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On the vibrations of a string with a concentrated mass and rapidly oscillating density

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ABSTRACT

The paper is devoted to the vibrations of a string I with a concentrated mass $\varepsilon^{-1}\varkappa Q\left(\varepsilon^{-1}x\right)$ and rapidly oscillating density $q\left(x,\mu^{-1}x\right)$, where $q(x,\zeta)$ is a 1-periodic in ζ function, $Q(\xi)$ is a function with compact support, the integral of which is equal to one, $0\in I$, μ,ε are small positive parameters, $\varkappa\in\mathbb{R}$. By combining homogenization and the method of matched asymptotic expansions, we construct solutions to the problems up to $O(\varepsilon+\mu)$.

1. Introduction

Studies on the vibrations of a string with inhomogeneous rapidly oscillating density have been conducted for a long time (see, for instance, [1–3]). In these monographs, using the homogenization method, the authors constructed asymptotics with respect to a small parameter (the period of the oscillations) of solutions to the boundary value problem.

The vibration of a string with a concentration of mass on a small set has been studied by means of other methods for a long time, too. The convergence of solutions to the respective boundary value problem was investigated in [4], and the asymptotics with respect to a small parameter was constructed in [5] by means of the method of matched asymptotic expansions [6]. The analogous spectral problem for the Laplace operator in 3D space was considered in [7]. Note that a close mathematical problem on the convergence of the Schrödinger operator with δ-type potential on the axis was studied in [8].

In the present paper, using the combination of homogenization and the method of matching of asymptotic expansions [9], we study the case when the string with a concentrated mass has an inhomogeneous rapidly oscillating density. We get the homogenized (limit) solution to a boundary value problem, up to $O(\mu+\varepsilon)$, where μ is a period of rapid oscillations of the density, and ε is the order of the length of the small part of the string, on which the concentrated mass is located.

2. Settings of the problem and main result

Let I be (a,b), $\{0\}\in I$. Consider the problem

$$\mathcal{L}_{\mu,\varepsilon}u_{\mu,\varepsilon}:=-\frac{\mathrm{d}^2u_{\mu,\varepsilon}}{\mathrm{d}x^2}+\left(q\left(x,\frac{x}{\mu}\right)+\varepsilon^{-1}\varkappa Q\left(\frac{x}{\varepsilon}\right)\right)u_{\mu,\varepsilon}=f(x),\quad x\in I$$
$$l_a u_{\mu,\varepsilon}:=h_a u_{\mu,\varepsilon}(a)-H_a u'_{\mu,\varepsilon}(a)=0\qquad l_b u_{\mu,\varepsilon}:=h_b u_{\mu,\varepsilon}(b)+H_b u'_{\mu,\varepsilon}(b)=0 \tag{1}$$

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where $q(x,\zeta)$ is a 1-periodic in ζ function belonging to $C^{2,0}(\bar{I}\times(-\infty,\infty))$, $q(x,\zeta)>0$, $Q(\xi)\in C_0(-\infty,\infty)$,

$$\int\limits_{-\infty}^{\infty} Q(\xi)\mathrm{d}\xi = 1$$

$0<\mu,\varepsilon\ll 1$, h_a, h_b, H_a, $H_b\geqslant 0$, $h_a+H_a>0$, $h_b+H_b>0$, $f\in L^2(I)$. Although, for the concentrated mass, $\varkappa>0$ and $Q(\xi)\geqslant 0$, we consider a more general case, when $\varkappa\in\mathbb{R}$ and $Q(\xi)$ can take negative values as well. We will assume that $\operatorname{supp} Q(\xi)\subset[-1,1]$.

