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On a singularly perturbed Steklov problem in a domain perforated along the boundary

Le problème Steklov singulièrement perturbé dans un domaine perforé le long de la frontière

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A B S T R A C T

We study the asymptotic behavior of solutions and eigenelements to a boundary value problem for the Laplace equation in a domain perforated along part of the boundary. On the boundary of holes, we set the homogeneous Dirichlet boundary condition and the Steklov spectral condition on the mentioned part of the outer boundary of the domain. Assuming that the boundary microstructure is periodic, we construct the limit problem and prove the homogenization theorem.

R É S U M É

Nous étudions le comportement asymptotique des solutions et des éléments propres à un problème aux limites pour l'équation de Laplace dans un domaine perforé le long d'une partie de la frontière. Sur la frontière de trous, nous posons la condition de Dirichlet homogène et la condition spectrale de Steklov sur la part mentionnée de la frontière extérieure du domaine. En supposant que la microstructure de la frontière est périodique, nous construisons le problème aux limites et prouvons le théorème d'homogénéisation.

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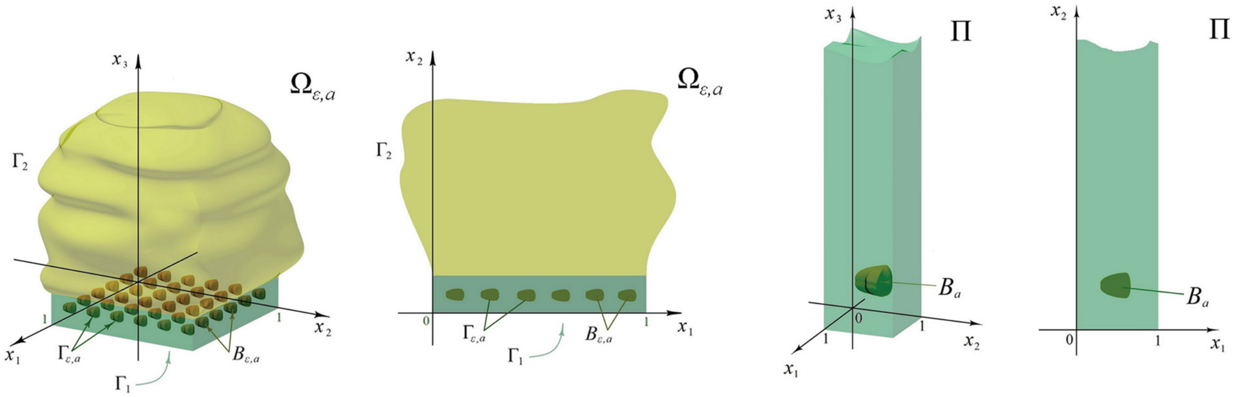

Fig. 1. Structure of the domain $\Omega_{\varepsilon,a}$ and the cell of periodicity.

1. Introduction

Steklov's spectral problem is well known and it has been studied for several decades (see, for instance [1–5]). There are numerous papers dealing with homogenization of problems in domains perforated along the boundary (e.g., see [6–15]).

In this paper, we study spectral problems with Steklov-type boundary conditions in 2D and 3D domains periodically perforated along part of the boundary, which are of interest in connection with the applications. Under the assumption that the ratio between the diameter of the cavities and the distance between them tends to zero, we present different cases of the limiting behavior of eigenpairs.

2. Setting of the problem and main results

Let Ω be a bounded domain in $\mathbb{R}^d$, $d=2,3$, situated in the semi-plane $x_2>0$ for $d=2$ and in the semi-space $x_3>0$ for $d=3$. Its boundary Γ consists of two parts: $\Gamma=\Gamma_1\cup\Gamma_2$, where Γ_1 is the segment $[0,1]$ in the axis $x_2=0$ for $d=2$ and square $[0,1]^2$ in the plane $x_3=0$ for $d=3$. For $d=2$, the part Γ_2 is infinitely differentiable and in a neighborhood of the points $(0,0)$ and $(1,0)$ coincides with the lines $x_1=0$ and $x_1=1$, respectively. For $d=3$, the part Γ_2 coincides with the lateral faces of the cube $[0,1]^3$ in a small neighborhood of the plane $x_3=0$ and, in addition, it is infinitely differentiable everywhere, except the vertical edges.

