C. R. Mecanique 344 (2016) 181–189

Contents lists available at ScienceDirect

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On scattering frequencies in homogenization problems. Critical cases

Sur les fréquences de diffusion dans les problèmes d'homogénéisation. Cas critiques

Rustem R. Gadyl'shin [a,b,*]

[a] *Bashkir State Pedagogical University, 3a October Revolution str., 450000 Ufa, Russia*
[b] *Bashkir State University, 32 Zaki Validi str., 450076 Ufa, Russia*

ARTICLE INFO

Article history:
Received 6 October 2015
Accepted 19 October 2015
Available online 11 February 2016

Keywords:
Asymptotics
Small parameter
Homogenization

Mots-clés :
Asymptotique
Petit paramètre
Homogénéisation

ABSTRACT

We consider a two-dimensional boundary value problems for the Helmholtz equation with Dirichlet and Neumann boundary conditions on a set of arcs. This set is obtained from a closed curve by cutting out small holes situated close to each other and having a locally periodic structure. We construct the asymptotics of the scattering frequencies (poles of the analytic continuation of solutions) with small imaginary parts, which converge to the square roots of multiple eigenvalues of limit problems.

RÉSUMÉ

On considère des problèmes aux limites bidimensionnels pour l'équation de Helmholtz avec conditions aux limites de Dirichlet ou de Neumann sur un ensemble d'arcs. Cet ensemble est obtenu à partir d'une courbe fermée en découpant des petits trous situés près les uns des autres, avec une structure localement périodique. Nous construisons le développement asymptotique des fréquences de diffusion (pôles du prolongement analytique des solutions) avec des parties imaginaires petites, qui convergent vers les racines carrées des valeurs propres multiples du problème limite.

1. Introduction

It is known that the scattering of E-polarized electromagnetic field on the ideally conducting cylindrical surface, the cross-section of which is the curve Γ_δ, and the vibrations of a membrane fixed on Γ_δ, are described by the solution to the following Dirichlet boundary value problem in $\Omega_\delta = \mathbb{R}^2 \backslash \overline{\Gamma_\delta}$:

$$(\Delta + k^2)u_\delta = f, \quad x \in \Omega_\delta, \quad u_\delta = 0, \quad x \in \Gamma_\delta, \quad \frac{\partial u_\delta}{\partial r} - \mathrm{i}ku_\delta = o(r^{-1/2}), \quad r \to \infty \tag{1}$$

* Correspondence to: Bashkir State Pedagogical University, 3a October Revolution str., 450000 Ufa, Russia.
E-mail address: gadylshin@yandex.ru.

http://dx.doi.org/10.1016/j.crme.2015.10.002
1631-0721/

where $x=(x_1,x_2)$, $r=|x|$, k is a positive number. In its turn, the following Neumann boundary value problem

$$(\Delta+k^2)u_\delta=f,\quad x\in\Omega_\delta,\quad \frac{\partial u_\delta}{\partial \mathbf{n}}=0,\quad x\in\Gamma_\delta,\quad \frac{\partial u_\delta}{\partial r}-\mathrm{i}ku_\delta=o(r^{-1/2}),\quad r\to\infty \tag{2}$$

where $\mathbf{n}$ is the normal to Γ_δ, describes an H-polarized electromagnetic field on the ideally conducting cylindrical surface, the cross-section of which is the curve Γ_δ, and the vibrations of a membrane not fixed on the cut Γ_δ.

Hereinafter, we assume that $\delta\geqslant 0$, $\Gamma_0\in C^\infty$ is a boundary of a bounded simply-connected domain Ω and, for $\delta=\varepsilon>0$, the curve Γ_ε is obtained from Γ_0 by cutting out a great number of openings of small diameter located close to each other. Namely, let ω be a unit circle with its center at the origin. Suppose that $\gamma_0=\partial\omega$, $N\gg 1$ is an integer number, $\varepsilon=2N^{-1}$. For the boundary value problem (1), we assume that $\gamma_\varepsilon=\{(r,\theta): r=1,\ \varepsilon(-a(\varepsilon)+m\pi)<\theta<\varepsilon(a(\varepsilon)+m\pi),\ m=0,1,\dots,N-1\}$, where θ is the polar angle and $0<a(\varepsilon)<\frac{\pi}{2}$, and for the boundary value problem (2), we assume that $\gamma_\varepsilon=\{(r,\theta): r=1,\ \varepsilon(a(\varepsilon)+m\pi)<\theta<\varepsilon(\pi(m+1)-a(\varepsilon)),\ m=1,\dots,N\}$. We denote $\Omega=\mathcal{P}(\omega)$, $\Gamma_\delta=\mathcal{P}(\gamma_\delta)$, $\mathcal{P}$ is a diffeomorphism $\mathbb{R}^2$ into $\mathbb{R}^2$ and assume that $\mathbf{n}$ is the outer normal to Ω. For $\delta=\varepsilon$, we will call the problems (1) and (2) the perturbed problems. Since $\Omega_0=\Omega\cup(\mathbb{R}^2\backslash\overline{\Omega})$, it follows that for $\delta=0$ the problem (1) decomposes into two Dirichlet problems, in Ω and in $\mathbb{R}^2\backslash\overline{\Omega}$, and the problem (2) decomposes into two Neumann problems, in Ω and in $\mathbb{R}^2\backslash\overline{\Omega}$. We will call them the internal limit and the external limit problems, respectively.

In the following, we assume that

$$\varepsilon\ln a(\varepsilon)\underset{\varepsilon\to 0}{\to}0 \tag{3}$$

for the boundary value problem (1) and

$$a(\varepsilon)=\exp\left(-\frac{1}{\varepsilon A(\varepsilon)}\right),\quad A(\varepsilon)>0,\quad \lim_{\varepsilon\to 0}A(\varepsilon)=0\quad \varepsilon\to 0 \tag{4}$$

for the boundary value problem (2).