For functions $G(x)$, we use the notation $\{G\}(0)=G(+0)-G(-0)$, and for 1-periodic in ζ functions $g(x,\zeta)$ we denote

$$[g](x) := \int\limits_0^1 g(x,\zeta)\mathrm{d}\zeta$$

Since $q>0$, the boundary value problems

$$\mathcal{L}u_0 := -\frac{\mathrm{d}^2u_0}{\mathrm{d}x^2} + [q](x)u_0 = f, \quad x\in I, \qquad l_a u_0 = l_b u_0 = 0$$

$$\mathcal{L}y = 0, \quad x\in I\backslash\{0\}, \qquad l_a y = l_b y = 0, \quad y(0)=1$$

have unique solutions in $H^2(I)$ and $H^1(I)\cap H^2(a,0)\cap H^2(0,b)$, respectively. It is easy to see that if $\varkappa\neq\varkappa_0 := \{y'\}(0)$, then the function

$$u^0(x) := u_0(x) + \frac{\varkappa u_0(0)}{\varkappa_0-\varkappa}y(x) \tag{2}$$

solves uniquely the boundary value problem

$$\mathcal{L}u^0 = f, \quad x\in I\backslash\{0\}, \qquad l_a u^0 = l_b u^0 = 0, \quad \left\{\left(u^0\right)'\right\}(0) = \varkappa u^0(0) \tag{3}$$

for any $f\in L^2(I)$ in $H^1(I)\cap H^2(a,0)\cap H^2(0,b)$.

The main result of the paper is the proof of the following proposition.

Theorem 2.1. *Assume that $\varkappa\neq\varkappa_0$. Then for sufficiently small ε and μ, the solution to boundary value problem (1) is uniquely determined in $H^2(I)$ and satisfies the uniform estimate*

$$\|u_{\mu,\varepsilon}-u^0\|_{L^2(I)} \leqslant (\mu+\varepsilon)C\|f\|_{L^2(I)} \tag{4}$$

where $u^0(x)$ is a solution to boundary value problem (3).

3. Preliminaries and auxiliary assertions

Consider the following boundary value problems:

$$\mathcal{L}_\mu u_\mu := -\frac{\mathrm{d}^2u_\mu}{\mathrm{d}x^2} + q\left(x,\frac{x}{\mu}\right)u_\mu = f, \quad x\in I, \qquad l_a u_\mu = l_b u_\mu = 0 \tag{5}$$

$$\mathcal{L}_\mu y_\mu = 0, \quad x\in I\backslash\{0\}, \qquad l_a y_\mu = l_b y_\mu = 0, \qquad y_\mu(0)=1 \tag{6}$$

$$\mathcal{L}_\mu u^\mu = f, \quad x\in I\backslash\{0\}, \qquad l_a u^\mu = l_b u^\mu = 0, \qquad \{(u^\mu)'\}(0) = \varkappa u^\mu(0) \tag{7}$$

Since $q>0$, the boundary value problems (5) and (6) have unique solutions from $H^2(I)$ and $H^1(I)\cap C^2[a,0]\cap C^2[0,b]$, respectively, and the estimate

$$\|u_\mu\|_{H^2(I)} \leqslant C\|f\|_{L^2(I)} \tag{8}$$

holds true.

It is easy to see that if $\varkappa\neq\varkappa_\mu := \{y'_\mu\}(0)$, then the function

$$u^\mu(x) := u_\mu(x) + \frac{\varkappa u_\mu(0)}{\varkappa_\mu-\varkappa}y_\mu(x) \in H^1(I)\cap H^2(a,0)\cap H^2(0,b) \tag{9}$$

is a unique solution to problem (7) for any $f\in L^2(I)$, in addition, because of (8) and the embedding of $H^1(I)$ in $C(\bar{I})$ (see, for instance, [10, Chapter III, §6]), which satisfies

$$\|u^\mu\|_{H^2(a,0)} + \|u^\mu\|_{H^2(0,b)} \leqslant C\frac{\|f\|_{L^2(I)}}{|\varkappa_\mu-\varkappa|} \tag{10}$$

Since $q(x,\zeta) \in C^{2,0}(\overline{I} \times (-\infty,\infty))$ and $q(x,\zeta) > 0$, then the construction of asymptotics of the functions $u_\mu(x)$ and $y_\mu(x)$ by the homogenization method leads to:

$$\|u_\mu - u_0\|_{H^1(I)} \leqslant C\mu \|f\|_{L^2(I)}, \qquad \|y_\mu - y\|_{C^1[a,0]} + \|y_\mu - y\|_{C^1[0,b]} = O(\mu) \tag{11}$$