Then assume that B is an arbitrary bounded domain with Lipschitz boundary. Denote $B_a=\{x:\big(a^{-1}(x_1-b_1), a^{-1}(x_2-c)\big)\in B\}$ for $d=2$, $B_a=\{x:\big(a^{-1}(x_1-b_1),a^{-1}(x_2-b_2),a^{-1}(x_3-c)\big)\in B\}$, $j=1,2$, for $d=3$, where $0<b_j<1$, $c>0$ are arbitrary fixed numbers, a is a sufficiently small positive parameter, such that $\overline{B_a}$ lies in the semi-strip $\Pi=(0,1)\times(0,\infty)$ for $d=2$ and in the semi-infinite parallelepiped $\Pi=(0,1)^2\times(0,\infty)$ for $d=3$.

Denote $B^{\mathbf{k}}_{\varepsilon,a}=\{x:(\varepsilon^{-1}x_1-\mathbf{k},\varepsilon^{-1}x_2)\in B_a\}$, $\mathbf{k}\in\mathbb{Z}$ for $d=2$, $B^{\mathbf{k}}_{\varepsilon,a}=\{x:(\varepsilon^{-1}x_1-k_1,\varepsilon^{-1}x_2-k_2,\varepsilon^{-1}x_3)\in B_a\}$, $\mathbf{k}=(k_1,k_2)$, $k_j\in\mathbb{Z}$ for $d=3$, $B_{\varepsilon,a}=\bigcup\limits_{\mathbf{k}}B^{\mathbf{k}}_{\varepsilon,a}$, $\Gamma_{\varepsilon,a}=\partial B_{\varepsilon,a}$. Hereafter ε is a small positive parameter, $\varepsilon=\varepsilon_N=\frac{1}{N}$, where $N\gg 1$ is a natural number. Define the domain $\Omega_{\varepsilon,a}$ as $\Omega\setminus\overline{B_{\varepsilon,a}}$ (see Fig. 1).

Consider the boundary value problems

$$\begin{array}{llll} -\Delta U_{\varepsilon,a}=0 & \text{in } \Omega_{\varepsilon,a}, & U_{\varepsilon,a}=0 & \text{on } \Gamma_{\varepsilon,a} \\ \dfrac{\partial U_{\varepsilon,a}}{\partial\nu}=\lambda U_{\varepsilon,a}+f & \text{on } \Gamma_1, & \dfrac{\partial U_{\varepsilon,a}}{\partial\nu}=0 & \text{on } \Gamma_2 \end{array} \tag{1}$$

$$-\Delta U_0=0 \text{ in } \Omega,\quad \frac{\partial U_0}{\partial\nu}+\sigma_d C_d(B)AU_0=\lambda U_0+f \text{ on } \Gamma_1,\quad \frac{\partial U_0}{\partial\nu}=0 \text{ on } \Gamma_2 \tag{2}$$

where ν is an outer normal, $\lambda\in\mathbb{R}$, and the respective spectral problems

$$\begin{array}{llll} -\Delta u_{\varepsilon,a}=0 & \text{in } \Omega_{\varepsilon,a}, & u_{\varepsilon,a}=0 & \text{on } \Gamma_{\varepsilon,a} \\ \dfrac{\partial u_{\varepsilon,a}}{\partial\nu}=\lambda_{\varepsilon,a}u_{\varepsilon,a} & \text{on } \Gamma_1, & \dfrac{\partial u_{\varepsilon,a}}{\partial\nu}=0 & \text{on } \Gamma_2 \end{array} \tag{3}$$

$$-\Delta u_0=0 \text{ in } \Omega,\quad \frac{\partial u_0}{\partial\nu}+\sigma_d C_d(B)Au_0=\lambda_0 u_0 \text{ on } \Gamma_1,\quad \frac{\partial u_0}{\partial\nu}=0 \text{ on } \Gamma_2 \tag{4}$$

Hereafter $\sigma_2=2\pi$, $\sigma_3=4\pi$, $C_2(B)=1$ and $C_3(B)>0$ is the capacity of the domain B. In particular, $C_3(B)=1$, if B is a unit ball.

We prove the following statements.

Theorem 2.1. *Suppose that*

$$-\frac{1}{\varepsilon \ln a} \longrightarrow A \neq \infty \quad \text{for } d=2, \qquad \frac{a}{\varepsilon} \longrightarrow A \neq \infty \quad \text{for } d=3 \tag{5}$$

$f \in L_2(\Gamma_1)$ and λ is not an eigenvalue of the problem (4).