For these cases, it is known [1] (see also [2]) that if $\lambda=k^2$ is not the eigenvalue of the limit internal problem, then a solution to the perturbed problem converges to solutions to the limit problems in Ω and $\mathbb{R}^2\backslash\overline{\Omega}$. Assume that $k_0>0$ and $\lambda_0=k_0^2$ is an eigenvalue of the limit internal problem. In [3] it is shown that the analytic continuation (with respect to k) of the solution to the perturbed problem has a pole τ^ε with small imaginary part (a scattering frequency), converging to k_0 as $\varepsilon\to 0$. This pole lies in the lower complex semi-plane $\operatorname{Im}k<0$, and the residue of the analytic continuation of the solution is a solution to the boundary value problem

$$\left(\Delta+\left(\tau^\varepsilon\right)^2\right)\psi^\varepsilon=0,\quad x\in\Omega_\varepsilon,\qquad \psi^\varepsilon=0,\quad x\in\Gamma_\varepsilon \tag{5}$$

for (1) and a solution to the boundary value problem

$$\left(\Delta+\left(\tau^\varepsilon\right)^2\right)\psi^\varepsilon=0,\quad x\in\Omega_\varepsilon,\qquad \frac{\partial\psi^\varepsilon}{\partial\mathbf{n}}=0,\quad x\in\Gamma_\varepsilon \tag{6}$$

for (2). Let us emphasize that for fixed ε the function ψ^ε increases exponentially as $r\to\infty$. We will call it a quasi-eigenfunction.

In the case when $\lambda_0=k_0^2$ is a simple eigenvalue of the limit internal problems, the leading terms of the asymptotics of pole converging to k_0 and of the associated quasi-eigenfunction were constructed in [4,5]. In the present paper, we constructed the leading terms of the asymptotics of poles converging to k_0 and of the associated quasi-eigenfunctions in cases when the multiplicity of an eigenvalue $\lambda_0=k_0^2$ of the limit internal problems is equal to $n\geqslant 2$.

2. Asymptotics of quasi-eigenelements for the boundary value problem (1)

In this section we assume that the condition (3) holds. Let λ_0 be an eigenvalue of multiplicity $n\geqslant 2$ of the limit interior boundary value problem for (1), and let $\psi_0^{(l)}$ $(l=1,2,\dots,n)$ be the corresponding orthonormal eigenfunctions in $L_2(\Omega)$, i.e.,

$$-\Delta\psi_0^{(l)}=\lambda_0\psi_0^{(l)},\quad x\in\Omega,\qquad \psi_0^{(l)}=0,\quad x\in\Gamma_0$$
$$\int_\Omega\left(\psi_0^{(l)}(x)\right)^2\mathrm{d}x=1,\qquad \int_\Omega\psi_0^{(l)}(x)\psi_0^{(p)}(x)\,\mathrm{d}x=0,\qquad l\neq p,\quad l,p=\overline{1,n}$$

The leading terms of the asymptotics for the poles $\tau^{\varepsilon,l}$ and associated quasi-eigenfunctions $\psi^{\varepsilon,l}$ outside a neighborhood of Γ_0 are constructed in the form

$$\tau^{\varepsilon,l}=k_0+\varepsilon\tau_1^{(l)}+\varepsilon^2\tau_2^{(l)}+\dots \tag{7}$$

$$\psi^{\varepsilon,l}(x)=\psi_0^{(l)}(x)+\varepsilon\psi_1^{(l)}(x)+\varepsilon^2\psi_2^{(l)}(x)+\dots,\quad x\in\Omega \tag{8}$$

$$\psi^{\varepsilon,l}(x)=\varepsilon\Psi_1^{(l)}(x;\tau^{\varepsilon,l})+\varepsilon^2\Psi_2^{(l)}(x;\tau^{\varepsilon,l})+\dots,\quad x\in\mathbb{R}^2\backslash\overline{\Omega} \tag{9}$$

where $\Psi_j^{(l)}(x;k)$ is a solution (and its analytic continuation) of the equation with radiation condition

$$(\Delta+k^2)\Psi_j^{(l)}=0,\quad x\in\mathbb{R}^2\backslash\overline{\Omega},\qquad \Psi_j^{(l)}=\alpha_j^{\mathrm{ex},l},\quad x\in\Gamma_0$$
$$\frac{\partial\Psi_j^{(l)}}{\partial r}-\mathrm{i}k\Psi_j^{(l)}=o(r^{-1/2}),\qquad r\to\infty \tag{10}$$

functions $\psi_j^{(l)}$ are solutions to the boundary value problems

$$(\Delta+\lambda_0)\psi_0^{(l)}=0,\quad x\in\Omega,\qquad \psi_0^{(l)}=0,\quad x\in\Gamma_0 \tag{11}$$
$$(\Delta+\lambda_0)\psi_1^{(l)}=-\lambda_1^{(l)}\psi_0^{(l)},\quad x\in\Omega,\qquad \psi_1^{(l)}=\alpha_1^{\mathrm{in},l},\quad x\in\Gamma_0 \tag{12}$$
$$(\Delta+\lambda_0)\psi_2^{(l)}=-\lambda_1^{(l)}\psi_1^{(l)}-\lambda_2^{(l)}\psi_0^{(l)},\ x\in\Omega,\quad \psi_2^{(l)}=\alpha_2^{\mathrm{in},l},\ x\in\Gamma_0 \tag{13}$$
$$\lambda_1^{(l)}=2k_0\tau_1^{(l)},\qquad \lambda_2^{(l)}=2k_0\tau_2^{(l)}+\left(\tau_1^{(l)}\right)^2 \tag{14}$$

and the boundary conditions $\alpha_j^{\mathrm{ex},l}$ and $\alpha_j^{\mathrm{in},l}$ in (10) and (12), (13) are unknown functions, yet. Due to such a definition of $\Psi_j^{(l)}$ series (9), the equation in (5) is obviously satisfied for any $\tau^\varepsilon=\tau^{\varepsilon,l}$. Equations in (11)–(13) are obtained by substituting (7), (8) in (5) and taking into account (14). Obviously, the eigenfunctions $\psi_0^{(l)}$ satisfy (11).