From (11) and the embedding theorems, it follows that

$$|u_\mu(0) - u_0(0)| \leqslant \mu C \|f\|_{L^2(I)}, \quad \varkappa_\mu = \varkappa_0 + O(\mu), \quad \|y_\mu - y\|_{H^1(I)} = O(\mu) \tag{12}$$

Due to (2), (9), (11) and (12), we derive for $\varkappa \neq \varkappa_0$ and sufficiently small μ the estimate

$$\|u^\mu - u^0\|_{H^1(I)} \leqslant C\mu \|f\|_{L^2(I)} \tag{13}$$

4. Existence and uniqueness of a solution to boundary value problem (1)

Let $\widetilde{h}_a$ be equal to $h_a H_a^{-1}$, if $H_a \neq 0$, and $\widetilde{h}_a = 0$, if $H_a = 0$. We define in an analogous way $\widetilde{h}_b$. The quadratic forms of the boundary value problems (5), (7) and (1) read as

$$\begin{aligned}
\mathfrak{h}_\mu[u] &= \int_I \left((u'(x))^2 + q\left(x, \frac{x}{\mu}\right) u^2(x) \right) \mathrm{d}x + \widetilde{h}_a u^2(a) + \widetilde{h}_b u^2(b), \\
\mathfrak{h}^\mu[u] &= \mathfrak{h}_\mu[u] + \varkappa u^2(0), \\
\mathfrak{h}_{\mu,\varepsilon}[u] &= \mathfrak{h}^\mu[u] + \varkappa \varepsilon^{-1} \int_I Q\left(\frac{x}{\varepsilon}\right) u^2(x)\mathrm{d}x - \varkappa u^2(0)
\end{aligned}$$

correspondingly, where $u(a) = 0$, if $H_a = 0$, and $u(b) = 0$, if $H_b = 0$.

Let us show the validity of the following estimate

$$\left|\mathfrak{h}_{\mu,\varepsilon}[u] - \mathfrak{h}^\mu[u]\right| \leqslant M_1 \varepsilon^{1/2} \|u\|_{L^2(I)} + M_2 \varepsilon^{1/2} \mathfrak{h}^\mu[u] \tag{14}$$

Denote $v(x) = u(x) - u(0)$. Then $v(0) = 0$, $v'(x) = u'(x)$,

$$\int_I \varepsilon^{-1} Q\left(\frac{x}{\varepsilon}\right) u^2(x)\mathrm{d}x - u^2(0) = 2\int_I \varepsilon^{-1} Q\left(\frac{x}{\varepsilon}\right) u(0)v(x)\mathrm{d}x + \int_I \varepsilon^{-1} Q\left(\frac{x}{\varepsilon}\right) v^2(x)\mathrm{d}x$$

The Friedrichs–Steklov inequality leads to the estimate

$$\begin{aligned}
2\left|\int_I \varepsilon^{-1} Q\left(\frac{x}{\varepsilon}\right) u(0)v(x)\mathrm{d}x\right| &\leqslant 2\int_{-\varepsilon}^{\varepsilon} \varepsilon^{-1} \max_{\mathbb{R}} |Q| \cdot |u(0)v(x)|\mathrm{d}x \\
&\leqslant \int_{-\varepsilon}^{\varepsilon} \varepsilon^{-1} \max_{\mathbb{R}} |Q| \left(\varepsilon^{1/2} u^2(0) + \varepsilon^{-1/2} v^2(x)\right) \mathrm{d}x \\
&\leqslant \max_{\mathbb{R}} |Q| \varepsilon^{1/2} \left(2u^2(0) + \int_{-\varepsilon}^{\varepsilon} \left(v'\right)^2(x)\mathrm{d}x \right)
\end{aligned}$$