Then:

1) the boundary value problem (1) has a unique solution in $W_2^1(\Omega_{\varepsilon,a})$ for any sufficiently small ε, and moreover the following uniform in ε estimate:

$$\|U_{\varepsilon,a}\|_{W_2^1(\Omega)} \leqslant C\|f\|_{L_2(\Gamma_1)} \tag{6}$$

holds true, where the function $U_{\varepsilon,a}$ is extended in $\overline{B_{\varepsilon,a}}$ by zero;

2) for the solution to problem (1) the following strong convergence

$$U_{\varepsilon,a} \underset{\varepsilon\to 0}{\to} U_0 \qquad \text{in } W_2^1(\Omega) \tag{7}$$

takes place, if $A=0$, and the weak convergence

$$U_{\varepsilon,a} \underset{\varepsilon\to 0}{\rightharpoonup} U_0 \qquad \text{in } W_2^1(\Omega) \tag{8}$$

holds true, if $A \neq 0$, where U_0 is a solution to the homogenized (limit) problem (2).

Theorem 2.2. *Suppose that the condition (5) holds, and the multiplicity of the eigenvalue λ_0 to the problem (4) equals to n. Then there exist n eigenfunctions $\lambda_{\varepsilon,a}^{(l)}$ of problem (3), $l=\overline{1,n}$ (with respect to their multiplicities) converging to λ_0 as $\varepsilon \to 0$.*

Let $u_{\varepsilon,a}^{(l)}$ be orthonormalized in $L_2(\Gamma_1)$ eigenfunctions of problem (3), corresponding to $\lambda_{\varepsilon,a}^{(l)}$. Then from the sequence $\left\{\varepsilon_k = \frac{1}{k}\right\}_{k=1}^{\infty}$ and any sequence $\{a_k\}_{k=1}^{\infty}$ (for which the convergence (5) takes place) one can choose subsequences $\{\varepsilon_{k'}\}$, $\{a_{k'}\}$ such that the convergence

$$u_{\varepsilon,a}^{(l)} \underset{\varepsilon\to 0}{\longrightarrow} u_*^{(l)} \qquad \text{in } W_2^1(\Omega)$$

holds, if $A=0$, and weak convergence

$$u_{\varepsilon,a}^{(l)} \underset{\varepsilon\to 0}{\rightharpoonup} u_*^{(l)} \qquad \text{in } W_2^1(\Omega)$$

holds, if $A \neq 0$, where the functions $u_{\varepsilon,a}^{(l)}$ are extended in $\overline{B_{\varepsilon,a}}$ by zero, and $u_*^{(l)}$ are orthonormalized in $L_2(\Gamma_1)$ eigenfunctions of problem (4), corresponding to λ_0 (they depend on the choice of the sequence $\{a_k\}_{k=1}^{\infty}$ and the subsequence).

The solution to the problems (1), (2), (3), and (4) is understood in a weak sense (in the sense of the integral identity, see, for instance, [16, Chapter IV]).

3. Construction of model functions in the semi-strip and in the semi-infinite parallelepiped

For $d=2$ we denote by $\mathbb{R}_+^2$ the semi-plane $x_2 \geqslant 0$, and by $x_0^{(\mathbf{k})}$ the points with coordinates $x_1 = b_1 + \mathbf{k}$, $x_2 = c$. Here $\mathbf{k} \in \mathbb{Z}$. For $d=3$ we denote by $\mathbb{R}_+^3$ the semi-space $x_3 \geqslant 0$, and by $x_0^{(\mathbf{k})}$ the points with coordinates $x_j = b_j + k_j$, $j=1,2$, $x_3 = c$. Here $\mathbf{k} = (k_1, k_2)$ and $k_j \in \mathbb{Z}$. Assume that $x_0 := x_0^{(\mathbf{0})}$, $y := x - x_0$ and $\Sigma := \{x : x_1 \in (0,1), x_2 = 0\}$ for $d=2$, $\Sigma := \{x : x_1, x_2 \in (0,1), x_3 = 0\}$ for $d=3$.

Define $G_2(t) := \ln t$, $G_3(t) := -t^{-1}$.

It is easy to prove the following lemma.

