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Thin hybrid linearly piezoelectric junctions

Les jonctions minces hybrides linéairement piézoélectriques

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ABSTRACT

We extend our previous study [1] devoted to thin linearly piezoelectric junctions to the case when the elastic, piezoelectric and dielectric coefficients of the junction are not of the same order of magnitude.

RÉSUMÉ

Nous étendons notre étude [1] consacrée aux jonctions minces linéairement piézoélectriques au cas où les coefficients élastiques, piézoélectriques et diélectriques de la jonction ne sont pas du même ordre de grandeur.

1. Introduction

Due to the wide range of values taken by the elastic, piezoelectric and dielectric coefficients of various devices, it is worthwhile to extend our previous study [1] devoted to thin linearly piezoelectric junctions to the case when the elastic, piezoelectric and dielectric coefficients of the junction *are not of the same order of magnitude*. Our various asymptotic models for a thin piezoelectric junction between two linearly piezoelectric or elastic bodies will be indexed by $p = (p_1, p_2, p_3)$ in $\{1, 2, 3, 4\}^3$. Indices p_1 and p_2 are respectively relative to the magnitude of the elastic and dielectric coefficients of the adhesive with respect to that of the constant thickness 2ε of the layer containing the adhesive. More precisely, we assume that $h := (\varepsilon, \mu) = (\varepsilon, \mu_{mm}, \mu_{ee}, \mu_{me})$ takes values in a countable set with a sole cluster point $\bar{h} \in \{0\} \times [0, +\infty]^3$, so that

$$\begin{cases} p_1 = 1: \ \bar{\mu}^1_{mm} := \lim_{h\to\bar{h}}(2\varepsilon\mu_{mm}) \in (0, +\infty) \\ p_1 = 2: \ \bar{\mu}^1_{mm} := \lim_{h\to\bar{h}}(2\varepsilon\mu_{mm}) = 0 \\ \qquad\quad \bar{\mu}^2_{mm} := \lim_{h\to\bar{h}}(\mu_{mm}/2\varepsilon) = +\infty \\ p_1 = 3: \ \bar{\mu}^2_{mm} := \lim_{h\to\bar{h}}(\mu_{mm}/2\varepsilon) \in (0, +\infty) \\ p_1 = 4: \ \bar{\mu}^2_{mm} := \lim_{h\to\bar{h}}(\mu_{mm}/2\varepsilon) = 0 \end{cases} \qquad \begin{cases} p_2 = 1: \ \bar{\mu}^1_{ee} := \lim_{h\to\bar{h}}(2\varepsilon\mu_{ee}) \in (0, +\infty) \\ p_2 = 2: \ \bar{\mu}^1_{ee} := \lim_{h\to\bar{h}}(2\varepsilon\mu_{ee}) = 0 \\ \qquad\quad \bar{\mu}^2_{ee} := \lim_{h\to\bar{h}}(\mu_{ee}/2\varepsilon) = +\infty \\ p_2 = 3: \ \bar{\mu}^2_{ee} := \lim_{h\to\bar{h}}(\mu_{ee}/2\varepsilon) \in (0, +\infty) \\ p_2 = 4: \ \bar{\mu}^2_{ee} := \lim_{h\to\bar{h}}(\mu_{ee}/2\varepsilon) = 0 \end{cases} \tag{1}$$

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The parameters μ_{mm}, μ_{ee}, μ_{me} respectively characterize the order of magnitude of the elastic, dielectric and piezoelectric coefficients of the adhesive. The case $p_1 = p_2$ being already treated in [1], in the following we assume $p_1 \neq p_2$. As in [1], index p_3 characterizes the status of the adherents but also that of the interfaces between adherents and adhesive:

$$\begin{cases} p_3 = 1: \text{ the two interfaces are electromechanically perfectly permeable} \\ p_3 = 2: \text{ the two interfaces are electrically permeable} \\ p_3 = 3: \text{ one interface is electrically permeable while the other one bears an electrode} \\ p_3 = 4: \text{ the two interfaces bear an electrode} \end{cases} \tag{2}$$

The physical situation is that of [1], which we recall as follows. Let Ω be a domain, with Lipschitz-continuous boundary, of $\mathbb{R}^3$, assimilated with the physical Euclidean space with basis $\{e_1, e_2, e_3\}$, whose intersection S with $\{x_3 = 0\}$ is a domain of $\mathbb{R}^2$ of positive two-dimensional Hausdorff measure $\mathcal{H}_2(S)$. Let $\Omega_\pm := \Omega \cap \{\pm x_3 > 0\}$ and ε be a small positive number, then adhesive and adherents occupy $B^\varepsilon := S \times (-\varepsilon, \varepsilon)$, $\Omega^\varepsilon_\pm := \Omega_\pm \pm \varepsilon e_3$, respectively; let $\Omega^\varepsilon = \Omega^\varepsilon_+ \cup \Omega^\varepsilon_-$, $S^\varepsilon_\pm := S \pm \varepsilon e_3$, $\mathcal{O}^\varepsilon := \Omega^\varepsilon \cup B^\varepsilon \cup_\pm S^\varepsilon_\pm$. Let $(\Gamma_{\mathrm{mD}}, \Gamma_{\mathrm{eD}})$, $(\Gamma_{\mathrm{eD}}, \Gamma_{\mathrm{eN}})$ be two partitions of $\partial\Omega$ with $\mathcal{H}_2(\Gamma_{\mathrm{mD}}), \mathcal{H}_2(\Gamma_{\mathrm{eD}}) > 0$ and $0 < \delta := \mathrm{dist}(\Gamma_{\mathrm{eD}}, S)$. For all Γ in $\{\Gamma_{\mathrm{mD}}, \Gamma_{\mathrm{mN}}, \Gamma_{\mathrm{eD}}, \Gamma_{\mathrm{eN}}\}$, $\Gamma_\pm$, $\Gamma^\varepsilon_\pm$, Γ^ε denotes $\Gamma \cap \{\pm x_3 > 0\}$, $\Gamma_\pm \pm \varepsilon e_3$, $\cup_\pm \Gamma^\varepsilon_\pm$, respectively; if $(\gamma_{\mathrm{D}}, \gamma_{\mathrm{N}})$ is a partition of $\gamma := \partial S$, we denote $\{\gamma_{\mathrm{D}}, \gamma_{\mathrm{N}}, \gamma\} \times (-\varepsilon, \varepsilon)$ by $\{\Gamma^\varepsilon_{\mathrm{DI}}, \Gamma^\varepsilon_{\mathrm{NI}}, \Gamma^\varepsilon_{\mathrm{lat}}\}$. The structure made of the adhesive and the two adherents, perfectly stuck together along $S^\varepsilon_\pm$, is clamped on $\Gamma^\varepsilon_{\mathrm{mD}}$ and subjected to body forces of density f^ε and to surface forces of density F^ε on $\Gamma^\varepsilon_{\mathrm{mD}}$ that vanishes on $\Gamma^\varepsilon_{\mathrm{lat}}$. Moreover, a given electric potential $\varphi^h_{p_0}$ is applied on $\Gamma^\varepsilon_{\mathrm{DI}}$ (and also on $\Gamma^\varepsilon_{\mathrm{eD}}$ when $p_3 = 1$), while electric charges of density d^ε appear on $\Gamma^\varepsilon_{\mathrm{NI}}$ (and also on $\Gamma^\varepsilon_{\mathrm{eN}}$ when $p_3 = 1$).