So, we derive

$$\begin{aligned}
\left|\varepsilon^{-1} \int_I Q\left(\frac{x}{\varepsilon}\right) u^2(x)\mathrm{d}x - u^2(0)\right| &\leqslant 2\max_{\mathbb{R}} |Q| \varepsilon^{1/2} u^2(0) + \max_{\mathbb{R}} |Q| \left(\varepsilon^{1/2} + \varepsilon\right) \int_{-\varepsilon}^{\varepsilon} \left(u'\right)^2(x)\mathrm{d}x \\
&\leqslant C\varepsilon^{1/2} \|u\|^2_{H^1(I)}, \quad u \in H^1(I)
\end{aligned} \tag{15}$$

Due to the definition of the quadratic forms and estimate (15), we get that the forms $\mathfrak{h}_{\mu,\varepsilon}$, $\mathfrak{h}^\mu$ are bounded from below, and for $\varkappa \geqslant 0$, inequality (14) holds.

Let $\varkappa < 0$. Consider the case $H_a + H_b \neq 0$. It is known (see, for instance, [10, Chapter, §4]) that for any fixed interval $I' = (a', b')$ such that $\overline{I} \subset I'$ and any $u \in H^1(I)$, there exists a continuation $U \in H^1(I')$ such that

$$U(a') = U(b') = 0, \qquad \|U\|_{H^1(I')} \leqslant M\|u\|_{H^1(I)}, \qquad \|U\|_{L^2(I')} \leqslant M\|u\|_{L^2(I)} \tag{16}$$

Since $\alpha\|u\|^2_{H^1(I)} \leqslant \mathfrak{h}_\mu[u]$, $\alpha > 0$, due to (16) we derive

$$h^{\mu}[u]=h_{\mu}[u]+\varkappa u^2(0)\geqslant\alpha\|u\|^2_{H^1(I)}+\varkappa u^2(0)\geqslant\frac{\alpha}{M}\|U\|^2_{H^1(I')}+\varkappa U^2(0)$$
$$=\frac{\alpha}{M}\|U\|^2_{H^1(I')}+2\varkappa\int\limits_{a'}^{0}U'(x)U(x)\mathrm{d}x\geqslant\frac{\alpha}{M}\|U\|^2_{H^1(I')}+\varkappa\left(\gamma\|U\|^2_{L^2(I')}+\gamma^{-1}\|U'\|^2_{L^2(I')}\right)$$
$$\geqslant\left(\frac{\alpha}{M}+\varkappa\gamma^{-1}\right)\|U\|^2_{H^1(I')}+\varkappa\gamma\|U\|^2_{L^2(I')}\geqslant\left(\frac{\alpha}{M}+\varkappa\gamma^{-1}\right)\|u\|^2_{H^1(I)}+\varkappa\gamma M\|u\|^2_{L^2(I)}$$

So, for large $\gamma>0$ we have

$$\|u\|^2_{H^1(I)}\leqslant K_1\|u\|^2_{L^2(I)}+K_2h^{\mu}[u],\qquad K_i>0$$

This estimate and inequality (15) imply estimate (14). In case $\varkappa<0$, $H_a=H_b=0$ the proof is similar and more simple.

Since $\varkappa_\mu \underset{\mu\to 0}{\to} \varkappa_0$ (see, (12)), then estimate (14) by means of [11, Chapter VI, Theorem 3.9] shows that, for $\varkappa\neq\varkappa_0$ and sufficiently small μ, boundary value problem (1) has a unique solution in $H^2(I)$. Moreover, for solutions to boundary value problems (1) and (7), the following uniform estimate in μ:

$$\|u_{\mu,\varepsilon}-u^{\mu}\|_{L^2(I)}\leqslant C\varepsilon^{1/2}\|f\|_{L^2(I)} \tag{17}$$

holds true. From this estimate and (13) we derive

$$\|u_{\mu,\varepsilon}-u^{0}\|_{L^2(I)}\leqslant C(\varepsilon^{1/2}+\mu)\|f\|_{L^2(I)} \tag{18}$$

Thus, to complete the proof of Theorem 2.1, it is sufficient to improve estimate (18) to get (4).

