ which is given by taking $\bar{s} = 0$ in (18), and its solution depends only on the geometric and martial properties of the shell.

3.2. Modal radius of curvature

Since the solution to Eq. (19) strongly depends on the variable curvature $R(s)$, it follows that the radius of curvature should represent the actual geometry of the middle surface of the shell. Using [7], the radius of the corrugated curvature is modified to take the form:

$$R(\bar{s})/r_0 = 1/\big(1 + \varepsilon\cos(4\pi\bar{s}) + \delta\sin(4N\pi\bar{s})\big), \quad \bar{s} = s/L_s \tag{20}$$

representing the dimensionless radius of corrugated curvature of the shell, C, and the same curvature approximates an ellipse having the same semi-major-to-semi-minor axis ratio as the oval. The range of eccentricity parameter is taken to be $(0 \le \varepsilon \le 1)$, which corresponds to a semi-major-to-semi-minor axis ratio a/b verifying $1 \le a/b \le 2.065$. As $\delta = A/r_0$ is a small parameter depending on the corrugation amplitude A, the relation between the tangential angle, α and $\bar{s}$, is found to be:

$$\alpha = \tan^{-1}\big((b/a)^2\tan(2\pi\bar{s})\big), \quad 0 \le \bar{s} \le 1 \tag{21}$$

3.3. Mode types of vibration

For a plane passing through the central axis in a shell with structural symmetry, symmetric and antisymmetric profiles can be obtained, and then, only one-half of the shell's circumference is considered, with the boundary conditions at the ends taken to be the symmetric or antisymmetric types of vibration modes. Therefore, the boundary conditions for symmetric-type modes are:

$$\tilde{V} = \tilde{\psi}_s = 0, \qquad \tilde{S}_s = \tilde{N}_{sx} = 0 \tag{22}$$

whereas for the antisymmetric-type modes, they are:

$$\tilde{U} = \tilde{W} = 0, \qquad \tilde{N}_s = \tilde{M}_s = 0 \tag{23}$$

The natural modes of vibration of an oval cylindrical shell which has a doubly symmetric profile may be classified as four different types of vibration modes (SS, SA, AA, AS) based on whether the modes are symmetric, S, or antisymmetric, A, with respect to the minor and major axes of the shell at $\bar{s} = 0$ or $\bar{s} = 0.5$, and $\bar{s} = 0.25$ or $\bar{s} = 0.75$, respectively. The substitution of (22) and (23) into (18) yields the following vibration equations:

$$\begin{bmatrix} T_{21} & T_{23} & T_{25} & T_{27} \\ T_{41} & T_{43} & T_{45} & T_{47} \\ T_{61} & T_{63} & T_{65} & T_{67} \\ T_{81} & T_{83} & T_{85} & T_{87} \end{bmatrix}_{(0.5)} \begin{pmatrix} \tilde{U} \\ \tilde{W} \\ \tilde{M}_s \\ \tilde{N}_s \end{pmatrix}_{(0)} = 0 \quad \text{for a symmetric vibration, SS, SA} \tag{24}$$

$$\begin{bmatrix} T_{12} & T_{14} & T_{16} & T_{18} \\ T_{32} & T_{34} & T_{36} & T_{38} \\ T_{52} & T_{45} & T_{56} & T_{58} \\ T_{72} & T_{74} & T_{76} & T_{78} \end{bmatrix}_{(0.5)} \begin{pmatrix} \tilde{V} \\ \tilde{\psi}_s \\ \tilde{S}_s \\ \tilde{N}_{sx} \end{pmatrix}_{(0)} = 0 \quad \text{for an antisymmetric vibration, AA, AS} \tag{25}$$

Eqs. (24) and (25) give a set of linear homogeneous equations with unknown coefficients $(\tilde{U}, \tilde{W}, \tilde{M}_s, \tilde{N}_s)^T_{(0)}$ and $(\tilde{V}, \tilde{\psi}_s, \tilde{S}_s, \tilde{N}_{sx})^T_{(0)}$, respectively, at $\bar{s} = 0$, and the determinant of the coefficient matrix should be made vanish for a non-trivial solution to exist. The non-trivial solution is found by searching the values λ in matrix $[T(\bar{s} = 0.5)]$ that make its determinant zero by using the Lagrange interpolation procedure. The vibration deformations (*axial, transverse and circumference deflection displacements*) at any point of the cross section of the shell are determined by calculating the eigenvectors corresponding to λ by using a Gaussian elimination procedure, when $X = 1$.

4. Presentation of results and discussion

A computer program, based on the analysis described herein, was written by the author to find the frequency parameters and the corresponding mode shapes of the vibration of an inhomogeneous corrugated oval cylindrical shell. Our study is divided into three parts, in which the Poisson ratio ν_s and the axial rigidity μ take the values 0.3 and 0.35, respectively.