If σ^h_p, u^h_p, $e(u^h_p)$, D^h_p, φ^h_p respectively stand for the fields of stress, displacement, strain, electric displacement and electric potential, the constitutive equations of the structure, for all $\hat{p} := (p_1, p_2)$, read as:

$$\begin{cases} (\sigma^h_p, D^h_p) = M^\mu_{\mathrm{I}}\big(e(u^h_p), \nabla\varphi^h_p\big) & \text{in } B^\varepsilon \ \forall p_3 \in \{1,2,3,4\} \\ \begin{cases} (\sigma^h_p, D^h_p) = M^\varepsilon_{\mathrm{E}}\big(e(u^h_p), \nabla\varphi^h_p\big) & \text{in } \Omega^\varepsilon \text{ if } p_3 = 1 \\ \sigma^h_p = a^\varepsilon_{\mathrm{E}} e(u^h_p) & \text{in } \Omega^\varepsilon \text{ if } p_3 > 1 \end{cases} \end{cases} \tag{3}$$

where

$$(M^\varepsilon_{\mathrm{E}}, a^\varepsilon_{\mathrm{E}})(x) = (M_{\mathrm{E}}, a_{\mathrm{E}})(x \mp \varepsilon e_3) \quad \forall x \in \Omega^\varepsilon_\pm \tag{4}$$

$$\begin{cases} (M_{\mathrm{I}}, M_{\mathrm{E}}) \in L^\infty\big(S \times \Omega; \mathrm{Lin}(\mathbb{K})\big) \text{ such that} \\ M^\mu_{\mathrm{I}} := \begin{bmatrix} \mu_{mm} a_{\mathrm{I}} & -\mu_{me} b_{\mathrm{I}} \\ \mu_{me} b^T_{\mathrm{I}} & \mu_{ee} c_{\mathrm{I}} \end{bmatrix}, \quad M_{\mathrm{E}} := \begin{bmatrix} a_{\mathrm{E}} & -b_{\mathrm{E}} \\ b^T_{\mathrm{E}} & c_{\mathrm{E}} \end{bmatrix} \\ M_{\mathrm{P}} := \begin{bmatrix} a_{\mathrm{P}} & -b_{\mathrm{P}} \\ b^T_{\mathrm{P}} & c_{\mathrm{P}} \end{bmatrix}; \exists \kappa > 0 \quad \kappa |k|^2 \le M_{\mathrm{P}}(x) k \cdot k \quad \forall k \in \mathbb{K} := \mathbb{S}^3 \times \mathbb{R}^3 \text{ a.e. } x \in \Omega, \ \forall \mathrm{P} \in \{\mathrm{I}, \mathrm{E}\} \end{cases} \tag{5}$$

and $\mathrm{Lin}(\mathbb{K})$ is the space of linear operators on $\mathbb{K}$ whose inner product and norm are noted $\cdot$ and $|\cdot|$ as in $\mathbb{R}^3$ (the same notations for the norm and inner product also stand for $\mathbb{S}^N$ the space of $N \times N$ symmetric matrices).

Lastly we have to add the following conditions on $S^\varepsilon_\pm$:

$$\begin{cases} p_3 = 2 & D^h_p \cdot e_3 = 0 & \text{on } S^\varepsilon_\pm \\ p_3 = 3 & D^h_p \cdot e_3 = 0 & \text{on } S^\varepsilon_+, \quad \varphi^h_p = \varphi^h_{p_0} \text{ on } S^\varepsilon_- \\ p_3 = 4 & \varphi^h_p = \varphi^h_{p_0} & \text{on } S^\varepsilon_\pm \end{cases} \tag{6}$$

the electric potential $\varphi^h_{p_0}$ being given on S^ε_+ or $S^\varepsilon_\pm$.

It will be convenient to use the following notations:

$$\begin{cases} \hat{k} := (\hat{e}, \hat{g}) \quad \hat{e} := e_{\alpha\beta}, \ 1 \le \alpha, \beta \le 2, \quad \hat{g} := (g_1, g_2), \quad \forall k = (e, g) \in \mathbb{K} \\ k(r) = k(v, \psi) := \big(e(v), \nabla\psi\big) \quad \forall r \in H^1(\mathcal{O}; \mathbb{R}^3 \times \mathbb{R}) \\ e(v) \in \mathcal{D}'(S; \mathbb{S}^2); \quad \big(e(v)\big)_{\alpha\beta} = \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha), \ 1 \le \alpha, \beta \le 2, \quad \forall v \in \mathcal{D}'(S; \mathbb{R}^3) \end{cases} \tag{7}$$

and the *same* symbol $e(\cdot)$ shall *also* stand for the symmetrized gradient in the sense of distributions of $\mathcal{D}'(\mathcal{O}; \mathbb{R}^3)$, $\mathcal{O} \in \{\mathcal{O}^\varepsilon, \Omega, \Omega \setminus S, B^\varepsilon, \Omega^\varepsilon\}$ or $\mathcal{D}'(S; \mathbb{R}^2)$. An electromechanical state with vanishing electric potential on $\Gamma^\varepsilon_{\mathrm{DI}}$ and on $\Gamma^\varepsilon_{\mathrm{eD}}$ when $p_3 = 1$ will belong to $V^\varepsilon_p := H^1_{\Gamma^\varepsilon_{\mathrm{mD}}}(\mathcal{O}^\varepsilon; \mathbb{R}^3) \times \Phi^\varepsilon_{p_3}$, with

$$\begin{cases} \Phi_1^\varepsilon = H^1_{\Gamma^\varepsilon_{\mathrm{DI}} \cup \Gamma^\varepsilon_{\mathrm{eD}}}(\mathcal{O}^\varepsilon) \\ \Phi_2^\varepsilon = H^1_{\Gamma^\varepsilon_{\mathrm{DI}}}(B^\varepsilon) \text{ if } \mathcal{H}_2(\Gamma^\varepsilon_{\mathrm{DI}}) > 0,\ H^1_m(B^\varepsilon) \text{ if } \mathcal{H}_2(\Gamma^\varepsilon_{\mathrm{DI}}) = 0 \\ \Phi_3^\varepsilon = H^1_{\Gamma^\varepsilon_{\mathrm{DI}} \cup S^\varepsilon_-}(B^\varepsilon) \\ \Phi_4^\varepsilon = H^1_{\Gamma^\varepsilon_{\mathrm{DI}} \cup_\pm S^\varepsilon_\pm}(B^\varepsilon) \end{cases} \tag{8}$$

where, for any domain $\mathcal{O}$ of $\mathbb{R}^N$, $N = 2, 3$, $H^1_\Sigma(\mathcal{O}; \mathbb{R}^M)$ denotes the subspace of $H^1(\mathcal{O}; \mathbb{R}^M)$, $M = 1$ or 3, of all elements with vanishing traces on a part Σ of the boundary of $\mathcal{O}$, while $H^1_m(\mathcal{O}; \mathbb{R}^M)$ denotes the subspace of all elements with vanishing average.