From inequalities (17) and (10), it follows that, for $\varkappa\neq\varkappa_0$ and sufficiently small μ, the solutions to (1) satisfy the uniform in μ and ε estimate $\|u_{\mu,\varepsilon}\|_{L^2(I)}\leqslant C\|f\|_{L^2(I)}$. On the other hand, this inequality leads to the estimate

$$\|w\|_{L^2(I)}\leqslant C\left(\|F\|_{L^2(I)}+|\widetilde{A}|+|\widetilde{B}|\right) \tag{19}$$

for solutions to the boundary value problem

$$\mathcal{L}_{\mu,\varepsilon}w=F,\quad x\in I,\qquad l_a w=\widetilde{A}\qquad l_b w=\widetilde{B}$$

outside a vicinity of the point $\varkappa_0$ for sufficiently small μ.

5. Derivation of estimate (4)

The construction of a formal approximation $Z_\mu(x,\varepsilon)$ of solutions to the boundary value problem (1) is based on the well-known method of matched asymptotic expansions [9,12]. Hence, we omit trivial explanations on getting the structure of $Z_\mu(x,\varepsilon)$.

Denote

$$v_{1,\mu}(\xi)=\varkappa u^{\mu}(0)\left(\xi\int\limits_{-\infty}^{\xi}Q(\tau)\mathrm{d}\tau-\int\limits_{-\infty}^{\xi}\tau Q(\tau)\mathrm{d}\tau\right)+\left(u^{\mu}\right)'(-0)\xi$$
$$\widetilde{v}_{1,\mu}(\xi)=v_{1,\mu}(\xi)-\left(u^{\mu}\right)'(\mp 0)\xi,\quad \mp\xi>0$$
$$Z_\mu(x,\varepsilon)=u^{\mu}(x)+\varepsilon\widetilde{v}_{1,\mu}\left(\frac{x}{\varepsilon}\right) \tag{20}$$

Then, $Z_\mu\in H^2(I)$ due to (9) and since $v''_{1,\mu}(\xi)=\varkappa u^{\mu}(0)Q(\xi)$, then due to (7) we have

$$\mathcal{L}_{\mu,\varepsilon}Z_\mu(x,\varepsilon)=f(x)+\mathcal{I}^1_\mu(x,\varepsilon)+\mathcal{I}^2_\mu(x,\varepsilon)+\mathcal{J}_\mu(x,\varepsilon),\quad x\in I$$
$$l_aZ_\mu=0,\qquad l_bZ_\mu=-\varepsilon u^{\mu}(0)h_b\varkappa\int\limits_{-\infty}^{\infty}\tau Q(\tau)\mathrm{d}\tau \tag{21}$$

where

$$\mathcal{I}^1_\mu(x,\varepsilon)=\varepsilon^{-1}\varkappa Q\left(\frac{x}{\varepsilon}\right)\left(u^{\mu}(x)-u^{\mu}(0)-\left(u^{\mu}\right)'(\mp 0)x\right),\quad \mp x>0$$
$$\mathcal{I}^2_\mu(x,\varepsilon)=\varepsilon q\left(x,\frac{x}{\mu}\right)\widetilde{v}_{1,\mu}\left(\frac{x}{\varepsilon}\right),\qquad \mathcal{J}_\mu(x,\varepsilon)=\varkappa Q\left(\frac{x}{\varepsilon}\right)v_{1,\mu}\left(\frac{x}{\varepsilon}\right)$$

From (20), (10) and the embedding theorems it follows that

$$\|\mathcal{I}_\mu^2\|_{L^2(I)} + |l_b Z_\mu| \leqslant \varepsilon C \|f\|_{L^2(I)} \tag{22}$$

If $f \in C(\overline{I})$, then, using for $\mathcal{I}_\mu^1(x,\varepsilon)$ the formula for the remainder term of the Taylor series in the integral form and the equation from (7), we get

$$\mathcal{I}_\mu^1(x,\varepsilon) = \varepsilon^{-1}\varkappa Q\left(\frac{x}{\varepsilon}\right)\int_0^x (x-t)\left(q\left(t,\frac{t}{\mu}\right)u^\mu(t) - f(t)\right)\mathrm{d}t$$

Then, keeping in mind the Cauchy–Bunjakovski–Schwarz inequality and estimate (10), we derive

$$\|\mathcal{I}_\mu^1\|_{L^2(I)} \leqslant \varepsilon C \|f\|_{L^2(I)} \tag{23}$$

uniformly in μ. Since the set $C(\overline{I})$ is dense in $L^2(I)$, then this estimate is valid for any $f \in L^2(I)$.