The boundary conditions $\alpha_j^{\mathrm{in},l}$ and $\alpha_j^{\mathrm{ex},l}$ in (12), (13) and (10) are determined from the matching of expansions [6] of (8) and (9) with another expansion in a neighborhood of Γ_0. To do this in a small neighborhood of Γ_0, we consider local coordinates (s,t), where t denotes the distance from the point up to Γ_0, measured in the direction of the inner normal to Ω, going through this point, and s denotes the natural parameter of the curve Γ_0. As $\Gamma_0=\mathcal{P}(\gamma_0)$, then the angle θ, parameterizing the curve γ_0, can be expressed in terms of s: $\theta=\theta(s)$, $\theta(0)=0$. Without loss of generality, we will assume that $\theta'(s)>0$.

According to the matching method [6], in a neighborhood of Γ_0, the asymptotics of $\psi^{\varepsilon,l}$ will be constructed in the form

$$\psi^{\varepsilon,l}(x)=\varepsilon v_1^{(l)}(\xi;s)+\varepsilon^2v_2^{(l)}(\xi;s)+\dots \tag{15}$$

where $\xi=(\xi_1,\xi_2)=\left(\theta(s)\varepsilon^{-1},t\theta'(s)\varepsilon^{-1}\right)$. Since the operator Δ in a neighborhood of the boundaries appears in the form

$$\Delta_x=\frac{1}{H}\left(\frac{\partial}{\partial t}\left(H\frac{\partial}{\partial t}\right)+\frac{\partial}{\partial s}\left(\frac{1}{H}\frac{\partial}{\partial s}\right)\right) \tag{16}$$

where $H=H(s,t)=1-t\kappa(s)$, $\kappa(s)=-\mathbf{r}''(s)\mathbf{n}(s)$, $\mathbf{r}(s)$ being the vector-function that defines the curve Γ_0, then substituting (16), (15) and (7) in (5) and coming back to the variables ξ, we obtain the boundary value problems for $v_j^{(l)}$:

$$\Delta_\xi v_1^{(l)}=0,\quad \xi\in\mathbb{R}^2\backslash\overline{\Gamma^a},\qquad v_1^{(l)}=0,\quad \xi\in\Gamma^a \tag{17}$$
$$\Delta_\xi v_2^{(l)}=\frac{1}{\theta'}\left(\left(1+2\xi_2\frac{\partial}{\partial\xi_2}\right)\left(\kappa\frac{\partial}{\partial\xi_2}-\frac{\theta''}{\theta'}\frac{\partial}{\partial\xi_1}\right)\right.$$
$$\left.-2\frac{\partial^2}{\partial\xi_1\partial s}\right)v_1^{(l)},\quad \xi\in\mathbb{R}^2\backslash\overline{\Gamma^a},\qquad v_2^{(l)}=0,\quad \xi\in\Gamma^a \tag{18}$$

where $\Gamma^a=\{\xi:\ \xi_2=0,\ |\xi_1-\pi m|<a,\ m\in\mathbb{Z}\}$.

Rewriting the asymptotics of the functions $\psi_j^{(l)}(x)$ and $\Psi_j^{(l)}(x;\tau^{\varepsilon,l})$ for $t\to0$ in the variables ξ, with the given boundary conditions in (10)–(13), we obtain the asymptotics for the coefficients of v_j for $|\xi_2|\to\infty$:

$$v_1^{(l)}(\xi;s)\sim\frac{\beta_0^{\mathrm{in},l}(s)}{\theta'(s)}\xi_2+\alpha_1^{\mathrm{in},l}(s),\quad \xi_2\to\infty$$
$$v_1^{(l)}(\xi;s)\sim\alpha_1^{\mathrm{ex},l}(s),\quad \xi_2\to-\infty, \tag{19}$$
$$v_2^{(l)}(\xi;s)\sim\frac{\beta_0^{\mathrm{in},l}(s)\kappa(s)}{2\,(\theta'(s))^2}\xi_2^2+\frac{\beta_1^{\mathrm{in},l}(s)}{\theta'(s)}\xi_2+\alpha_2^{\mathrm{in},l}(s),\quad \xi_2\to\infty$$
$$v_2^{(l)}(\xi;s)\sim\frac{\beta_1^{\mathrm{ex},l}(s)}{\theta'(s)}\xi_2+\alpha_2^{\mathrm{ex},l}(s),\quad \xi_2\to-\infty \tag{20}$$

where

$$\beta_j^{\mathrm{in},l}(s) = -\frac{\partial \psi_j^{(l)}}{\partial \mathbf{n}}\bigg|_{\Gamma_0}, \qquad \beta_1^{\mathrm{ex},l}(s) = -\frac{\partial \Psi_1^{(l)}}{\partial \mathbf{n}}\bigg|_{\Gamma_0, k=k_0} \tag{21}$$

Denote

$$X(\xi) = \operatorname{Re}\ln\left(\sin z + \left(\sin^2 z - \sin^2 a\right)^{1/2}\right) - \ln\sin a$$

where $z = \xi_1 + \mathrm{i}\xi_2$. By definition, the function X is harmonic in a half-strip $\Pi = \{\xi : -\pi/2 < \xi_1 < \pi/2,\ \xi_2 > 0\}$. It is continued, with period π, on the whole half-plane $\xi_2 > 0$, and satisfies the boundary condition

$$X = 0, \quad \xi \in \Gamma^a \tag{22}$$

and, even after continuation in the lower half plane $\xi_2 < 0$, it belongs to $W^1_{2,\mathrm{loc}}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \backslash \overline{\Gamma^a})$. Let us keep the notation X for this continuation. We obtain that the function X has the asymptotics

$$X(\xi) = \pm\xi_2 - \ln\sin a + O(\exp\{\mp\xi_2\}), \quad \xi_2 \to \pm\infty \tag{23}$$

and the function

$$v_1^{(l)}(\xi; s) = \frac{\beta_0^{\mathrm{in},l}(s)}{2\theta'(s)}(X(\xi) + \xi_2) \tag{24}$$

due to (22) is a solution to (17) and

$$\alpha_1^{\mathrm{in},l}(s) = \alpha_1^{\mathrm{ex},l}(s) = -\frac{\beta_0^{\mathrm{in},l}(s)}{2\theta'(s)}\ln\sin a \tag{25}$$

due to (23) and (24) has the asymptotics (19) with accuracy up to exponentially small terms.