4.1. Verification of solution method

In references [29–32], Khalifa presented a number of comparisons between numerical results to verify the validity of the used framework and approach method and the accuracy of the results. The comparison studies are calculated by different approaches and have confirmed the correctness of the proposed shell theory and of the method used for studying the mechanical behavior of such shells.

4.2. Fundamental vibrations

The first three natural frequency parameters are obtained; they correspond to the axial, circumferential, and transverse displacements. The fundamental frequencies, whose minimum one is of practical interest, since it is associated with the transverse direction. In fact, the study of the vibration finds the frequency parameter λ and the corresponding deflection displacements at each value of X. The numerical results present the fundamental vibration for the cases of (SS, SA, AA, AS) type modes. Generally, the corrugations in shells lead to more significant changes in the vibration behavior than the smooth ones, and the frequency parameters λ increase/decrease with an increase of the corrugation parameters (N, δ) regardless of the isotropy/orthotropy and homogeneity/non-homogeneity of the materials. Also, the frequency parameters λ decrease with an increase in the value of the ovality parameter, regardless of the smooth/corrugation of the shell, and this result confirms the drastic change in the membrane and bending energies of the shell by increasing the non-circularity due to a reduction in the area of the oval centrode. The effects of the corrugation parameters (N, δ), homogeneity variation H, ovality parameter ε, and orthotropy (η, μ) of the shell on the vibration characteristics, Table 1, govern the behavior of fundamental frequency parameters λ for different models of homogeneity and corrugations (H, N, δ). From this table, we can obtain the following opinions:

i. the values of the natural frequency parameters λ of S- and A-modes for the $(H, N, \delta = 1, 0, 0)$ model may be equal in the case of cylindrical shells ($\varepsilon = 0$), and they are almost close together in the case when $\varepsilon > 1$, but the corresponding vibration modes will be different due to the independence of the governing equations;
ii. the values of λ for $(H = 1, N, \delta)$ model oscillate between increasing and decreasing based on the (N, δ) values, and for the shells with corrugations of small amplitudes, they are less than the ones for circular cylindrical shells;
iii. the values of λ for the model where $H > 1, N, \delta$ model decrease slowly, but for higher H values they take oscillation mode, regardless of the circularity/ellipticity of shell as well as the rate of oscillations for the circular shell is more than one for the oval shell;
iv. the values of λ for non-homogeneous, orthotropic oval cylindrical shells change irregularly with the corrugation characteristic of the shell;
v. orthotropy does not change the behavior of λ values, but affects the relative increase in the fundamental frequencies with increasing the parameters of corrugations.

Fig. 2 presents the effect of the ovality parameter (ε) of homogeneous and non-homogeneous corrugated shells on the λ curves. It can be observed from these figures that the increase in the values of the ovality parameter leads to a decrease in the frequency curves for the shells of homogeneous materials. On the contrary, the λ curves increase initially and decrease later for the shells of non-homogeneous materials in the SS- and AS-modes, and this situation is accompanied by an increase in the flexural stiffness in the circumferential direction. Also, it can be seen that the λ curves of the AA- and SA-modes decrease linearly in the homogeneous and non-homogeneous cases, regardless of the isotropy/orthotropy of the materials. Fig. 3 investigate how the frequency parameter λ depends on the homogeneity variation, regardless of the smooth/corrugation of the shells. For all type modes, the exact curves have two main sections depending on the H values. The first section corresponds to a small homogeneity ratio, and in this section, the non-homogeneity of materials has not effect on the λ curves, and these situations may be considered as homogeneous cases. In the second section, the frequency parameter curve decreases and increases, taking wavy shapes. It can be seen that, for oval shells, the curves corresponding to AS and SA modes are linear ones with increasing H values, and the changes in the behavior of the curves evidenced in Fig. 3(c, d) are due to the dependence of the vibration modes on their cross-section. On the contrary, for a circular cylindrical shell, the non-homogeneity of the materials strongly influences the natural frequency curves, regardless of the smoothness or the corrugations of the shell, see Fig. 3(a, b). Fig. 4 presents the effects of corrugations number N of circular/oval cylindrical shells when they are made vibrate according to the first mode of vibrations. It is found that the increase in the values of N leads to a decrease and an increase in the λ curves, which take wavy shapes after they coincide at $N = 2$ for circular cylindrical shells. On the other hand, the increase in the values of N makes the λ curves separate waving ones, which do not coincide for oval cylindrical shells. Also, it can be seen that the λ curves of the AA and SS modes engage in behaviors similar to those of the SA and AS modes, regardless of the isotropy/orthotropy and the circularity/ovality of the shells. One can consider that the increase in the number of corrugation N, for each value of the amplitude parameter δ, makes stiffer the non-homogeneous shell that has a noncircular profile. Fig. 5 demonstrates the effect of the corrugation amplitude δ on the behavior of the λ curves for a non-homogeneous circular/oval cylindrical shell. One can see from these figures that the frequency parameters slowly decrease with δ, and that they intersect together at a certain value of δ. Furthermore, the corrugation effect becomes unless later. Also, it can be observed that the increase in δ leads to an increase in the SA modes