We make the following assumptions on the data:

$$\begin{cases} \text{Given } (f, F, d_{\mathrm{E}}, d_{\mathrm{I}}) \text{ in } L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma_{\mathrm{mN}}; \mathbb{R}^3) \times L^2(\Gamma_{\mathrm{eN}}) \times L^2\big(\gamma_{\mathrm{N}} \times (-1, 1)\big) \\ \text{with } \int_{\Gamma_{\mathrm{lat}}} d_{\mathrm{I}}\, \mathrm{d}\mathcal{H}_2 = 0 \text{ when } p_3 = 2 \text{ and } \mathcal{H}_2\big(\gamma_{\mathrm{D}} \times (-1, 1)\big) = 0 \\ \varphi_{o\mathrm{I}} \text{ in } H^{3/2}(\mathbb{R}) \text{ vanishing in } \{\, |x_3| > 1 + \delta/2 \,\}, \text{ and } \varphi_{o\mathrm{E}} \text{ in } H^1(\Omega) \text{ vanishing on } S, \text{ then:} \\ f^\varepsilon(x) = f(x \mp \varepsilon e_3) \text{ a.e. } x \in \Omega^\varepsilon_\pm, \quad f^\varepsilon(x) = 0 \text{ a.e. } x \in B^\varepsilon \\ F^\varepsilon(x) = F(x \mp \varepsilon e_3) \text{ a.e. } x \in \Gamma^\varepsilon_{\mathrm{mN}\pm} \\ d^\varepsilon(x) = (2\mu_{ee})^{1/2} d_{\mathrm{I}}(\hat{x}, x_3/\varepsilon) \text{ a.e. } x \in \Gamma^\varepsilon_{NI} \\ d^\varepsilon(x) = d_{\mathrm{E}}(x \mp \varepsilon e_3) \text{ a.e. } x \in \Gamma^\varepsilon_{\mathrm{eN}\pm} \quad \text{if } p_3 = 1 \\ \varphi^h_{p_o}(x) = \begin{cases} \varphi_{o\mathrm{E}}(x \mp \varepsilon e_3) + \varepsilon^{p_{\mathrm{DI}}} \varphi_{o\mathrm{I}}(x \pm (1 - \varepsilon) e_3) \text{ a.e. } x \in \Omega^\varepsilon_\pm \\ \varepsilon^{p_{\mathrm{DI}}} \varphi_{o\mathrm{I}}(\hat{x}, x_3/\varepsilon) \text{ a.e. } x \in B^\varepsilon \end{cases} \end{cases} \tag{9}$$

where p_{DI} is such that $p_{\mathrm{DI}} = 0$ if $\partial_3 \varphi_{o\mathrm{I}} = 0$ in $S \times (-1, 1)$, $p_{\mathrm{DI}} = 1$ if $\partial_3 \varphi_{o\mathrm{I}} \neq 0$ in $S \times (-1, 1)$. We also introduce the element φ_o of $H^{1,1}(\Omega, S) := \big\{ \psi \in H^1(\Omega) \text{ whose trace } \gamma_o(\psi) \text{ on } S \text{ belongs to } H^1(S) \big\}$ defined by $\varphi_o(x) = \varphi_{o\mathrm{E}}(x) + (1 - p_{\mathrm{DI}})\varphi_{o\mathrm{I}}(x \pm e_3)$ a.e. $x \in \Omega_\pm$. We note $\overline{\varphi}_o$ the trace on γ_{D} of φ_o and set $\triangle \varphi_{o\mathrm{I}} = \frac{1}{2}\big(\varphi_{o\mathrm{I}}(\cdot, 1) - \varphi_{o\mathrm{I}}(\cdot, -1)\big)$.

Then, if $\mathcal{M}_p$ and $\mathcal{L}_p$ are defined by:

$$\begin{cases} \mathcal{M}_p(s, r) := \begin{cases} \int_{\Omega^\varepsilon} M^\varepsilon_{\mathrm{E}} k(s) \cdot k(r)\, \mathrm{d}x + \int_{B^\varepsilon} M^\mu_{\mathrm{I}} k(s) \cdot k(r)\, \mathrm{d}x, & \text{if } p_3 = 1 \\ \int_{\Omega^\varepsilon} a^\varepsilon_{\mathrm{E}} e(u) \cdot e(v)\, \mathrm{d}x + \int_{B^\varepsilon} M^\mu_{\mathrm{I}} k(s) \cdot k(r)\, \mathrm{d}x, & \text{if } p_3 > 1 \end{cases} \\ \mathcal{L}_p(r) := \int_\Omega f^\varepsilon \cdot v\, \mathrm{d}x + \int_{\Gamma^\varepsilon_{\mathrm{mN}}} F^\varepsilon \cdot v\, \mathrm{d}\mathcal{H}_2 + \int_{\Gamma^\varepsilon_{NI} \cup \Gamma^\varepsilon} d^\varepsilon \psi\, \mathrm{d}\mathcal{H}_2 \quad \Gamma^\varepsilon = \Gamma^\varepsilon_{\mathrm{eN}} \text{ if } p_3 = 1, \Gamma^\varepsilon = \varnothing \text{ if } p_3 > 1 \end{cases} \tag{10}$$

seeking an equilibrium state leads to the problem

$$(\mathcal{P}^h_p): \quad \text{Find } s^h_p \text{ in } (0, \varphi^h_{p_0}) + V^\varepsilon_p \text{ such that} \quad \mathcal{M}_p(s^h_p, r) = \mathcal{L}_p(r), \quad \forall r \in V^\varepsilon_p$$

which, by Stampacchia's theorem, has a unique solution.

2. The asymptotic models

By proceeding as in [1], we will determine the asymptotic behavior of the structure when h goes to $\bar{h}$ under the following assumption on the behavior of μ_{me}, whose rationale will clearly appear in Step 3 below.

$$(H_p) \quad \begin{cases} \text{There exists } \bar{\mu}_{me} \text{ in } [0, +\infty) \text{ such that} \quad \bar{\mu}_{me} = \lim_{h \to \bar{h}} \mu_{me}, \text{ with} \\ \lim_{h \to \bar{h}} \frac{\mu^2_{me}}{\mu_{ee}} \frac{1}{\varepsilon} = 0 \quad \text{when } p = (4, 1, 4) \text{ or } p = (3, 2, p_3), (3, 4, p_3), (4, 2, p_3), 1 \le p_3 \le 4 \\ \lim_{h \to \bar{h}} \frac{\mu^2_{me}}{\mu_{ee}} \varepsilon = 0 \quad \text{when } p = (2, 1, 4) \text{ or } p = (1, 2, p_3), (1, 4, p_3), (2, 4, p_3), 1 \le p_3 \le 4 \\ \lim_{h \to \bar{h}} \frac{\mu^2_{me}}{\mu_{mm}} \frac{1}{\varepsilon} = 0 \quad \text{when } p = (2, 3, p_3), (2, 4, p_3), (4, 3, p_3), 1 \le p_3 \le 4 \\ \lim_{h \to \bar{h}} \frac{\mu^2_{me}}{\mu_{mm}} \varepsilon = 0 \quad \text{when } p = (4, 2, 1) \text{ or } p = (2, 1, p_3), (4, 1, p_3), 1 \le p_3 \le 4 \\ \lim_{h \to \bar{h}} \frac{\mu_{me}}{\varepsilon} = 0 \quad \text{when } p = (4, 3, 4) \\ \lim_{h \to \bar{h}} \mu_{me} = 0 \quad \text{when } p = (1, 3, 4) \text{ or } p = (2, 3, 4) \\ \lim_{h \to \bar{h}} \mu_{me} = \bar{\mu}_{me} \quad \text{when } p = (1, 3, p_3), 1 \le p_3 \le 3, \text{ or } p = (3, 1, p_3), 1 \le p_3 \le 4 \end{cases}$$

In the following, C will denote various constants independent of h which may vary from line to line. It will be convenient in the cases $p_3 > 1$ to use the same symbol s_p^h for $(u_p^h, \tilde{\varphi}_p^h)$ where $\tilde{\varphi}_p^h$ denotes the extension into Ω^ε of φ_p^h by 0. Without loss of generality, we suppose $\mathcal{H}_2(\Gamma_{\mathrm{mD}+}) > 0$; moreover, we assume $\mathcal{H}_2(\Gamma_{\mathrm{mD}-}) > 0$ when $p_1 = 4$, and $\mathcal{H}_2(\Gamma_{\mathrm{eD}\pm}) > 0$ when $p_2 = 4$.