Defining $\alpha_1^{\mathrm{ex},l}$, we find $\Psi_1^{(l)}(x; k)$ by (10).

The necessary and sufficient conditions of solvability of the boundary value problem (12) are the following equalities

$$\lambda_1^{(l)} = \int_{\Gamma_0} \alpha_1^{\mathrm{in},l}\frac{\partial \psi_0^{(l)}}{\partial \mathbf{n}}\,\mathrm{d}s, \qquad \int_{\Gamma_0} \alpha_1^{\mathrm{in},l}\frac{\partial \psi_0^{(p)}}{\partial \mathbf{n}}\,\mathrm{d}s = 0, \quad p \neq l \tag{26}$$

Due to (26), (25), (21) and (14), we can rewrite these conditions as follows:

$$\tau_1^{(l)} = \frac{\ln\sin a}{4k_0}\int_{\Gamma_0}\left(\frac{\partial \psi_0^{(l)}}{\partial \mathbf{n}}\right)^2\frac{\mathrm{d}s}{\theta'} \tag{27}$$

$$\int_{\Gamma_0}\frac{\partial \psi_0^{(l)}}{\partial \mathbf{n}}\frac{\partial \psi_0^{(p)}}{\partial \mathbf{n}}\frac{\mathrm{d}s}{\theta'} = 0, \qquad l \neq p, \quad l, p = \overline{1, n} \tag{28}$$

So, we obtained the main perturbed terms (27) of the pole and the additional conditions (28) for the main terms of the quasi-eigenfunctions. For simplicity, we will assume that

$$\int_{\Gamma_0}\left(\frac{\partial \psi_0^{(l)}}{\partial \mathbf{n}}\right)^2\frac{\mathrm{d}s}{\theta'} \neq \int_{\Gamma_0}\left(\frac{\partial \psi_0^{(p)}}{\partial \mathbf{n}}\right)^2\frac{\mathrm{d}s}{\theta'}, \quad l \neq p \tag{29}$$

Denote by $\widetilde{\psi}_1^{(l)}$ the solution to the boundary value problem (12), which is orthogonal to $\psi_0^{(p)}$, $p = \overline{1, n}$, in $L_2(\Omega)$. Let

$$\psi_1^{(l)}(x) := \widetilde{\psi}_1^{(l)}(x) + \sum_{p \neq l}\mu_{l,p}\psi_0^{(p)}(x), \quad l, p = \overline{1, n} \tag{30}$$

where $\mu_{l,p}$ are some arbitrary constant, yet.

Since due to (19) the right-hand sides in (20) are asymptotic solutions to equation (18) when $\xi_2 \to \pm\infty$, then it is easy to show that there exists a solution to the boundary value problem (18), having asymptotics (20) for some values of $\alpha_2^{\mathrm{in},l}$ (which depend on $\mu_{l,p}$) and $\alpha_2^{\mathrm{ex},l}$. Defining $\alpha_2^{\mathrm{ex},l}$, we find $\Psi_2^{(l)}(x; k)$ by (10). Taking in account (20), (21) and (29), one can obtain the solvability of (13) for some $\lambda_2^{(l)}$ by choosing $\mu_{l,p}$ in (30). Defining $\lambda_2^{(l)}$, we find $\tau_2^{(l)}$ by (14).

Let us calculate $\operatorname{Im}\tau_2^{(l)}$.

Denote by $\Psi^{(l)}(x; k)$ the solution (and its analytical continuation) of the model boundary value problem

$$(\Delta + k^2)\Psi^{(l)} = 0, \quad x \in \mathbb{R}^2\setminus\overline{\Omega}, \qquad \Psi^{(l)} = \frac{1}{2\theta'}\frac{\partial \psi_0^{(l)}}{\partial \mathbf{n}}, \quad x \in \Gamma_0$$
$$\frac{\partial \Psi^{(l)}}{\partial r} - \mathrm{i}k\Psi^{(l)} = o(r^{-1/2}), \qquad r \to \infty \tag{31}$$

Then by (25), (21), the function

$$\Psi_1^{(l)}(x;k) = \ln\sin a\,\Psi^{(l)}(x;k) \tag{32}$$

is the solution to the boundary value problem (10) (and its analytical continuation with complex k).