Table 1
The first three fundamental frequencies parameters λ versus the circumferential rigidity η and non-homogeneity parameter H of a corrugated oval/circular cylindrical shell, ($h_0 = 0.02,\ X = 1,\ \mu = 0.35$).

H	(N, δ)	$\eta =$	Symmetric vibration			Antisymmetric vibration		
			0.5	1	2	0.5	1	2
$\varepsilon = 0$								
1	(0, 0)		0.084871	0.102491	0.112014	0.084871	0.102485	0.111988
			0.096120	0.104323	0.134042	0.096119	0.104323	0.134042
			0.109052	0.145494	0.171190	0.109052	0.145492	0.171190
	(2, 0.1)		0.083730	0.098746	0.108876	0.083568	0.098732	0.108847
			0.092095	0.103665	0.134006	0.092093	0.103549	0.134002
			0.109028	0.145778	0.166043	0.108997	0.145777	0.165870
	(8, 0.1)		0.084762	0.100283	0.109992	0.084941	0.104336	0.114486
			0.093819	0.104130	0.133647	0.098960	0.105250	0.133954
			0.109181	0.145441	0.172167	0.108788	0.145113	0.171915
	(4, 0.3)		0.085022	0.104141	0.117377	0.083358	0.098067	0.107861
			0.100536	0.108476	0.133361	0.090957	0.101826	0.129465
			0.110846	0.144751	0.170761	0.109398	0.144589	0.182630
	(7, 0.3)		0.085978	0.095508	0.105647	0.082268	0.101990	0.119773
			0.088786	0.104372	0.131833	0.101624	0.110643	0.131871
			0.108082	0.143780	0.171352	0.112129	0.145526	0.179376
	(6, 0.8)		0.078312	0.097212	0.104509	0.078348	0.096976	0.104375
			0.091567	0.093056	0.115642	0.091543	0.097125	0.125292
			0.098948	0.131069	0.134607	0.098807	0.130947	0.178476
3.5	(0, 0)		0.058626	0.092298	0.107565	0.064102	0.083004	0.105593
			0.084196	0.094026	0.128498	0.102579	0.099582	0.130610
			0.110663	0.145329	0.163247	0.112079	0.147529	0.170481
	(2, 0.1)		0.057612	0.089164	0.104506	0.064632	0.081203	0.103490
			0.083506	0.094093	0.128418	0.099955	0.099092	0.129917
			0.110622	0.140970	0.158667	0.111069	0.147944	0.161556
	(8, 0.1)		0.057027	0.090631	0.105987	0.062919	0.085014	0.108146
			0.084379	0.095711	0.131047	0.105531	0.098913	0.129639
			0.110287	0.143097	0.161458	0.112614	0.147346	0.172286
	(4, 0.3)		0.057412	0.094176	0.114887	0.063139	0.081278	0.104445
			0.084179	0.099096	0.129901	0.103896	0.101393	0.133594
			0.108188	0.145320	0.168301	0.107721	0.144501	0.170713
	(7, 0.3)		0.057003	0.087563	0.102942	0.063038	0.087524	0.112023
			0.085413	0.097867	0.134091	0.108590	0.096441	0.126711
			0.110417	0.136448	0.155213	0.114832	0.146055	0.181150
	(6, 0.8)		0.064114	0.076755	0.102483	0.065174	0.070994	0.089745
			0.073651	0.094031	0.107560	0.088158	0.094391	0.124307
			0.102246	0.124291	0.139157	0.101709	0.133634	0.181953
$\varepsilon = 0.5$								
1	(0, 0)		0.078515	0.096447	0.107105	0.080566	0.091051	0.101330
			0.087920	0.096680	0.121110	0.082887	0.099426	0.127354
			0.112500	0.146245	0.159951	0.111861	0.146081	0.178339
	(2, 0.1)		0.078033	0.093265	0.104092	0.079586	0.088790	0.099547
			0.084949	0.096412	0.121068	0.080845	0.098879	0.126977
			0.111621	0.146536	0.155661	0.111095	0.146341	0.167998
	(8, 0.1)		0.079708	0.094859	0.105569	0.079824	0.092684	0.102710
			0.086187	0.097955	0.123021	0.084776	0.098692	0.126563
			0.112379	0.146280	0.155854	0.111213	0.145884	0.181397
	(4, 0.3)		0.079445	0.098267	0.110925	0.081326	0.088906	0.100042
			0.093015	0.100735	0.124441	0.082372	0.102599	0.133291
			0.110734	0.146377	0.160542	0.109216	0.145179	0.164312
	(7, 0.3)		0.081018	0.093537	0.104851	0.077920	0.092644	0.102082
			0.083959	0.098704	0.122206	0.085761	0.097268	0.125970
			0.112944	0.142650	0.149430	0.109305	0.145083	0.191194
	(6, 0.8)		0.068899	0.081870	0.099584	0.072802	0.079283	0.087473
			0.089006	0.101878	0.110772	0.076760	0.095438	0.123715
			0.108306	0.132649	0.140609	0.100238	0.131321	0.178165
3.5	(0, 0)		0.077893	0.081997	0.102640	0.074492	0.074947	0.088712
			0.088969	0.091872	0.104032	0.101992	0.095476	0.122311
			0.115131	0.147673	0.162672	0.122681	0.149335	0.184803
	(2, 0.1)		0.075593	0.082171	0.101174	0.073925	0.073811	0.087809
			0.088172	0.088859	0.102218	0.101742	0.095002	0.121691
			0.114192	0.142784	0.159054	0.119686	0.149457	0.175088
	(8, 0.1)		0.076486	0.083301	0.102746	0.073728	0.075955	0.089683
			0.089690	0.090352	0.103912	0.103962	0.094698	0.121486
			0.114975	0.143913	0.159242	0.122481	0.149206	0.187935
	(4, 0.3)		0.081619	0.085086	0.107554	0.077412	0.075727	0.090806
			0.089949	0.095972	0.108212	0.105950	0.100173	0.129978
			0.113468	0.148443	0.163989	0.118598	0.146815	0.167856
	(7, 0.3)		0.075397	0.082146	0.101655	0.072677	0.074702	0.088126
			0.090490	0.089583	0.102568	0.106423	0.093698	0.121282
			0.115463	0.139360	0.155190	0.118752	0.147946	0.196701
	(6, 0.8)		0.076094	0.066043	0.082529	0.073043	0.062504	0.073705
			0.081584	0.099021	0.111174	0.086785	0.093561	0.121708
			0.110513	0.134780	0.148602	0.109512	0.134031	0.180539