However, it easy to see that $\|J_\mu\|_{L^2(I)} = O(\varepsilon^{1/2})$. Consequently we need a corrector term for the function $Z_\mu(x,\varepsilon)$. Let us define that

$$\widetilde{v}_{2,\mu}(\xi) := \varkappa\left(\xi\int_{-\infty}^{\xi} v_{1,\mu}(\tau)Q(\tau)\mathrm{d}\tau - \int_{-\infty}^{\xi}\tau v_{1,\mu}(\tau)Q(\tau)\mathrm{d}\tau\right) \tag{24}$$

$$W_\mu(x,\varepsilon) := Z_\mu(x,\varepsilon) + \varepsilon^2\widetilde{v}_{2,\mu}\left(\frac{x}{\varepsilon}\right) \tag{25}$$

Then $W_\mu \in H^2(I)$ and since $v''_{2,\mu}(\xi) = v_{1,\mu}(\xi)Q(\xi)$, then by means of (21) we have

$$\begin{gathered}\mathcal{L}_{\mu,\varepsilon}W_\mu(x,\varepsilon) = f(x) + \mathcal{I}_\mu^1(x,\varepsilon) + \mathcal{I}_\mu^2(x,\varepsilon) + \mathcal{I}_\mu^3(x,\varepsilon), \quad x \in I \\ l_a W_\mu = 0, \qquad l_b W_\mu = l_b Z_\mu + \varepsilon\varkappa\int_{-\infty}^{\infty} v_{1,\mu}(\tau)Q(\tau)\Big(h_b(b-\varepsilon\tau) + H_b\Big)\mathrm{d}\tau\end{gathered} \tag{26}$$

where

$$\mathcal{I}_\mu^3(x,\varepsilon) = \varepsilon\varkappa Q\left(\frac{x}{\varepsilon}\right)\widetilde{v}_{2,\mu}\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 q\left(x,\frac{x}{\mu}\right)\widetilde{v}_{2,\mu}\left(\frac{x}{\varepsilon}\right) \tag{27}$$

From (27), (24), (20), (22), (23) and (10) we get

$$\|\mathcal{I}_\mu^1\|_{L^2(I)} + \|\mathcal{I}_\mu^2\|_{L^2(I)} + \|\mathcal{I}_\mu^3\|_{L^2(I)} + |l_b W_\mu| \leqslant \varepsilon C\|f\|_{L^2(I)}$$

Then, due to the problems (1) and (26) we get

$$\|\mathcal{L}_{\mu,\varepsilon}(u_{\mu,\varepsilon} - W_\mu)\|_{L^2(I)} + |l_a\left(u_{\mu,\varepsilon} - W_\mu\right)| + |l_b\left(u_{\mu,\varepsilon} - W_\mu\right)| \leqslant \varepsilon C\|f\|_{L^2(I)}$$

uniformly in μ; and using (19), we derive

$$\|u_{\mu,\varepsilon} - W_\mu\|_{L^2(I)} \leqslant \varepsilon C\|f\|_{L^2(I)} \tag{28}$$

From (20) and (10), it follows that

$$\|u^\mu - Z_\mu\|_{L^2(I)} \leqslant \varepsilon C\|f\|_{L^2(I)} \tag{29}$$

In an analogous way, using (25), (24), (20) and (10), we get

$$\|W_\mu - Z_\mu\|_{L^2(I)} \leqslant \varepsilon C\|f\|_{L^2(I)} \tag{30}$$

The inequalities (28), (29), (30) and (13) lead to estimate (4).

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