By construction, $v_1^{(l)}$, $\beta_0^{\mathrm{in},l}$ are the real values. Therefore, the function $v_2^{\mathrm{Im},l} = \mathrm{Im}\, v_2^{(l)}$ due to (20) has the asymptotics

$$v_2^{\mathrm{Im},l}(\xi;s) \sim \mathrm{Im}\,\alpha_2^{\mathrm{in},l}(s), \quad \xi_2 \to \infty$$
$$v_2^{\mathrm{Im},l}(\xi;s) \sim \frac{\mathrm{Im}\,\beta_1^{\mathrm{ex},l}(s)}{\theta'(s)}\xi_2 + \mathrm{Im}\,\alpha_2^{\mathrm{ex},l}(s), \quad \xi_2 \to -\infty \tag{33}$$

and due to (18) it is the solution to the boundary value problem

$$\Delta_\xi v_2^{\mathrm{Im},l} = 0, \quad \xi \in \mathbb{R}^2\setminus\overline{\Gamma^a}, \qquad v_2^{\mathrm{Im},l} = 0, \quad \xi \in \Gamma^a \tag{34}$$

Since the boundary value problem (34) is uniquely solvable in the class of functions bounded as $|\xi_2| \to \infty$, then

$$v_2^{\mathrm{Im},l}(\xi;s) = \frac{\mathrm{Im}\,\beta_1^{\mathrm{ex},l}(s)}{2\theta'(s)}(\xi_2 - X(\xi_1,-\xi_2))$$

due to (33) and (23), moreover,

$$\mathrm{Im}\,\alpha_2^{\mathrm{in},l}(s) = \frac{\mathrm{Im}\,\beta_1^{\mathrm{ex},l}(s)}{2\theta'(s)}\ln\sin a \tag{35}$$

Denote

$$\sigma^{(l)} = \lim_{R\to\infty}\int\limits_{|x|=R}\left|\Psi^{(l)}(x,k_0)\right|^2 \mathrm{d}S > 0 \tag{36}$$

It is easy to show that

$$k_0\sigma^{(l)} = \ln^{-1}\sin a\int\limits_{\Gamma_0}\frac{\beta_0^{\mathrm{in},l}}{2\theta'}\,\mathrm{Im}\,\beta_1^{\mathrm{ex},l}\,\mathrm{d}s \tag{37}$$

due to (31), (32) and (21). Finally, from (26), (35)–(37), it follows that

$$\mathrm{Im}\,\tau_2^{(l)} = -\frac{1}{2k_0}\int\limits_{\Gamma_0}\beta_0^{\mathrm{in},l}\left(\frac{\mathrm{Im}\,\beta_1^{\mathrm{ex},l}}{2\theta'}\ln\sin a\right)\mathrm{d}s = -\frac{\sigma^{(l)}}{2}\ln^2\sin a < 0 \tag{38}$$

We obtain that the quasi-eigenelements have the asymptotics (7)–(9), (15), where for the coefficients the equalities (27), (38), (28), (36), (24), (21), (32) hold and $\Psi^{(l)}$ is the solution to the model boundary value problem (31).

3. Asymptotics of quasi-eigenelements for the boundary value problem (2)

In this section, we assume that the condition (4) holds. Let λ_0 be an eigenvalue of multiplicity $n \geqslant 2$ of the limit interior boundary value problem for (2), and let $\psi_0^{(l)}$ $(l = 1,2,\dots,n)$ be the associated orthonormal eigenfunctions in $L_2(\Omega)$, i.e.,

$$-\Delta\psi_0^{(l)} = \lambda_0\psi_0^{(l)}, \quad x \in \Omega, \qquad \frac{\partial\psi_0^{(l)}}{\partial\mathbf{n}} = 0, \quad x \in \Gamma_0$$
$$\int\limits_\Omega\left(\psi_0^{(l)}(x)\right)^2\mathrm{d}x = 1, \qquad \int\limits_\Omega\psi_0^{(l)}(x)\psi_0^{(p)}(x)\,\mathrm{d}x = 0, \qquad l \neq p, \quad l,p = \overline{1,n} \tag{39}$$

The leading terms of the asymptotics for the pole $\tau^{\varepsilon,l}$ and the associated quasi-eigenfunction $\psi^{\varepsilon,l}$ outside a neighborhood of Γ_0 are constructed in the form

$$\tau^{\varepsilon,l} = \tau_0^{(l)}(A(\varepsilon)) + \dots, \quad \tau_0^{(l)} \underset{A\to 0}{\to} k_0 \tag{40}$$

$$\psi^{\varepsilon,l}(x) = \Psi_0^{+,l}(x; A(\varepsilon)) + \dots, \quad x \in \Omega, \quad \Psi_0^{+,l} \underset{A\to 0}{\to} \psi_0^{(l)} \tag{41}$$

$$\psi^{\varepsilon,l}(x) = A(\varepsilon)\Psi_0^{-,l}(x; \tau^{\varepsilon}; A(\varepsilon)) + \dots, \quad x \in \mathbb{R}^2\backslash\overline{\Omega} \tag{42}$$

where $\Psi_0^{-,l}(x;k;A)$ is a solution (and its analytic continuation) of the equation and radiation condition

$$(\Delta + k^2)\Psi_0^{-,l} = 0, \quad x \in \mathbb{R}^2\backslash\overline{\Omega}, \quad \frac{\partial \Psi_0^{-,l}}{\partial r} - \mathrm{i}k\Psi_0^{-,l} = o(r^{-1/2}), \qquad r \to \infty \tag{43}$$

Denote $\Lambda_0^{(l)}(A) = \left(\tau_0^{(l)}(A)\right)^2$. Substituting (40), (41) in (6), we obtain the following equation for $\Psi_0^{+,l}$:

$$\left(\Delta + \Lambda_0^{(l)}\right)\Psi_0^{+,l} = 0, \quad x \in \Omega \tag{44}$$

In a small neighborhood of Γ_0 (moreover, outside smaller neighborhood of the openings), the asymptotics of $\psi^{\varepsilon,l}$ are constructed in another form, employing the method of matching asymptotic expansions [6]. In order to do that, we introduce above mentioned local coordinates (s,t) in a neighborhood of Γ_0. We denote

$$\phi_0^{+,l} = \Psi_0^{+,l}\big|_{\Gamma_0}, \;\; \phi_1^{+,l} = \frac{\partial \Psi_0^{+,l}}{\partial \mathbf{n}}\big|_{\Gamma_0}, \;\; \phi_0^{-,l} = \Psi_0^{-,l}\big|_{\Gamma_0, k=\tau_0^{(l)}}, \;\; \phi_1^{-,l} = \frac{\partial \Psi_0^{-,l}}{\partial \mathbf{n}}\big|_{\Gamma_0, k=\tau_0^{(l)}} \tag{45}$$