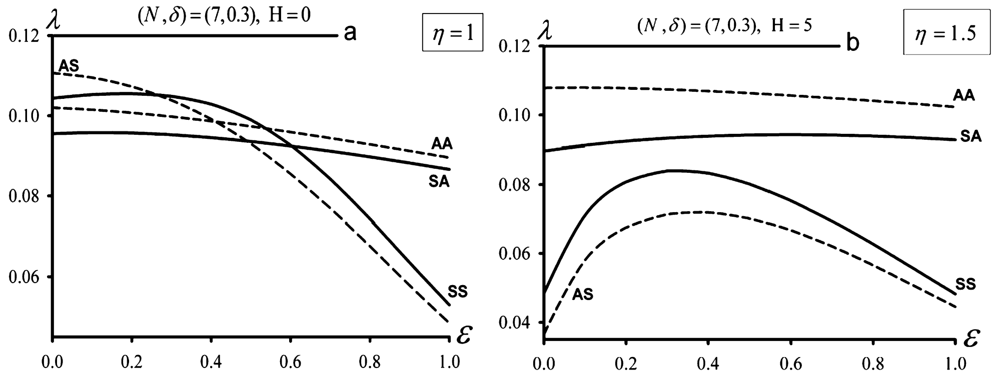

Fig. 2. First 1st frequency parameter versus ovality parameter for homogeneous/inhomogeneous, isotropic/orthotropic corrugated circular and oval cylindrical shells for S and A modes ($X = 1$, $h_0 = 0.02$, $\mu = 0.35$).