Lemma 3.1. *There exists a function $g_d \in C^\infty\left(\mathbb{R}_+^d \setminus \bigcup_{\mathbf{k}} \{x_0^{(\mathbf{k})}\}\right)$, 1-periodic in x_1 for $d=2$ and in x_1, x_2 for $d=3$, which satisfies the problem*

$$\begin{cases} \Delta g_d = 0 & \text{if } x \in \Pi \setminus \{x_0\} \\ \frac{\partial g_d}{\partial \nu} = \sigma_d & \text{if } x \in \Sigma \end{cases}$$

has the differentiable asymptotics

$$g_d(x) = O\left(\mathrm{e}^{-2\pi x_d}\right), \quad x_d \to +\infty$$

and in a neighborhood of x_0 has the representation

$$g_d(x) = G_d(|y|) + g_d^{(1)}(x)$$

where $g_d^{(1)}(x)$ is an infinitely smooth function in the neighborhood of this point.

Corollary 3.2. *The differentiable asymptotics*

$$g_d(x) = G_d(|y|) + c_{\Pi,d} + P_1^{\Pi,d}(y) + O\left(|y|^2\right), \qquad y \to 0$$

holds, where $c_{\Pi,d}$ is a constant and $P_1^{\Pi,d}(y)$ is a homogeneous polynomial of the first order.

Using [17, § 5.8] we prove the following lemma.

Lemma 3.3. *There exist functions $V_0^{(d)}, V_1^{(d)} \in C^\infty\left(\mathbb{R}^d \backslash B\right)$, being solutions to the problems*

$$\Delta V_i^{(d)} = 0, \quad x \in \mathbb{R}^d \backslash \overline{B}, \qquad V_i^{(d)} = 0, \quad x \in \partial B$$

and having differentiable asymptotics

$$V_0^{(2)}(x) = \ln|x| + c_B + O(|x|^{-1}), \qquad V_0^{(3)}(x) = 1 - C_3(B)|x|^{-1} + P_1^{B,3}(x)|x|^{-3} + O(|x|^{-3})$$
$$V_1^{(2)}(x) = P_1^{\Pi,2}(x) + \widetilde{c} + O(|x|^{-1}), \qquad V_1^{(3)}(x) = P_1^{\Pi,3}(x) - C_3(P,B)|x|^{-1} + O(|x|^{-2})$$

as $|x| \to \infty$, where $P_1^{B,3}(y)$ is a homogeneous polynomial of the first order.

Lemma 3.4. *Let $d = 3$. Then there exists a 1-periodic in x_1, x_2 function $\widehat{g}_3 \in C^\infty\left(\mathbb{R}^3_+ \backslash \bigcup\limits_{\mathbf{k}} \left\{x_0^{(\mathbf{k})}\right\}\right)$, which satisfies the problem*

$$\begin{cases} \Delta \widehat{g}_3 = 0 & \text{if } x \in \Pi \setminus \{x_0\} \\ \frac{\partial \widehat{g}_3}{\partial \nu} = 0 & \text{if } x \in \Sigma \end{cases}$$

has the differentiable asymptotics

$$\widehat{g}_3(x) = O\left(\mathrm{e}^{-2\pi x_3}\right), \quad x_3 \to +\infty$$

and in a neighborhood of x_0 has the representation

$$\widehat{g}_3(x) = P_1^{B,3}(y)|y|^{-3} + \widehat{g}_3^{(1)}(x)$$

where $\widehat{g}_3^{(1)}(x)$ is an infinitely smooth function in the neighborhood of this point including this point.

Corollary 3.5. *The differentiable asymptotics*

$$\widehat{g}_3(x) = P_1^{B,3}(y)|y|^{-3} + \widehat{c}_{B,\Pi} + O(|y|), \qquad y \to 0 \tag{9}$$

holds.

Denote $\Pi_a = \Pi \backslash \overline{B_a}$ (see Figure) and define in $\overline{\Pi_a}$ the following function:

$$\begin{aligned} W_a(x) &:= \left(1 - \chi\left(\frac{|y|}{a^\beta}\right)\right)\left(1 - \frac{1}{\ln a}\left(g_2(x) + c_B - c_{\Pi,2}\right)\right) \\ &\quad - \frac{1}{\ln a}\chi\left(\frac{|y|}{a^\beta}\right)\left(V_0^{(2)}\left(\frac{y}{a}\right) + aV_1^{(2)}\left(\frac{y}{a}\right)\right) && \text{for } d = 2 \\ W_a(x) &:= \left(1 - \chi\left(\frac{|y|}{a^\beta}\right)\right)\left(1 + aC_3(B)\left(g_3(x) - c_{\Pi,3}\right) + a^2\left(C_3(P,B)g_3(x) + \widehat{g}_3(x)\right)\right) \\ &\quad + \chi\left(\frac{|y|}{a^\beta}\right)\left(V_0^{(3)}\left(\frac{y}{a}\right) + aV_1^{(3)}\left(\frac{y}{a}\right)\right) && \text{for } d = 3 \end{aligned} \tag{10}$$

where $\beta \in (0,1)$.