Step 1 (a priori estimates): By taking $r = s_p^h - (0, \varphi_{p_0}^h)$ in the variational formulation of $(\mathcal{P}_p^h)$, one has:

$$\mu_{mm}|e(u_p^h)|^2_{L^2(B^\varepsilon;\mathbb{S}^3)} + \mu_{ee}|\nabla\varphi_p^h|^2_{L^2(B^\varepsilon;\mathbb{R}^3)} + |k(s_p^h)|^2_{L^2(\Omega^\varepsilon;\mathbb{K})} \le C \tag{11}$$

Step 2 (convergence of (s_p^h)): As in [1], the two following tools are suitable to describe the asymptotic behavior of the electromechanical state in the adherents and adhesive, respectively. First, let T^ε be the mapping from $H^1(\Omega^\varepsilon;\mathbb{R}^3\times\mathbb{R})$ into $H^1(\Omega\setminus S;\mathbb{R}^3\times\mathbb{R})$ defined by:

$$(T^\varepsilon r)(x) = \big(T^\varepsilon(v,\psi)\big)(x) = (T_1^\varepsilon v, T_2^\varepsilon\psi)(x) := (v,\psi)(x \pm \varepsilon e_3) \quad \forall x \in \Omega_\pm \tag{12}$$

Note that $T^\varepsilon s_p^h = (T_1 u_p^h, 0)$ if $p_3 > 1$! For any w in $H^1(\Omega\setminus S;\mathbb{R}^N)$, $N \in \{1,3\}$, if $\gamma_o^\pm(w^\pm)$ denotes the trace on S of its restriction $w^\pm$ to $\Omega_\pm$, $[\![w]\!]$ stands for $\gamma_o^+(w^+) - \gamma_o^-(w^-)$.

Next for all $r = (v,\psi)$ in $H^1(B^\varepsilon;\mathbb{R}^3\times\mathbb{R})$, we set the following element of $L^2(S;\mathbb{K})$:

$$k_p(\varepsilon,r) = \big(e_p(\varepsilon,v), g_p(\varepsilon,\psi)\big) := \left(\frac{1}{(2\varepsilon)^{q_1}}\int_{-\varepsilon}^{\varepsilon} e(v)(\cdot,x_3)\,\mathrm{d}x_3, \frac{1}{(2\varepsilon)^{q_2}}\int_{-\varepsilon}^{\varepsilon}\nabla\psi(\cdot,x_3)\,\mathrm{d}x_3\right)$$
$$q_i = \max(2-p_i,0),\ i=1,2 \tag{13}$$

and there holds

$$\widehat{k_p(\varepsilon,r)} = (e(\widehat{U_p}),\nabla\Phi_p),\quad (U_p^h,\Phi_p^h) := \left(\frac{1}{(2\varepsilon)^{q_1}}\int_{-\varepsilon}^{\varepsilon} u_p^h(\cdot,x_3)\,\mathrm{d}x_3, \frac{1}{(2\varepsilon)^{q_2}}\int_{-\varepsilon}^{\varepsilon}\varphi_p^h(\cdot,x_3)\,\mathrm{d}x_3\right) \tag{14}$$

So (11) and standard estimates in Sobolev spaces (see [1]) imply:

$$\begin{cases}|k(T^\varepsilon s_p^h)|_{L^2(\Omega\setminus S;\mathbb{K})} \le C, \quad |[\![T^\varepsilon s_p^h]\!]|^2_{L^2(S;\mathbb{R}^3\times\mathbb{R})} \le C\varepsilon\left(1+\dfrac{1}{\mu_{mm}}+\dfrac{1}{\mu_{ee}}\right)\\ |e_p(\varepsilon,u_p^h)|^2_{L^2(S;\mathbb{S}^3)} \le C\varepsilon^{-2q_1}\dfrac{\varepsilon}{\mu_{mm}}, \quad |g_p(\varepsilon,\varphi_p^h)|^2_{L^2(S;\mathbb{R}^3)} \le C\varepsilon^{-2q_2}\dfrac{\varepsilon}{\mu_{ee}}\\ |U_p^h|^2_{L^2(S;\mathbb{R}^3)} \le C\varepsilon^{2(1-q_1)}\left(1+\dfrac{\varepsilon}{\mu_{mm}}\right)\\ \varepsilon^{2(q_2-1)}|\Phi_p^h|^2_{L^2(S)} \le Cc^*(h), \quad c^*(h) = \left(1+\dfrac{\varepsilon}{\mu_{ee}}\right) \text{ if } p_3=1, \dfrac{1}{\varepsilon\mu_{ee}} \text{ if } p_3=2, \varepsilon^2+\dfrac{\varepsilon}{\mu_{ee}} \text{ if } p_3>2\\ |e(\widehat{U}_p^h)|^2_{L^2(S;\mathbb{S}^2)} \le C\dfrac{1}{\varepsilon^{2q_1}}\cdot\dfrac{\varepsilon}{\mu_{mm}}, \quad |\nabla\Phi_p^h|^2_{L^2(S;\mathbb{R}^2)} \le C\dfrac{1}{\varepsilon^{2q_2}}\cdot\dfrac{\varepsilon}{\mu_{ee}}\\ |U_p^h - \gamma_o^\pm((T_1^\varepsilon u_p^h)^\pm)|^2_{L^2(S;\mathbb{R}^3)} \le C\left(\varepsilon+\dfrac{\varepsilon}{\mu_{mm}}\right) \quad \text{if } p_1=1\\ |\Phi_p^h - \gamma_o^\pm((T_2^\varepsilon\varphi_p^h)^\pm)|^2_{L^2(S)} \le C\left(\varepsilon+\dfrac{\varepsilon}{\mu_{ee}}\right) \quad \text{if } p_2=1\end{cases} \tag{15}$$

Thus, if $a\otimes_S b$ denotes the symmetrized tensor product of a and b in $\mathbb{R}^3$, we deduce:

Proposition 2.1.