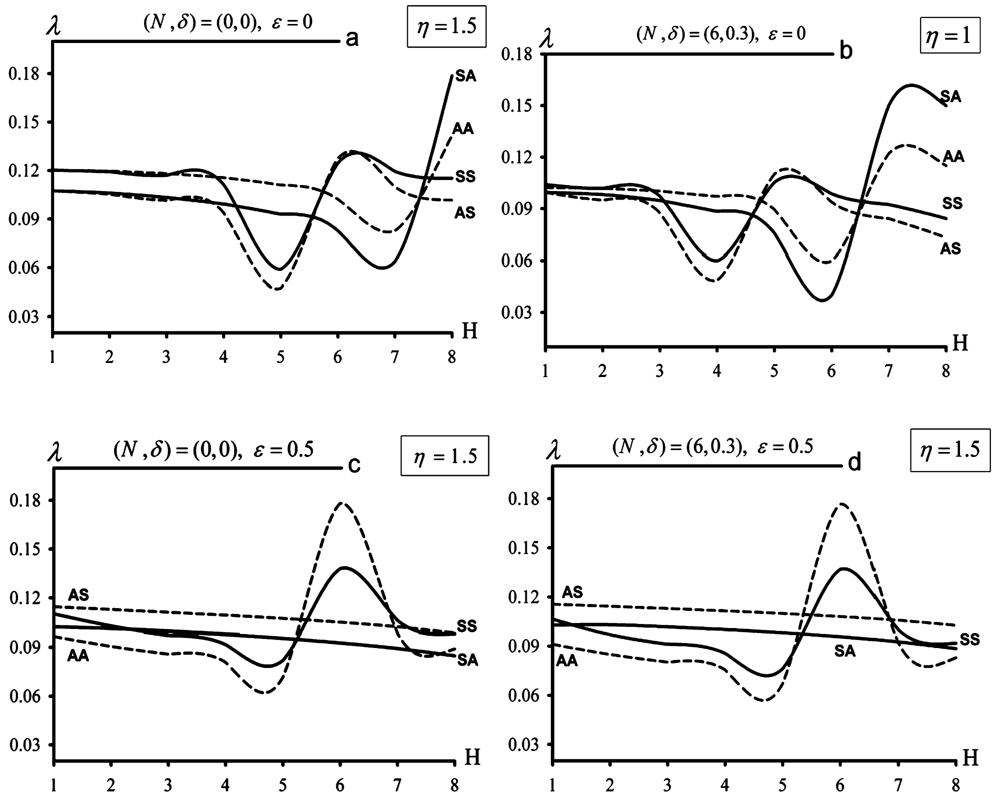

Fig. 3. First 1st frequency parameter versus homogeneity parameter for isotropic and orthotropic corrugated circular and oval cylindrical shells for S and A modes.

until they become the same as in the AA modes, and to a decrease in the SS modes until they become the same as in AS modes. One can think that, for a certain value of δ, the natural frequencies become antisymmetric, see Fig. 5(b).

4.3. Shapes of the vibration modes

We can define the vibration mode of the shell structure that is the deflection of a structure at a certain frequency λ, which depends on the geometrical form and on the material properties of the structure. When the shell has a corrugation form (longitudinally or circumferentially), it has high bending and compressive stiffness in the corrugation direction; then its vibration is more stable than that of its smooth form. To obtain the (SS, AA) vibration modes corresponding to a particular frequency parameter of the shell, we substitute λ into Eqs. (24) and (25) and, using a Gaussian elimination technique, we obtain the state vector at any point of the cross section. It is found that non-homogeneity and the corrugation model extremely affect the vibration behavior of such shells. Figs. 6 and 7 depict the transverse, circumferential and longitudinal

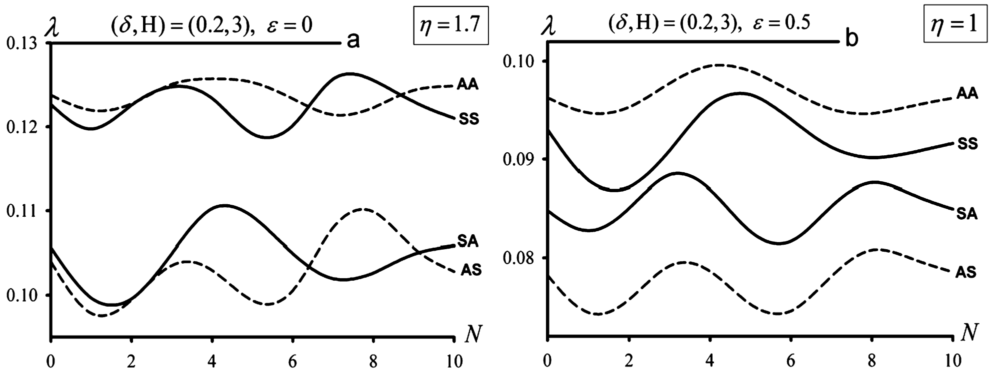

Fig. 4. Effects of the corrugation number on the vibration frequency parameter of inhomogeneous elastic circular/oval cylindrical shells with a sine-shaped hoop for S and A modes.