$\xi = (\xi_1, \xi_2) = \left(\theta(s)\varepsilon^{-1}, t\theta'(s)\varepsilon^{-1}\right)$. By the matching condition of the series (41), (42) with new series in the ξ variables in a neighborhood of Γ_0, we deduce that these series have the form

$$\psi^{\varepsilon,l}(x) = V_0^{\pm,l}(\xi; s; A) + \varepsilon V_1^{\pm,l}(\xi; s; A) + \dots, \quad \pm\xi_2 > 0 \tag{46}$$

$$\begin{aligned} &V_0^{+,l}(\xi; s; A) \sim \phi_0^{+,l}(s; A), \qquad V_1^{+,l}(\xi; s; A) \sim -\frac{\phi_1^{+,l}(s; A)}{\theta'(s)}\xi_2, \qquad \xi_2 \to +\infty \\ &V_0^{-,l}(\xi; s; A) \sim A\phi_0^{-,l}(s; A), \quad V_1^{-,l}(\xi; s; A) \sim -A\frac{\phi_1^{-,l}(s; A)}{\theta'(s)}\xi_2, \quad \xi_2 \to -\infty \end{aligned} \tag{47}$$

Here s is a "slow" variable. All the openings expressed in ξ being exponentially small, we will introduce another expansion in a each opening. Therefore, the coefficients of the series (46) should obey the homogeneous boundary condition

$$\frac{\partial V_j^{\pm,l}}{\partial \xi_2} = 0, \qquad \xi_2 = 0, \quad \xi_1 \neq \pi m, \quad m \in \mathbb{Z} \tag{48}$$

Substituting (46) and (40) in (6) and passing to ξ, we obtain the equation for $V_j^{\pm}$ and, in particular:

$$\Delta_\xi V_j^{\pm,l} = 0, \quad \pm\xi_2 > 0 \tag{49}$$

for $j = 0$. It is clear that the functions

$$V_0^{+,l}(\xi; s; A) \equiv \phi_0^{+,l}(s; A), \quad V_0^{-,l}(\xi; s; A) \equiv A\phi_0^{-,l}(s; A) \tag{50}$$

satisfy (49) and have the asymptotics (47). Keeping in mind (50), we conclude that the equations for $V_1^{\pm,l}$ have the form (49) where $j = 1$. Denote $Y(\xi) = \mathrm{Re}\,\ln\sin z + \ln 2$. By definition, the function Y is harmonic in a half-strip $\Pi = \{\xi : -\pi/2 < \xi_1 < \pi/2,\ \xi_2 > 0\}$, can be continued over the period π on the half-plane $\xi_2 > 0$ and satisfies the condition:

$$\frac{\partial Y}{\partial \xi_2} = 0, \quad \xi_2 = 0, \;\; \xi_1 \neq \pi m \tag{51}$$

Let us continue this function on the half-plane $\xi_2 < 0$ in a even way and keep the same notation Y for this continuation. As a result, we conclude that the function Y has the asymptotics

$$Y(\xi) = \pm\xi_2 + O(\exp\{\mp 2\xi_2\}), \quad \xi_2 \to \pm\infty, \qquad Y(\xi) = \ln\rho + \ln 2 + O(\rho^2), \quad \rho = |\xi| \to 0 \tag{52}$$

Using (51) and (52), we obtain that the functions

$$V_1^{+,l}(\xi; s; A) = -\frac{\phi_1^{+,l}(s; A)}{\theta'(s)}Y(\xi), \qquad V_1^{-,l}(\xi; s; A) = A\frac{\phi_1^{-,l}(s; A)}{\theta'(s)}Y(\xi) \tag{53}$$

are the solutions to equations (49) satisfying the boundary condition (48) and have the asymptotics (47).

The coefficients of $V_j^{\pm,l}$ bear the jump on the openings, which is why in a neighborhood of the m-th opening, the asymptotics of the quasi-eigenfunction $\psi^{\varepsilon,l}$ is sought as a new series whose coefficients depend on the variables $\zeta^{(m)} = (\zeta_1^{(m)}, \zeta_2) = ((\xi_1 - \pi m)a^{-1}, \xi_2 a^{-1})$, where $a(\varepsilon)$ is defined by formula (4). Rewriting the asymptotics of the series (46) as $(\xi_1 - \pi m, \xi_2) \to 0$ in the variables $\zeta^{(m)}$ and according to the method of matching asymptotic expansions, by (50), (52) and (53) we obtain that in the neighborhood of the m-th opening the asymptotics of the function $\psi^{\varepsilon,l}$ has the form

$$\psi^{\varepsilon,l}(x) = W_{0,l}^{(m)}(\zeta^{(m)}; s; A) + \varepsilon W_{1,l}^{(m)}(\zeta^{(m)}; A) + \dots \tag{54}$$

where $W_{j,l}^{(m)}$ satisfies the asymptotics as $|\zeta^{(m)}| \to \infty$

$$\begin{aligned} &W_{0,l}(\zeta; s; A) \sim \phi_0^{+,l}(s; A) + A^{-1}\frac{\phi_1^{+,l}(s; A)}{\theta'}, \quad W_{1,l}(\zeta; s; A) \sim -\frac{\phi_1^{+,l}(s; A)}{\theta'}(\ln|\zeta| + \ln 2), \quad \zeta_2 > 0 \\ &W_{0,l}(\zeta; s; A) \sim A\phi_0^{-,l}(s; A) - \frac{\phi_1^{-,l}(s; A)}{\theta'}, \quad W_{1,l}(\zeta; s; A) \sim A\frac{\phi_1^{-,l}(s; A)}{\theta'}(\ln|\zeta| + \ln 2), \quad \zeta_2 < 0 \end{aligned} \tag{55}$$