Now, denote $B_a^{\mathbf{k}} = \{x : (x_1 - \mathbf{k}, x_2) \in B_a\}$, $\mathbf{k} \in \mathbb{Z}$ for $d = 2$, $B_a^{\mathbf{k}} = \{x : (x_1 - k_1, x_2 - k_2, x_3) \in B_a\}$, $\mathbf{k} = (k_1, k_2)$, $k_j \in \mathbb{Z}$ for $d = 3$, $B^a = \bigcup\limits_{\mathbf{k}} B_a^{\mathbf{k}}$ and extend the function $W_a(x)$ 1-periodically in x_1 for $d = 2$ and in x_1, x_2 for $d = 3$, keeping the same notation $W_a(x)$.

Using Lemmas 3.1 and 3.3, we prove the following theorem.

Theorem 3.6. *The function $W_a(x) \in C^\infty\left(\mathbb{R}^d_+ \backslash B^a\right)$ is 1-periodic in x_1 for $d=2$, and in x_1, x_2 for $d=3$, has the differentiable asymptotics*

$$W_a(x) = 1 - \frac{1}{\ln a}\left(c_B - c_{\Pi,2} + O\left(\mathrm{e}^{-2\pi x_2}\right)\right) \quad \text{as } x_2 \to \infty \text{ for } d=2$$
$$W_a(x) = 1 - a\left(C_3(B)c_{\Pi,3} + O\left(\mathrm{e}^{-2\pi x_3}\right)\right) \quad \text{as } x_3 \to \infty \text{ for } d=3$$

uniform in a, and satisfies the problem

$$\begin{cases} \Delta W_a = F_a \quad \text{if } x \in \Pi_a \\ \frac{\partial W_a}{\partial \nu} = -\frac{2\pi}{\ln a} \quad \text{for } d=2, \qquad \frac{\partial W_a}{\partial \nu} = a4\pi C_3(B) \quad \text{for } d=3, \quad \text{if } x \in \Sigma \\ W_a = 0 \quad \text{if } x \in \partial B_a \end{cases}$$

where $F_a \in C_0^\infty(\Pi_a)$.

Moreover,

$$\|1 - W_a\|_{L_2(\Sigma)} = O\left(\frac{1}{|\ln a|}\right) \quad \text{for } d=2, \qquad \|1 - W_a\|_{L_2(\Sigma)} = O\left(a\right) \quad \text{for } d=3$$
$$\left\|1 - \left(W_a + \frac{1}{\ln a}\left(c_B - c_{\Pi,2}\right)\right)\right\|_{L_2(\Pi_a)} = O\left(\frac{1}{|\ln a|}\right) \quad \text{for } d=2$$
$$\|1 - \left(W_a + aC_3(B)c_{\Pi,3}\right)\|_{L_2(\Pi_a)} = O\left(a + a^{\frac{3\beta}{2}}\right) \quad \text{for } d=3$$
$$\|F_a\|_{L_2(\Pi_a)} = O\left(\frac{1}{|\ln a|}\left(a^\beta + a^{1-2\beta}\right)\right) \quad \text{for } d=2$$
$$\|F_a\|_{L_2(\Pi_a)} = O\left(a^{1+\frac{3}{2}\beta} + a^{2-\frac{1}{2}\beta} + a^{3-\frac{7}{2}\beta}\right) \quad \text{for } d=3$$

4. Proof of Theorem 2.1

Before proving Theorem 2.1, we prove some auxiliary propositions. Using Theorem 3.6, we prove the following lemma.