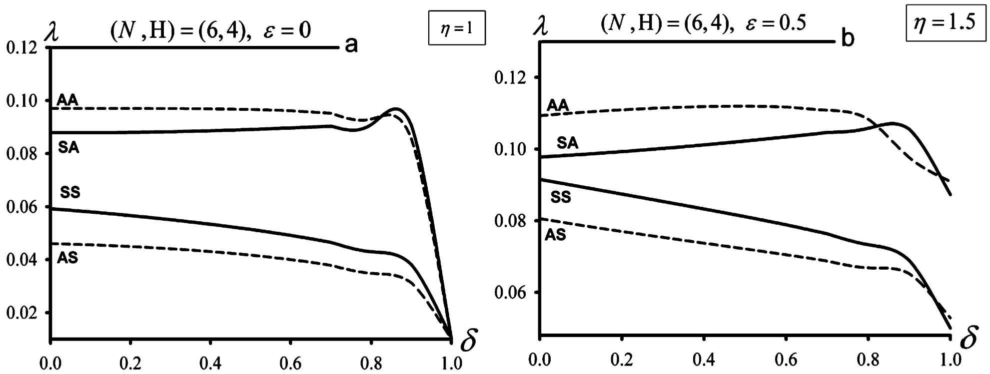

Fig. 5. Effects of corrugation amplitude parameter on vibration frequency parameter of a corrugated elastic circular and oval cylindrical shells for S and A modes.

deflections, with thick curves, along the circumference of smooth/corrugated, isotropic/orthotropic circular/oval cylindrical shells with homogeneous/non-homogeneous materials for (SS, SA, AA, AS) type modes that correspond to some natural frequency parameters λ, listed in Table 1, whereas the dotted curves show the original corrugated shell shape based on (N, δ) values before deformation. There are considerable differences between the modes of homogeneous, smooth and corrugated shells corresponding to the models $(H = 1,\ N = 0,\ \delta = 0)$ and $(H = 1,\ N > 0,\ \delta > 0)$, respectively, where the vibration modes are distributed regularly over the shell surface for the smooth shells, whereas most modes become more pronounced around the regions $(\bar{s} = 0, 0.5)$ than the corrugated ones due to high bending and compressive stiffness in the hoop direction, regardless of the circularity/ovality of the shells. For the corrugated shells of small corrugation parameters, the mode shapes of deflections are distributed regularly for smooth ones, whereas the situation is different for big corrugation parameters; especially, for the axial deflection $(u/u_{\max})$, the modes are more reduced around the area $(\bar{s} = 0.25)$ due to low flexural stiffness in the longitudinal direction, see Figs. 6 and 7, when $H = 1$. On the contrary, for non-homogeneous corrugated shells, the transverse, circumferential and longitudinal deflections are not distributed regularly on the shell circumference, and may be concentrated around the area $(\bar{s} = 0.25)$, where the homogeneity variation is the maximum, as well as they become more pronounced by increasing the corrugation parameters, regardless of the circularity/ovality of the shells, see Fig. 7, when $H = 3.5$. From these figures, one may be observe that the three vibration modes become clearer in the area of the oval centrode when the ovality parameter increases, regardless of the homogeneity/non-homogeneity of the materials. One may conclude that the effects of the corrugation parameters and of the homogeneity of the materials are more significant than that of the smooth shell on the vibration behavior, regardless of the fact shells are isotropic or orthotropic, and no difference is observed between the isotropic and orthotropic cases of the type modes. The orthotropy of the material, which is represented by the ratios μ and η, does not considerably influence the shape of the vibration modes, regardless of the circularity or of the ovality of the shells.

5. Conclusions

A new approach to the analysis of the vibration behavior of circumferentially, corrugated circular, and oval cylindrical shells with non-homogeneous material is presented. Basic relations have been obtained for orthotropic cylindrical shells; it has been evidenced that Young's moduli and density vary continuously in the circumferential direction, depending on the tangential angle. Due to the Flügge's shell theory and the transfer matrix of the shell, the governing equations are formulated