Hereinafter, for the sake of brevity, we omit the indices m of $W_{j,l}^{(m)}$ and $\zeta^{(m)}$. Substituting (54) and (40) in (6) and passing to ζ, we obtain the boundary value problems for $W_{j,l}$, and, in particular:

$$\Delta_\zeta W_{0,l} = 0, \quad \zeta \in \mathbb{R}^2 \backslash \overline{\Gamma^0}, \qquad \frac{\partial W_{0,l}}{\partial \zeta_2} = 0, \quad \zeta \in \Gamma^0 \tag{56}$$

where Γ^0 is the axis $O\zeta_1$ without the segment $[-1, 1]$. It is clear that under the condition

$$\phi_0^{+,l}(s; A) + \left(A\theta'(s)\right)^{-1}\phi_1^{+,l}(s; A) = A\phi_0^{-,l}(s; A) - \left(\theta'(s)\right)^{-1}\phi_1^{-,l}(s; A) \tag{57}$$

the solution to the boundary value problem (56), having the asymptotics (55), has the form

$$W_{0,l}(\zeta; s; A) \equiv \phi_0^{+,l}(s; A) + \frac{\phi_1^{+,l}(s; A)}{A\theta'(s)} \tag{58}$$

and it is independent of ζ. By (58) the boundary value problem for W_1 is

$$\Delta_\zeta W_{1,l} = 0, \quad \zeta \in \mathbb{R}^2 \backslash \overline{\Gamma^0}, \qquad \frac{\partial W_{1,l}}{\partial \zeta_2} = 0, \quad \zeta \in \Gamma^0 \tag{59}$$

Denote $Z(\zeta) = \operatorname{Re} \ln\left(w + \sqrt{w^2 - 1}\right)$, where $w = \zeta_1 + i\zeta_2$ is a complex variable. It is easy to verify that the harmonic in $\mathbb{R}^2 \backslash \overline{\Gamma^0}$ function $Z \in H^1_{\text{loc}}(\mathbb{R}^2 \backslash \overline{\Gamma^0}) \cap C^\infty(\mathbb{R}^2 \backslash \overline{\Gamma^0})$ satisfies the boundary condition

$$\frac{\partial Z}{\partial \zeta_2} = 0, \quad \zeta \in \Gamma^0 \tag{60}$$

and has the following asymptotic at infinity:

$$Z(\zeta) = \pm(\ln|\zeta| + \ln 2) + O(|\zeta|^{-2}), \quad \pm\zeta_2 > 0 \tag{61}$$

By (60), (61) the function

$$W_{1,l}(\zeta; s; A) = -\frac{\phi_1^{+,l}(s; A)}{\theta'(s)} Z(\zeta)$$

is a solution to the boundary value problem (59) and meets the asymptotics (55) if

$$\frac{\phi_1^{+,l}(s; A)}{\theta'} = A\frac{\phi_1^{-,l}(s; A)}{\theta'} \tag{62}$$

From (62), (57) and (45) for equations (43), (44) we obtain the boundary conditions of the conjugation type:

$$\begin{aligned} &\frac{\partial}{\partial \mathbf{n}}\left(\Psi_0^{+,l}(x; A) + A\Psi_0^{-,l}(x; \tau_0^{(l)}(A); A)\right) + A\theta'(s)\left(\Psi_0^{+,l}(x; A) - A\Psi_0^{-,l}(x; \tau_0^{(l)}(A); A)\right) = 0, \\ &\frac{\partial}{\partial \mathbf{n}}\left(\Psi_0^{+,l}(x; A) - A\Psi_0^{-,l}(x; \tau_0^{(l)}(A); A)\right) = 0, \qquad x \in \Gamma_0. \end{aligned} \tag{63}$$

From the solvability conditions for the boundary value problem (43), (44), (63) and the conditions $\tau_0^{(l)} \to k_0$ and $\Psi_0^{+,l} \to \psi_0$ as $A \to 0$ by (40) and (41), we determine $\tau_0^{(l)}(A)$, $\Psi_0^{\pm,l}$ and, in particular, we get the additional orthogonality conditions for $\psi_0^{(l)}$ on Γ_0:

$$\int_{\Gamma_0} \psi_0^{(l)}\psi_0^{(p)}\mathrm{d}\theta(s)=0, \qquad l\neq p, \quad l,p=\overline{1,n} \tag{64}$$

which are analogous to (28), and

$$\tau_0^{(l)}(A)=k_0+A\tau_{0,1}^{(l)}+A^2\tau_{0,2}^{(l)}+O(A^3), \qquad \Psi_0^{-,l}(x;k;A)\underset{A\to 0}{\to}\psi_0^{-,l}(x;k) \tag{65}$$

where $\psi_0^{-,l}(x;k)$ is a solution to the boundary value problem

$$\begin{aligned}&(\Delta+k^2)\psi_0^{-,l}=0, \quad x\in\mathbb{R}^2\backslash\overline{\Omega}, \quad \frac{\partial\psi_0^{-,l}}{\partial\mathbf{n}}=-\frac{\theta'\psi_0^{(l)}}{2}, \quad x\in\Gamma_0\\ &\frac{\partial\psi_0^{-,l}}{\partial r}-\mathrm{i}k\psi_0^{-,l}=o(r^{-1/2}), \quad r\to\infty\end{aligned} \tag{66}$$

if $k>0$ and its analytic continuation in the complex plane with the cut along the imaginary axis,

$$\begin{aligned}&\tau_{0,1}^{(l)}=\frac{1}{4k_0}\int_{\Gamma_0}\left(\psi_0^{(l)}(x(s))\right)^2\mathrm{d}\theta(s)>0, \quad \operatorname{Im}\tau_{0,2}^{(l)}=-\frac{\sigma^{(l)}}{2}<0\\ &\sigma^{(l)}=\lim_{R\to\infty}\int_{|x|=R}\left|\psi_0^{-,l}(x;k_0)\right|^2\mathrm{d}s>0\end{aligned} \tag{67}$$