1. *There exists $\bar{s}_p = (\bar{u}_p,\bar{\varphi}_p)$ in $H^1_{\Gamma_{mD}}(\Omega\setminus S;\mathbb{R}^3)\times H^1_{\Gamma_{eD}}(\Omega\setminus S)$ such that $T^\varepsilon s_p^h$ weakly converges in $H^1(\Omega\setminus S;\mathbb{R}^3\times\mathbb{R})$ toward some $\bar{s}_p = (\bar{u},\bar{\varphi}_p)$; $\bar{\varphi}_p = 0$ when $p_3 > 1$, and $\bar{u}_p$ belongs to $H^1(\Omega;\mathbb{R}^3)$ when $p_1 \le 2$, $\bar{\varphi}_p$ belongs to $H^1(\Omega)$ when $p_2 \le 2$ and $p_3 = 1$.*
2. *When $p_1 \ne 4$, $e_p(\varepsilon,u_p^h)$ weakly converges in $L^2(S;\mathbb{S}^3)$ toward some $\bar{e}_p$, and there exists $\overline{U}_p$ in $H^1(S;\mathbb{R}^3)$ such that $\widehat{U_p^h}$ weakly converges in $H^1(S;\mathbb{R}^2)$ toward $\widehat{\overline{U}_p}$, $(U_p^h)_3$ strongly converges in $L^2(S)$ toward $(\overline{U}_p)_3$, moreover*
   i) *when $p_1 = 1$, $\overline{U}_p = \gamma_0(\bar{u}_p)$, $\widehat{\bar{e}_p} = \widehat{e(\bar{u}_p)}$;*
   ii) *when $p_1 > 1$, $\overline{U}_p = 0$, and $\bar{e}_p = [\![u_p]\!]\otimes_S e_3$.*
3. *When $p_2 \ne 4$, $g_p(\varepsilon,\varphi_p^h)$ weakly converges in $L^2(S;\mathbb{R}^3)$ toward some $\bar{g}_p$ and there exists $\overline{\Phi}_p$ in $H^1(S)$ such that Φ_p^h weakly converges in $H^1(S)$ toward $\overline{\Phi}_p$; moreover,*

i) *when* $p_2=1$, $\overline{\Phi}_p$ *is equal to* $\gamma_o(\bar\varphi_p)$ *when* $p_3=1$ *or to* $\gamma_o(\varphi_o)$ *when* $p_3\geq 3$, *furthermore the trace on* $\gamma_{\rm D}$ *of* $\overline{\Phi}_p$ *is equal to* $\bar\varphi_o$ *while* $\widehat{\bar g_p}=\nabla\overline{\Phi}_p$ *and* $(\bar g_p)_3=\triangle\varphi_{o\rm I}$ *when* $p_3=4$;
ii) *when* $p_2=2$, $\overline{\Phi}_p=0$ *and* $\bar g_p=0$;
iii) *when* $p_2=3$, $\overline{\Phi}_p$ *and* $\widehat{\bar g_p}$ *vanish only when* $p_3\neq 2$, *while* $(\bar g_p)_3=[\![\varphi_p]\!]$ *when* $p_3=1$, $(\bar g_p)_3=0$ *when* $p_3=3$.

As in the next step, we will show that $(\bar u_p,\bar\varphi_p)$ is the unique solution of a variational problem, note that the whole sequences converge. When both $\bar e_p$ and $\bar g_p$ are defined we set $\bar k_p=(\bar e_p,\bar g_p)$.

Step 3 (identification of $(\bar s_p,\bar e_p,\bar g_p)$): We proceed in two different ways depending on whether $\bar k_p$ does not exist, $\bar k_p$ is fully or partially identified.

When $p=(3,1,4)$ or $1<p_1,p_2\leq 4$, $p_1\neq p_2$, $1\leq p_3\leq 4$, it suffices to go to the limit in the variational formulation of $(\mathcal{P}_p^h)$ by using the following test functions $r_p^\varepsilon=(v_p^\varepsilon,\psi_p^\varepsilon)$ and taking duly account of the estimates (15), Proposition 2.1, Cauchy Schwarz inequality and (H_p) which are constructed in order that

$$\lim_{h\to\bar h}\int_{B^\varepsilon} b_I\nabla\varphi_p^h\cdot e(v_p^\varepsilon)\,{\rm d}x=\lim_{h\to\bar h}\int_{B^\varepsilon} b_I^T e(u_p^h)\cdot\nabla\psi_p^\varepsilon\,{\rm d}x=0$$

The test functions r_p^ε reads as:

$$\begin{cases} v_p^\varepsilon=w^{\min(p_1-1,2),\varepsilon}, & 1\leq p_3\leq 4\\ \psi_p^\varepsilon=\begin{cases}\zeta^{\min(p_2-1,2),\varepsilon} & p_3=1\\ (\theta_1+x_3\theta_2)/\varepsilon,\quad \theta_1,\theta_2\in C_0^\infty(S) & p_2=3,4,\ p_3=2\\ (1+x_3/\varepsilon)\theta,\quad \theta\in C_0^\infty(S) & p_2=3,4,\ p_3=3\\ 0 & \text{if } (2\leq p_2\leq 4,\ p_3=4) \text{ or } (p_2=2,\ p_3=2,3)\end{cases}\end{cases}$$

where for all w^1 in $H^1_{\Gamma_{\rm mD}}(\Omega;\mathbb{R}^3)$ and all ζ^1 in $H^1_{\Gamma_{\rm eD}}(\Omega)$ vanishing in a neighborhood of $\gamma_{\rm D}$, let $(w^{1,\varepsilon},\zeta^{1,\varepsilon})$ be defined by

$$\big(w^{1,\varepsilon},\zeta^{1,\varepsilon}\big)(x)=\begin{cases}(w^1,\zeta^1)(x\mp\varepsilon e_3) & \text{a.e. } x\in\Omega_\pm^\varepsilon\\ (w^1,\zeta^1)(\hat x,0) & \text{a.e. } x\in B^\varepsilon\end{cases}$$

For all w^2 in $H^1_{\Gamma_{\rm mD}}(\Omega\setminus S;\mathbb{R}^3)$ and all ζ^2 in $H^1_{\Gamma_{\rm eD}}(\Omega\setminus S)$ vanishing in a neighborhood of $\gamma_{\rm D}$, let $(w^{2,\varepsilon},\zeta^{2,\varepsilon})$ be defined by

$$(w^{2,\varepsilon},\zeta^{2,\varepsilon})(x)=\begin{cases}(w^2,\zeta^2)(x\mp\varepsilon e_3) & \text{a.e. } x\in\Omega_\pm^\varepsilon\\ (w^a,\zeta^a)(\hat x,x_3/\varepsilon)+\frac{|x_3|}{\varepsilon}(w^s,\zeta^s)(\hat x,x_3/\varepsilon) & \text{a.e. } x\in B^\varepsilon\end{cases}$$

with

$$(w^a,\zeta^a)(x)=\frac{1}{2}[(w^2,\zeta^2)(\hat x,x_3)-(w^2,\zeta^2)(\hat x,-x_3)]$$
$$(w^s,\zeta^s)(x)=\frac{1}{2}[(w^2,\zeta^2)(\hat x,x_3)+(w^2,\zeta^2)(\hat x,-x_3)]$$

When $\hat p=(1,3)$ or $(3,1)$, with $1\leq p_3\leq 3$, let

$$\overline{M}_{\rm Ip}:=\begin{bmatrix}\bar\mu_{mm}^{L_1}a_{\rm I} & -\bar\mu_{me}b_{\rm I}\\ \bar\mu_{me}b_{\rm I}^T & \bar\mu_{ee}^{L_2}c_{\rm I}\end{bmatrix},\quad L_i=1+\lfloor p_i/2\rfloor,\quad i=1,2 \tag{16}$$

with $\lfloor\cdot\rfloor$ the floor function. We first prove

$$(\overline{M}_{\rm Ip}\bar k_p)_p^2=0 \tag{17}$$

where k_p^i denotes the projection on $\mathbb{K}_p^i$ of any element k of $\mathbb{K}$ with