Lemma 4.1. *There exists a function $W_{\varepsilon,a}(x)$ from $W_2^1\left(\Omega_{\varepsilon,a}; \Gamma_{\varepsilon,a}\right) \cap W_2^2\left(\Omega_{\varepsilon,a}\right)$, such that the relations*

$$\left.\frac{\partial W_{\varepsilon,a}}{\partial \nu}\right|_{\Gamma_1} = -\frac{2\pi}{\varepsilon \ln a} \quad \text{for } d=2, \qquad \left.\frac{\partial W_{\varepsilon,a}}{\partial \nu}\right|_{\Gamma_1} = \frac{4\pi C_3(B)a}{\varepsilon} \quad \text{for } d=3$$
$$\|1 - W_{\varepsilon,a}\|_{L_2(\Omega)} \underset{\varepsilon\to 0}{\longrightarrow} 0$$
$$\|\Delta W_{\varepsilon,a}\|_{L_2(\Omega_{\varepsilon,a})} \underset{\varepsilon\to 0}{\longrightarrow} 0$$
$$\|1 - W_{\varepsilon,a}\|_{L_2(\Gamma)} \underset{\varepsilon\to 0}{\longrightarrow} 0$$
$$\left\|\frac{\partial W_{\varepsilon,a}}{\partial \nu}\right\|_{L_2(\Gamma_2)} \underset{\varepsilon\to 0}{\longrightarrow} 0$$

hold if (5) is true.

Next lemma follows from Lemma 4.1.

Lemma 4.2. *Let condition (5) hold. Assume also that the function $U_{\varepsilon,a} \in W_2^1\left(\Omega_{\varepsilon,a}; \Gamma_{\varepsilon,a}\right)$ converges weakly*

$$U_{\varepsilon,a} \underset{\varepsilon\to 0}{\rightharpoonup} U^* \quad \text{in } W_2^1(\Omega)$$

Then for any functions $v \in C^\infty(\overline{\Omega})$ the convergences

$$\int\limits_{\Omega_{\varepsilon,a}} U_{\varepsilon,a}(vW_{\varepsilon,a})\,\mathrm{d}x \to \int\limits_{\Omega} U^* v\,\mathrm{d}x$$
$$\int\limits_{\Omega_{\varepsilon,a}} \nabla U_{\varepsilon,a}\nabla(vW_{\varepsilon,a}), \mathrm{d}x \to \int\limits_{\Omega} \nabla U^* \nabla v\,\mathrm{d}x + \sigma_d C_d(B) A \int\limits_{\Gamma_1} U^* v\,\mathrm{d}x_1$$

take place.

Proof of Theorem 2.1. Since the Fredholm alternative for problem (1) holds (see, for instance, [16, Chapter II, § 3]), then it is sufficient to show the estimate (6) to prove *1)*.

The standard norm in $W_2^1(\Omega)$ is equivalent to the norm in $\|u\|_{H^1(\Omega)}$, generated by the following scalar product:

$$(u,v)_{H^1(\Omega)} = \int_\Omega \nabla u \nabla v \, dx + \int_{\Gamma_1} uv \, ds$$

(see, for instance, [16, Ch. III, §5.6]).

Using the identity of the problems (1), (2), and applying the Cauchy inequality, we get, for any fixed $\lambda \in \mathbb{R}$, the a priori estimate, uniform in ε and a, for the solution to problem (1), of the form

$$\|U_{\varepsilon,a}\|_{H^1(\Omega)} \le C\left(\|U_{\varepsilon,a}\|_{L_2(\Gamma_1)} + \|f\|_{L_2(\Gamma_1)}\right) \tag{11}$$

If $U_{\varepsilon_{m(k)},a_k} = 0$ on Γ_1, then estimate (6) follows from (11).

Then, applying Lemma 4.2, we prove the statement *2)* and complete the proof. □

In an analogous way, using Lemma 4.2, one can derive the following assertion.

Lemma 4.3. *Let the condition (5) hold, assume also that λ is not an eigenvalue of problem (4), $U_{\varepsilon,a}$ is the solution to problem (1) for $f = f_{\varepsilon,a}$, U_0 is the solution to problem (2) for $f = f_0$ and the weak convergence:*

$$f_{\varepsilon,a} \underset{\varepsilon\to 0}{\rightharpoonup} f_0 \qquad \text{in } L_2(\Gamma_1) \tag{12}$$

holds.

Then the convergence (7) and (8) take place.

Obviously the following proposition holds true.

Lemma 4.4. *Suppose that λ is not an eigenvalue of the problem (4), $U^{\varepsilon,a}$ be is the solution to problem (2) for $f = f_{\varepsilon,a}$, U_0 is the solution to problem (2) for $f = f_0$ and the weak convergence (12) holds.*

Then the weak convergence

$$U^{\varepsilon,a} \underset{\varepsilon\to 0}{\rightharpoonup} U_0 \qquad \text{in } W_2^1(\Omega)$$

takes place.