Thus, from (40)–(42) and (65), it follows that the poles of the analytic continuation and the associated quasi-eigenfunction have the asymptotics

$$\tau^{\varepsilon,l}\sim k_0+A\tau_{0,1}^{(l)}, \quad \operatorname{Im}\tau^{\varepsilon,l}\sim A^2\tau_{0,2}^{(l)}, \quad \psi^{\varepsilon,l}(x)\sim\psi_0^{(l)}(x), \quad x\in\Omega, \quad \psi^{\varepsilon,l}(x)\sim A\psi_0^{-,l}(x;k_0), \quad x\in\mathbb{R}^2\backslash\overline{\Omega} \tag{68}$$

where $\tau_{0,1}^{(l)}$ and $\operatorname{Im}\tau_{0,2}^{(l)}$ are defined by (67), $\psi_0^{(l)}$ is a eigenfunction of the Neumann problem for $-\Delta$ in Ω, associated with the eigenvalue $\lambda_0=k_0^2>0$ and normalized as (39), (64), and $\psi_0^{-,l}$ is the solution to (66).

4. Conclusion remarks

Similarly [3,7], one can obtain that for f with finite support and k close to k_0, the solutions to the problems (1), (2) and their analytic continuations meet the representation

$$u_\varepsilon(x,k)=\sum_{l=1}^{n}\frac{\psi^{\varepsilon,l}(x)}{k^2-\left(\tau^{\varepsilon,l}\right)^2}\int_{\mathbb{R}^2}\psi^{\varepsilon,l}(y)\,f(y)\,\mathrm{d}y+\widetilde{u}_\varepsilon(x,k) \tag{69}$$

$$\widetilde{u}_\varepsilon\underset{\varepsilon\to 0}{\to}u_0 \quad \text{in } H^1_{\mathrm{loc}}(\mathbb{R}^2\backslash\overline{\Omega}), \quad \|\widetilde{u}_\varepsilon\|_{H^1(\Omega)}\le C\|f\|_{L_2(\Omega)} \tag{70}$$

From (69), it follows that for the real frequencies k, the first term in the right-hand side has the strongest impact in the case

$$k=k^{(l)}(\varepsilon)=\operatorname{Re}\tau^{\varepsilon,l}+O(\operatorname{Im}\tau^{\varepsilon,l}) \tag{71}$$

For real valued t, from (71), (7), (27), (38), (65), (67) it follows that the peak frequencies have the form

$$k^{(l)}(\varepsilon)=k_0+\varepsilon\tau_1^{(l)}+\varepsilon^2\ln^2\sin a\,t+o\left(\varepsilon^2\ln^2\sin a\right), \qquad \text{for (1)} \tag{72}$$

$$k^{(l)}(\varepsilon)=k_0+A(\varepsilon)\tau_{0,1}^{(l)}+A^2(\varepsilon)t+o\left(A^2(\varepsilon)\right), \qquad \text{for (2)} \tag{73}$$

Denote $T^{(l)}(t)=2k_0\left(t-\ln^{-2}\sin a\,\tau_2^{(l)}\right)$ for the boundary value problems (1) and $T^{(l)}(t)=2k_0\left(t-\tau_{0,2}^{(l)}\right)$ for the boundary value problems (2). Let $\operatorname{supp} f\subset\mathbb{R}^2\backslash\overline{\Omega}$ (the scattering problem).

Then substituting (72), (7), (8), (9), (32) and (73), (68) in (69), and using (70) we obtain that for real k satisfying (73) the leading terms of the asymptotics of the solutions to the boundary value problems (1), (2) have the forms

$$u_\varepsilon(x;k^{(l)}(\varepsilon))\sim-\frac{\psi_0^{(l)}(x)}{2\varepsilon T^{(l)}(t)}\int_{\Gamma_0}\frac{\partial}{\partial\mathbf{n}}\psi_0^{(l)}(x(s))\frac{\partial}{\partial\mathbf{n}}u_0(x(s);k_0)\frac{\mathrm{d}s}{\theta'(s)}, \quad x\in\Omega,$$

$$u_\varepsilon(x,k^{(l)}(\varepsilon))\sim-\frac{\Psi^{(l)}(x;k_0)}{2T^{(l)}(t)}\int_{\Gamma_0}\frac{\partial}{\partial\mathbf{n}}\psi_0^{(l)}(x(s))\frac{\partial}{\partial\mathbf{n}}u_0(x(s);k_0)\frac{\mathrm{d}s}{\theta'(s)}+u_0(x;k_0), \quad x\in\mathbb{R}^2\backslash\overline{\Omega}$$

for (1), and

$$u_\varepsilon(x;k^{(l)}(\varepsilon)) \sim -\frac{\psi_0^{(l)}(x)}{2A(\varepsilon)T^{(l)}(t)}\int\limits_{\Gamma_0}\psi_0^{(l)}(x(s))u_0(x(s);k_0)\,\mathrm{d}\theta(s), \quad x\in\Omega,$$

$$u_\varepsilon(x,k^{(l)}(\varepsilon)) \sim -\frac{\psi_0^{-,l}(x;k_0)}{2T^{(l)}(t)}\int\limits_{\Gamma_0}\psi_0^{(l)}(x(s))u_0(x(s);k_0)\,\mathrm{d}\theta(s)+u_0(x;k_0), \quad x\in\mathbb{R}^2\backslash\overline{\Omega}$$

for (2).

In the scattering problems on the peak frequencies, the solutions increase in Ω only, while in $\mathbb{R}^2\backslash\overline{\Omega}$ it differs from the solution to the external limit problems up to $O(1)$. This phenomenon was discovered by Rayleigh for the classic acoustic resonator with one opening [8].

Acknowledgements

The work was supported by grant of RFBR-NCNI-a (13-01-91052).

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