$$\mathbb{K}=\mathbb{K}_p^1\oplus\mathbb{K}_p^2\oplus\mathbb{K}_p^3 \tag{18}$$

$\mathbb{K}_p^1$ being made of the elements $k=(e,g)$ whose nonvanishing components e_{ij}, g_l are the nonvanishing components $(\bar e_p)_{ij}$, $(\bar g_p)_l$ of $\bar k_p$ which are identified by Proposition 2.1 in terms of $\bar u_p$, $\overline{\Phi}_p$ or φ_o, $\mathbb{K}_p^2$ is made of the elements whose nonvanishing components are the components of $\bar k_p$ that are not identified by Proposition 2.1, and $\mathbb{K}_p^3$ is made of the elements whose nonvanishing components are the vanishing components of $\bar k_p$ identified by Proposition 2.1. By using suitable test functions ρ_p^ε, we may deduce (see [2,3]):

$$(\overline{M}_{\mathrm{I}p}\bar{k}_p)^1 = \widetilde{M}_{\mathrm{I}p}(\bar{k}_p)^1; \quad \widetilde{M}_{\mathrm{I}p} := (\overline{M}_{\mathrm{I}p})^{11} - (\overline{M}_{\mathrm{I}p})^{11}(\overline{M}_{\mathrm{I}p})^{12}\big((\overline{M}_{\mathrm{I}p})^{22}\big)^{-1}(\overline{M}_{\mathrm{I}p})^{21} \tag{19}$$

with $(\overline{M}_{\mathrm{I}p})^{ij}$, $1 \le i, j \le 3$, being the decomposition of $\overline{M}_{\mathrm{I}p}$ in linear operators mapping $\mathbb{K}_p^i$ into $\mathbb{K}_p^j$. That is obtained by using (H_p) and ρ_p^ε defined by:

$$\rho_p^\varepsilon(x) = \begin{cases} (x_3+\varepsilon)(I_{p_1}w, I_{p_2p_3}\psi(\hat{x})) & \text{a.e. } x \in B^\varepsilon \\ 2\varepsilon(I_{p_1}w^+, I_{p_2p_3}\psi^+)(x-\varepsilon e_3) & \text{in } \Omega_+^\varepsilon,\ 0 \text{ in } \Omega_-^\varepsilon \end{cases} \tag{20}$$

where, given (w,ψ) in $C_o^\infty(S;\mathbb{R}^3\times\mathbb{R})$, (w^+,ψ^+) is an extension into $H^1_{\Gamma_{\mathrm{mD}+}}(\Omega_+;\mathbb{R}^3) \times H^1_{\Gamma_{\mathrm{eD}+}}(\Omega_+)$, and $I_{p_1} = \max(0, 2-p_1)$, $I_{1p_3} = 1$ if $p_3 \le 2$, $I_{1p_3} = 0$ if $p_3 > 2$, $I_{3p_3} = 0$ if $p_3 = 1$, $I_{3p_3} = 1$ if $p_3 > 1$.

Second, given (v,ψ) in $\big(H^1_{\Gamma_{\mathrm{mD}}}(\Omega;\mathbb{R}^3) \times H^1_{\Gamma_{\mathrm{eD}}}(\Omega)\big) \cap H^2(\Omega;\mathbb{R}^3\times\mathbb{R})$, ψ vanishing in a neighborhood of γ_{D}, we define $r_p^\varepsilon = (v_p^\varepsilon, \psi_p^\varepsilon)$ by:

$$\text{when } p_1 = 1\colon \begin{cases} \widehat{v_p^\varepsilon}(x) = \hat{v}(\hat{x},0) - x_3\nabla v_3(\hat{x},0), \quad \big(v_3^\varepsilon(x) = v_3(\hat{x},0)\big) & \text{a.e. } x \in B^\varepsilon \\ v_p^\varepsilon(x) = v(x \mp \varepsilon e_3) \mp \varepsilon R^\pm\big(\nabla v_3(\cdot,0),0\big)(x\mp\varepsilon e_3) & \text{a.e. } x \in \Omega_\pm^\varepsilon \end{cases}, \quad \text{when } p_1 = 3\colon \quad v_p^\varepsilon = w^{2,\varepsilon}$$

$$\text{when } p_2 = 1\colon \begin{cases} \psi_p^\varepsilon(x) = \psi(x\mp\varepsilon e_3) \text{ in } \Omega_\pm^\varepsilon, \quad \psi(\hat{x},0) \text{ in } B^\varepsilon & \text{if } p_3 = 1 \\ \psi_p^\varepsilon(x) = \psi(\hat{x},0) \text{ in } B^\varepsilon & \text{if } p_3 = 2 \\ \psi_p^\varepsilon(x) = 0 \text{ in } B^\varepsilon & \text{if } p_3 \ge 3 \end{cases}, \quad \text{when } p_2 = 3 : \psi_p^\varepsilon = \begin{cases} \zeta^{2,\varepsilon} & \text{if } p_3 = 1 \\ \psi(\hat{x},0) & \text{if } p_3 = 2 \\ 0 & \text{if } p_3 = 3 \end{cases} \tag{21}$$

where $R^\pm$ is a continuous lifting operator from $H^{1/2}(S;\mathbb{R}^3)$ into $H^1_{\Gamma_{\mathrm{mD}\pm}}(\Omega_\pm;\mathbb{R}^3)$. Hence (H_p) and (19) yield:

$$\lim_{h\to\bar{h}} \int_{B^\varepsilon} M_{\mathrm{I}}^\mu k(s_p^h)\cdot k(r_p^\varepsilon)\,\mathrm{d}x = \int_S \widetilde{M}_{\mathrm{I}p}(\bar{k}_p)^1 \cdot (e', g')\,\mathrm{d}\hat{x} \tag{22}$$

where:

$$\begin{cases} \text{when } \hat{p} = (1,3): \ e' = e(\hat{v}), \quad g' = [\![\zeta]\!]e_3 \text{ if } p_3 = 1, \quad g' = \nabla\psi \text{ if } p_3 = 2, \quad g' = 0 \text{ if } p_3 = 3 \\ \text{when } \hat{p} = (3,1): \ e' = [\![w^2]\!] \otimes_S e_3, \quad g' = \nabla\psi \text{ for } 1 \le p_3 \le 3 \end{cases} \tag{23}$$

In the remaining cases, as, respectively, $\bar{g}_p = 0$ or does not exist, or $\bar{e}_p = 0$ or does not exist, we proceed in the same way, but with a suitable decomposition of $\mathbb{S}^3$ or $\mathbb{R}^3$, respectively, and $\widetilde{M}_{\mathrm{I}p}$ replaced by $\bar{\mu}^1_{mm}\tilde{a}_{\mathrm{I}}$ or $\bar{\mu}^1_{ee}\tilde{c}_{\mathrm{I}}$, respectively, $\tilde{a}_{\mathrm{I}}$ and $\tilde{c}_{\mathrm{I}}$ being defined in a similar way as $\widetilde{M}_{\mathrm{I}p}$.

Lastly, Jensen inequality and the previously established weak convergences achieve the proof of the following convergence result, which supports our asymptotic models in the form of variational problems $\big(\overline{\mathcal{P}}_p\big)$.

Theorem 2.1.