5. Proof of Theorem 2.2

Denote by $\mathcal{P}_{\varepsilon,a}$ and $\mathcal{P}_0$ operators $\mathcal{P}_{\varepsilon,a}, \mathcal{P}_0 : L_2(\Gamma_1) \to L_2(\Gamma_1)$, mapping f to the traces on Γ_1 of solutions to boundary value problems (1) and (2), respectively, for $\lambda = -1$. It follows from the definition that these operators are compact, selfadjoint and positive.

Lemmas 4.3 and 4.4 lead to the following statements.

Lemma 5.1. *If condition (5) and weak convergence (12) hold, then*

$$\mathcal{P}_{\varepsilon,a} f_{\varepsilon,a} \underset{\varepsilon\to 0}{\longrightarrow} \mathcal{P}_0 f_0, \quad \mathcal{P}_0 f_{\varepsilon,a} \underset{\varepsilon\to 0}{\longrightarrow} \mathcal{P}_0 f_0 \qquad \text{in} \quad L_2(\Gamma_1)$$

strongly.

Lemma 5.2. *If condition (5) holds, then*

$$\mathcal{P}_{\varepsilon,a} \underset{\varepsilon\to 0}{\rightrightarrows} \mathcal{P}_0$$

uniformly.

Denote by $\mathcal{L}_{\varepsilon,a}, \mathcal{L}_0 : L_2(\Gamma_1) \to L_2(\Gamma_1)$ the operators inverse to $\mathcal{P}_{\varepsilon,a}, \mathcal{P}_0 : L_2(\Gamma_1) \to L_2(\Gamma_1)$. From Lemma 5.2 and [18, Ch.IV, §3.4] (see also [19, Ch. 9, §4]), we easily derive the following proposition.

Lemma 5.3. *Suppose that the condition (5) holds, and the multiplicity of the eigenvalue Λ_0 to the operator $\mathcal{L}_0$ equals to n. Then there exist n eigenvalues $\Lambda_{\varepsilon,a}^{(l)}$ of the operator $\mathcal{L}_{\varepsilon,a}$, $l = \overline{1,n}$ (with respect to their multiplicities) converging to Λ_0 as $\varepsilon \to 0$.*

Since obviously $\Lambda^{(l)}_{\varepsilon,a} = \lambda^{(l)}_{\varepsilon,a} + 1$, $\Lambda_0 = \lambda_0 + 1$, then the next assertion follows.

Lemma 5.4. *Suppose that the condition (5) holds, and the multiplicity of the eigenvalue λ_0 to the problem (4) equals to n. Then there exist n eigenvalues $\lambda^{(l)}_{\varepsilon,a}$ of problem (3), $l = \overline{1,n}$ (with respect to their multiplicities) converging to λ_0 as $\varepsilon \to 0$.*

Using the trick from the proof of Theorem 2.1, we prove the next lemma.

Lemma 5.5. *Assume that condition (5) holds, $\lambda^{(l)}_{\varepsilon,a}$, $l = \overline{1,n}$ are eigenvalues of problem (3), converging to n-multiple eigenvalue λ_0 of the limit problem (4) and $u^{(l)}_{\varepsilon,a}$ are the respective normalized in $L_2(\Gamma_1)$ eigenfunctions. Then, from a sequence $\varepsilon_k = \frac{1}{k}$ and any sequence $a_k \longrightarrow 0$ as $k \to \infty$, one can chose subsequences $\{\varepsilon_{k'}\}$, $\{a_{k'}\}$, such that the strong convergence*

$$u^{(l)}_{\varepsilon,a} \underset{\varepsilon\to 0}{\longrightarrow} u^{(l)}_{*} \qquad in\ W^1_2(\Omega)$$

holds, if $A = 0$ and a weak convergence

$$u^{(l)}_{\varepsilon,a} \underset{\varepsilon\to 0}{\rightharpoonup} u^{(l)}_{*} \qquad in\ W^1_2(\Omega)$$

holds if $A \neq 0$, where $u^{(l)}_{*}$ are orthonormalized in $L_2(\Omega)$ eigenfunctions of the limit problem (4), corresponding to λ_0 (which in general depend on the choice of the sequence $a_k \longrightarrow 0$ as $k \to \infty$, and the subsequence).

Proof of Theorem 2.2. The proof lies on Lemmas 5.4 and 5.5. □

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