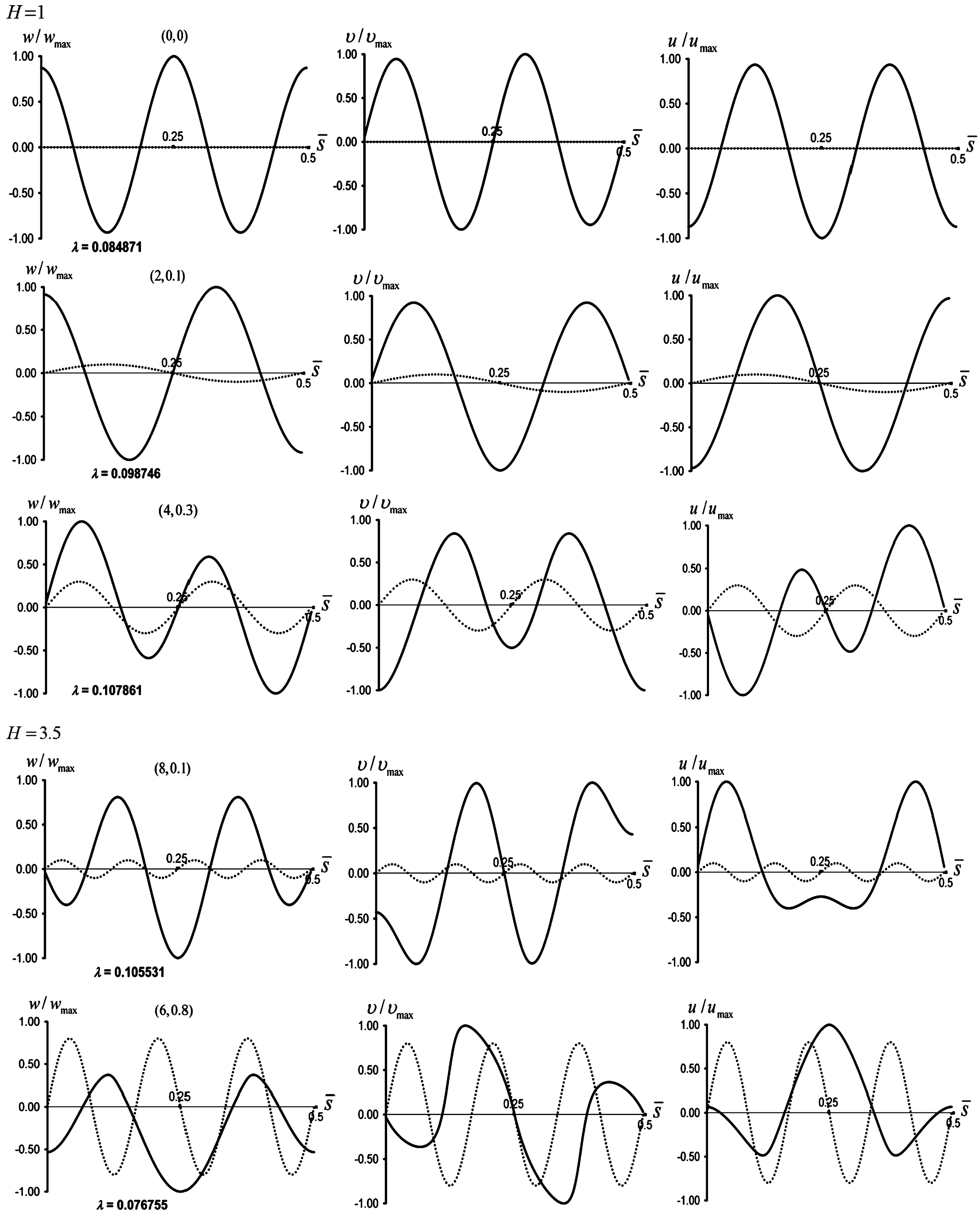

Fig. 6. The vibration modes of homogeneous/inhomogeneous, isotropic/orthotropic corrugated circular cylindrical shell for S and A modes ($\varepsilon = 0$, $X = 1$, $h_0 = 0.02$).

and reduced to eight first-order differential ones in the circumferential coordinate; they can be solved as a one-dimensional boundary-value problem using the Romberg integration method. The computed results allowed us to find the fundamental vibration frequencies and the corresponding mode shapes of the deflection displacements for four type modes of vibration (SS, SA, AA, AS). In detail, a parametric study is assumed to investigate the influence of various parameters, such as ovality, corrugations, orthotropy and non-homogeneity of materials, on the vibration behavior of the shell. From this study, it could be concluded that:

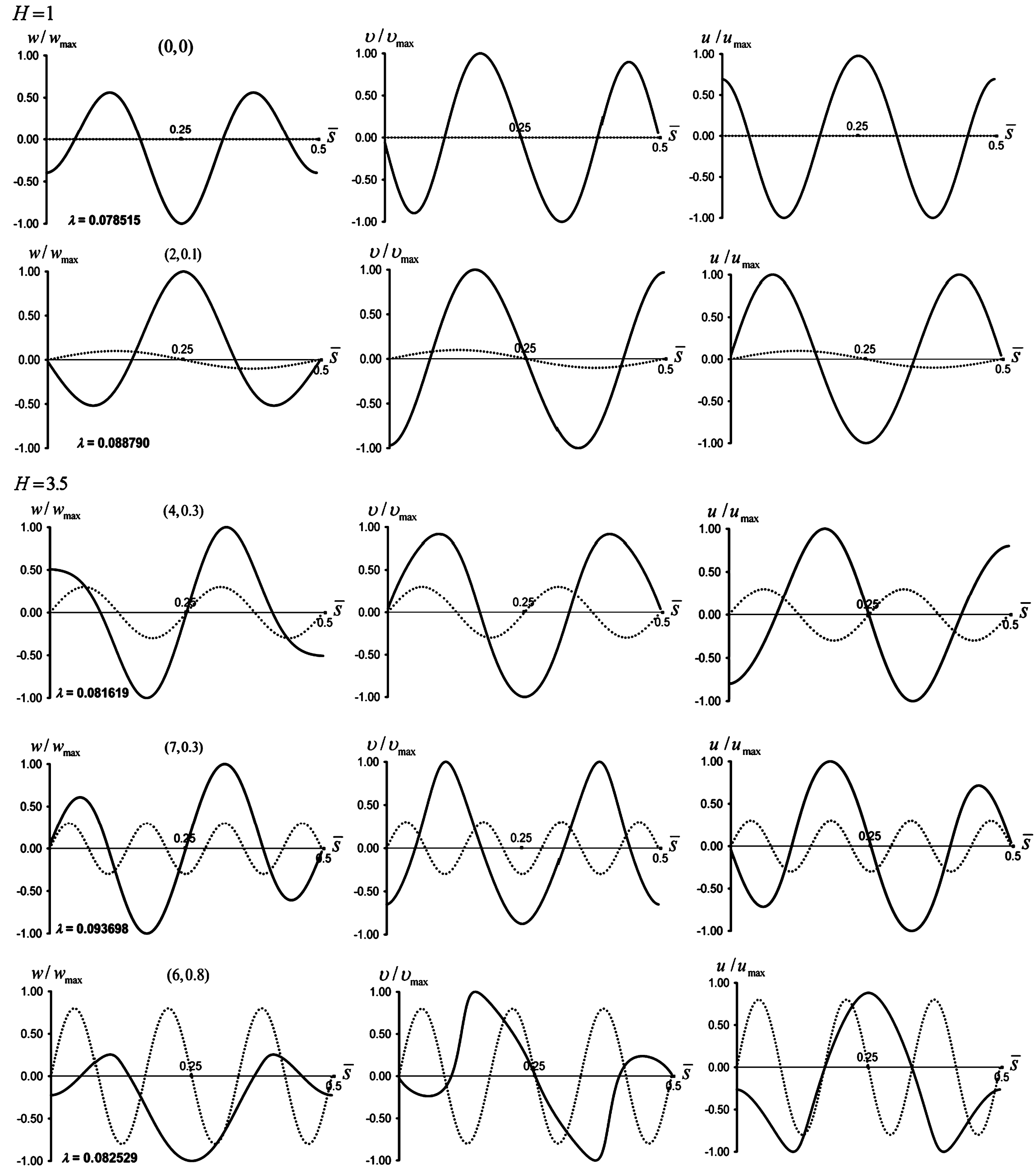

Fig. 7. The vibration modes of homogeneous/inhomogeneous, isotropic/orthotropic corrugated oval cylindrical shell for S and A modes ($\varepsilon = 0.5$, $X = 1$, $h_0 = 0.02$).

- the computed results confirm that the fundamental frequencies for shells with corrugations of small amplitudes are less than those for circular shells, and that the circumferential corrugated shells have a larger vibration margin than the smooth ones, regardless of the isotropy/orthotropy of the shell material;
- the effect of corrugation becomes stronger by increasing its length and amplitude due to the imperfection of the local radius of shell;
- when the amplitude of the corrugations reaches a certain value, the natural frequencies become almost equal for all type modes of vibrations, and the corrugations become useless, and an increase in the number of corrugations makes the shell stiffer, which leads to an increase in frequencies, regardless of the cross-section geometry of shell;
- the non-homogeneity of the material has a significant effect on the vibration behavior of circular and oval cylindrical shells, regardless of the isotropy or orthotropy of material, and for higher H values, the effects become more dominant for the corrugated shells than for the smooth ones;

- as the ovality parameter ε increases, the effect of non-homogeneity on the SS and AS modes of vibration increases initially and decreases with high values of ε, whereas the effect of non-homogeneity is insignificant for the AA and SA modes with ε values;
- the mode shapes of vibration for the corrugated shells are not distributed regularly over the shell surface as in the case of the smooth ones, and become more sensitive to an increase in the corrugation parameters (N, δ);
- the mode shapes of vibration are more sensitive to the corrugations parameters and ovality parameter than the non-homogeneity ratio and the orthotropy parameters of the shell;
- the vibration behavior of a corrugated, isotropic and orthotropic oval shell is found to have an almost identical behavior, regardless of the homogeneity/non-homogeneity of material, and the mode shapes vary significantly compared to those in the case of a circular cylinder;
- in general, the geometry of the shell and the properties of the material play an important role in the vibrational mechanical behavior.

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