- *If $p_3 = 1$, when h goes to $\bar{h}$, $T^\varepsilon s_p^h$ strongly converges in $H^1(\Omega\setminus S;\mathbb{R}^3\times\mathbb{R})$ toward $\bar{s}_p$ the unique solution to*

$$\big(\overline{\mathcal{P}}_{(\hat{p},1)}\big): \quad \begin{cases} \textit{Find } s = (u,\varphi) \textit{ in } (0,\varphi_o) + V_{p_1}\times\Psi_{p_21} \textit{ such that} \\ \overline{\mathcal{M}}_{(\hat{p},1)}(s,r) = \overline{\mathcal{L}}_{(\hat{p},1)}(r) \quad \forall r = (v,\psi) \in V_{p_1}\times\Psi_{p_21} \end{cases}$$

where

$$\overline{\mathcal{M}}_{(\hat{p},1)}(s,r) = \int_\Omega M_{\mathrm{E}}k(s)\cdot k(r)\,\mathrm{d}x + \overline{\mathcal{M}}_{\mathrm{I}(\hat{p},1)}(s,r)$$

$$\overline{\mathcal{M}}_{\mathrm{I}(\hat{p},1)}(s,r) := \begin{cases} 0 \textit{ if } \hat{p} = (2,4) \textit{ or } (4,2) \\ \int_S \bar{\mu}^1_{mm}\tilde{a}_{\mathrm{I}}e(\hat{u})\cdot e(\hat{v})\,\mathrm{d}\hat{x} \textit{ if } \hat{p} = (1,2) \textit{ or } (1,4) \\ \int_S \bar{\mu}^1_{ee}\tilde{c}_{\mathrm{I}}\widehat{\nabla}\varphi\cdot\widehat{\nabla}\psi\,\mathrm{d}\hat{x} \textit{ if } \hat{p} = (2,1) \textit{ or } (4,1) \\ \int_S \bar{\mu}^2_{ee}c_{\mathrm{I}}[\![\varphi]\!]e_3\cdot[\![\psi]\!]e_3\,\mathrm{d}\hat{x} \textit{ if } \hat{p} = (2,3) \textit{ or } (1,3) \\ \int_S \bar{\mu}^2_{mm}a_{\mathrm{I}}[\![u]\!]\otimes_S e_3\cdot[\![v]\!]\otimes_S e_3\,\mathrm{d}\hat{x} \textit{ if } \hat{p} = (3,2) \textit{ or } (3,4) \\ \int_S \widetilde{M_{\mathrm{I}p}}(e(\hat{u}),[\![\varphi]\!]e_3)\cdot(e(\hat{v}),[\![\psi]\!]e_3)\,\mathrm{d}\hat{x} \textit{ if } \hat{p} = (1,3) \\ \int_S \widetilde{M_{\mathrm{I}p}}\big([\![u]\!]\otimes_S e_3, \nabla\gamma_o(\varphi)\big)\cdot\big([\![v]\!]\otimes_S e_3,\nabla\gamma_o(\psi)\big)\,\mathrm{d}\hat{x} \textit{ if } \hat{p} = (3,1) \end{cases}$$

$$\overline{\mathcal{L}}_p(r) := \begin{cases} \int_\Omega f\cdot v\,\mathrm{d}x + \int_{\Gamma_{mN}} F\cdot v\,\mathrm{d}\mathcal{H}_2 + \int_{\Gamma_{eN}} d_{\mathrm{E}}\,\psi\,\mathrm{d}\mathcal{H}_2 + (\bar{\mu}^1_{ee})^{1/2}\int_{\gamma_{\mathrm{N}}}\left(\int_{-1}^1 d_{\mathrm{I}}(\cdot,x_3)\,\mathrm{d}x_3\right)\psi\,\mathrm{d}l & \text{if } p_2=1\\ \int_\Omega f\cdot v\,\mathrm{d}x + \int_{\Gamma_{mN}} F\cdot v\,\mathrm{d}\mathcal{H}_2 + \int_{\Gamma_{eN}} d_{\mathrm{E}}\,\psi\,\mathrm{d}\mathcal{H}_2 & \text{if } p_2\geq 2\end{cases}$$

$$V_1 := \left\{ v\in H^1_{\Gamma_{mD}}(\Omega;\mathbb{R}^3);\ \hat{v}\in H^1(S;\mathbb{R}^2)\right\},\quad V_2 := H^1_{\Gamma_{mD}}(\Omega;\mathbb{R}^3),\quad V_3 = V_4 := H^1_{\Gamma_{mD}}(\Omega\setminus S;\mathbb{R}^3)$$

$$\Psi_{11} := \left\{ \psi\in H^1_{\Gamma_{eD}}(\Omega);\ \gamma_o(\psi)\in H^1_{\gamma_{\mathrm{D}}}(S)\right\},\quad \Psi_{21} := H^1_{\Gamma_{eD}}(\Omega),\quad \Psi_{31}=\Psi_{34} := H^1_{\Gamma_{eD}}(\Omega\setminus S)$$

- *If $p_3>1$, when h goes to $\bar{h}$, $T_1^\varepsilon u_p^h$ strongly converges in $H^1(\Omega\setminus S;\mathbb{R}^3)$ toward $\bar{u}_p$ while Φ_p^h converges, strongly in $H^1(S)$ if $p_2\leq 3$ and strongly in $L^2(S)$ if $p_3=4$ and $\lim_{h\to\bar{h}}\varepsilon^3/\mu_{ee}=0$, toward $\overline{\Phi}_p$ the unique solution to*

$$\left(\overline{\mathcal{P}}_p\right):\quad \begin{cases}\text{Find } s=(u,\phi) \text{ in } \big(0,q_2\,\gamma_o(\varphi_o)\big)+V_{p_1}\times\Psi_{p_2p_3} \text{ such that}\\ \overline{\mathcal{M}}_p\big(s,r\big)=\overline{\mathcal{L}}_p(r)\quad \forall r=(v,\psi)\in V_{p_1}\times\Psi_{p_2p_3}\end{cases}$$

where

$$\overline{\mathcal{M}}_p\big(s,r\big) := \int_\Omega a\,e(u)\cdot e(v)\,\mathrm{d}x + \overline{\mathcal{M}}_{\mathrm{I}p}(s,r)$$

- $p_3=2$

$$\overline{\mathcal{M}}_{\mathrm{I}p}(s,r) := \begin{cases} 0 \text{ if } \hat{p}=(2,4) \text{ or } (4,2)\\ \int_S \bar{\mu}^2_{mm}a_{\mathrm{I}}[\![u]\!]\otimes_S e_3\cdot[\![v]\!]\otimes_S e_3\,\mathrm{d}\hat{x} \text{ if } \hat{p}=(3,2)\\ \int_S \bar{\mu}^2_{ee}c_{\mathrm{I}}\nabla\phi\cdot\nabla\psi\,\mathrm{d}\hat{x} \text{ if } \hat{p}=(2,3)\\ \int_S \bar{\mu}^1_{mm}\tilde{a}_{\mathrm{I}}e(\hat{u})\cdot e(\hat{v})\,\mathrm{d}\hat{x} \text{ if } \hat{p}=(1,2) \text{ or } (1,4)\\ \int_S \bar{\mu}^1_{ee}\tilde{c}_{\mathrm{I}}\nabla\phi\cdot\nabla\psi\,\mathrm{d}\hat{x} \text{ if } \hat{p}=(2,1) \text{ or } (4,1)\\ \int_S \widetilde{M_{\mathrm{I}p}}(e(\hat{u}),\nabla\phi)\cdot(e(\hat{v}),\nabla\psi)\,\mathrm{d}\hat{x} \text{ if } \hat{p}=(1,3)\\ \int_S \widetilde{M_{\mathrm{I}p}}\big([\![u]\!]\otimes_S e_3,\nabla\phi\big)\cdot\big([\![v]\!]\otimes_S e_3,\nabla\psi\big)\,\mathrm{d}\hat{x} \text{ if } \hat{p}=(3,1)\end{cases}$$

- $p_3=3$

$$\overline{\mathcal{M}}_{\mathrm{I}p}(s,r) := \begin{cases} 0 & \text{if } \hat{p}\in\{2,4\}\times\{1,3,4\}\\ \int_S \widetilde{M_{\mathrm{I}p}}([\![u]\!]\otimes_S e_3,\nabla\gamma_o(\varphi_o),\triangle\varphi_{o\mathrm{I}})\cdot([\![v]\!]\otimes_S e_3,0)\,\mathrm{d}\hat{x} & \text{if } \hat{p}=(3,1)\\ \int_S \bar{\mu}^2_{mm}a_{\mathrm{I}}[\![u]\!]\otimes_S e_3\cdot[\![v]\!]\otimes_S e_3\,\mathrm{d}\hat{x} & \text{if } \hat{p}=(3,2) \text{ or } (3,4)\\ \int_S \bar{\mu}^1_{mm}\tilde{a}_{\mathrm{I}}e(\hat{u})\cdot e(\hat{v})\,\mathrm{d}\hat{x} & \text{if } \hat{p}=(1,2) \text{ or } (1,4)\\ \int_S \widetilde{M_{\mathrm{I}p}}e(\hat{u})\cdot e(\hat{v})\,\mathrm{d}\hat{x} & \text{if } \hat{p}=(1,3)\end{cases}$$

- $p_3=4$

$$\overline{\mathcal{M}}_{\mathrm{I}p}(s,r) := \begin{cases} 0 & \text{if } \hat{p}\in\{2,4\}\times\{1,3,4\}\\ \int_S \widetilde{M_{\mathrm{I}p}}([\![u]\!]\otimes_S e_3,\nabla\gamma_o(\varphi_o),\triangle\varphi_{o\mathrm{I}})\cdot([\![v]\!]\otimes_S e_3,0)\,\mathrm{d}\hat{x} & \text{if } \hat{p}=(3,1)\\ \int_S \bar{\mu}^2_{mm}a_{\mathrm{I}}[\![u]\!]\otimes_S e_3\cdot[\![v]\!]\otimes_S e_3\,\mathrm{d}\hat{x} & \text{if } \hat{p}=(3,2) \text{ or } (3,4)\\ \int_S \bar{\mu}^1_{mm}\tilde{a}_{\mathrm{I}}e(\hat{u})\cdot e(\hat{v})\,\mathrm{d}\hat{x} & \text{if } \hat{p}\in\{1\}\times\{2,3,4\}\end{cases}$$

$$\overline{\mathcal{L}}_p(r) := \begin{cases} \int_\Omega f\cdot v\,\mathrm{d}x + \int_{\Gamma_{mN}} F\cdot v\,\mathrm{d}\mathcal{H}_2 + (\bar{\mu}^1_{ee})^{1/2}\int_{\gamma_{\mathrm{N}}}\left(\int_{-1}^1 d_{\mathrm{I}}(\cdot,x_3)\,\mathrm{d}x_3\right)\psi\,\mathrm{d}l & \text{if } p_2=1\\ \int_\Omega f\cdot v\,\mathrm{d}x + \int_{\Gamma_{mN}} F\cdot v\,\mathrm{d}\mathcal{H}_2 & \text{if } p_2\geq 2\end{cases}$$

$\Psi_{(p_2,2)} := H^1_{\gamma_{\mathrm{D}}}(S)$ *or* $H^1_m(S)$ *according to the positivity of the length of* γ_{D}

$\Psi_{(p_2,3)}=\Psi_{(p_2,4)} := \{0\},\ p_2\neq 2$

$\Psi_{(2,p_3)} := \{0\},\ 2\leq p_3\leq 4$

3. Concluding remarks

For piezoelectric adhesive *and* adherents, when the elastic and dielectric coefficients of the adhesive are not of the same order, the piezoelectric coupling remains in the asymptotic model only when $\hat{p} = (1, 3)$ or $(3, 1)$. More generally, when (necessarily only) one index p_1 or p_2 is equal to 1, the status of the limit model for the adhesive is *hybrid*. When $p_1 = 1$, the adhesive is replaced by both a material surface perfectly bonded to the adherents, from the mechanical point of view, and a constraint, from the electrical point view. On the contrary, when $p_2 = 1$, a mechanical constraint appears with an electrical material surface perfectly permeable. The mechanical material surface is an elastic membrane with a possible nonvanishing (only when $\hat{p} = (1, 3)$) residual stress *stemming from the possible discontinuity of the electrical potential induced by the limit electrical constraint*, which is perfect permeability, electric pull-back or impermeability, according to the magnitude of the dielectric coefficients. The electrical material surface is of linear conductor type with a possible nonvanishing (only when $\hat{p} = (3, 1)$) residual term *stemming from the possible nonvanishing relative displacement induced by the mechanical constraint*, which is perfect adhesion, elastic pull-back or free separation according to the magnitude of the stiffness of the adhesive. When both p_1 and p_2 are greater than 1, the adhesive is replaced by an electromechanical constraint. As the orders of magnitude of the elastic and dielectric coefficients differ, this electromechanical constraint reduces to two *independent* mechanical and electrical constraints of the types previously evocated according to the values of p_1 and p_2, respectively.

For a thin piezoelectric layer embedded between two purely elastic adherents through two electrically impermeable interfaces, the piezoelectric coupling remains in the asymptotic model only when $\hat{p} = (1, 3)$ or $(3, 1)$. When $\hat{p} = (1, 3)$, the adhesive layer is replaced by a piezoelectric material surface; when $\hat{p} = (3, 1)$, it is replaced by a material conductive surface and a mechanical constraint. This constraint is of elastic pull-back type with a residual term *stemming from the electrical potential in the conductive surface*. Actually, when $p_1 = 1$, the adhesive layer is replaced by a material elastic surface perfectly bonded to the adherents. When $p_2 = 3$, the material surface has a non-local elastic behavior since the electrical potential can be eliminated; in the other cases, the material's surface is a standard elastic membrane. When p_1 ranges from 2 to 4, the adhesive layer is replaced by a mechanical constraint, which is perfect adhesion, elastic pull-back or free separation. The elastic pull-back is nonlocal when $p_2 = 1$. When $p_2 = 2$, the electric potential vanishes, in the remaining cases the limit surface is a linear elastic conductor.

The limit models for a thin piezoelectric layer embedded between two elastic adherents, through either two electroded interfaces or one electroded and the other being impermeable, only differ when $\hat{p} = (1, 3)$. In all cases, there is a perfect decoupling between Electricity and Mechanics. When the magnitude of the stiffness is of the order of the inverse of the thickness, the adhesive is replaced by an elastic material membrane perfectly bonded to the adherents; when it is lesser, the adhesive is replaced by a mechanical constraint, which is perfect adhesion, elastic pull-back, free separation according to the magnitude of the stiffness. The limit surface is at a given potential φ_0 when $\hat{p} \in \{3, 4\} \times \{1\}$, at a vanishing one in the other cases. Actually, when $p = (1, 3, 3)$, the memory of Electricity remains because piezoelectric and dielectric coefficients enter the constitutive equations of the elastic membrane the adhesive layer reduces to.

Eventually, the previous method may work when the elastic and dielectric coefficients of the junction are of the same order of magnitude with piezoelectric coefficients of lesser order. Obviously the conclusions of [1] remain but with b_I replaced by 0, so that piezoelectric coupling disappears in the asymptotic models.

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