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A fifth-order approximation to gravity-capillary interfacial waves of infinite depth

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A B S T R A C T

Two-dimensional periodic gravity-capillary waves at the interface between two unbounded fluids with different density are analyzed. The lighter fluid is above the interface. The perturbation method is used to obtain solutions to the fifth order for interface profile, velocity potential and oscillation frequency. The solutions have been carefully controlled by other solutions (third-order surface gravity-capillary solutions, third-order interface gravity waves and fifth-order surface gravity waves). These solutions can be used to describe the qualitative nature of small-amplitude traveling waves and provide initial guesses for numerical solutions to the full Euler system. The results highlight the significant influence on wave profile and wave frequency. In addition, this study extends the Wilton singularity to interfacial waves.

1. Introduction

A great number of studies has been conducted on two-dimensional progressive gravity-capillary surface waves. The interested reader is referred to the excellent literature review article of Dias and Kharif (Dias & Kharif [1]). A similar study can also be carried out for two-dimensional progressive gravity-capillary waves, propagating at the interface between two fluids of different densities. Hunt [2] applied the Levi-Civita's method for progressive and standing interface waves and obtained formulae of the wave profile and phase velocity up to the third order. He showed that the existence of upper layer reduces the propagation velocity and the amplitude of the higher harmonics in the wave profile. Unfortunately, this paper is incorrect, since the boundary conditions are incorrectly applied. Tsuji & Nagata [3] used a perturbation expansion in wave amplitude to the fifth order, for interfacial gravity waves of infinite depths. Their results suggested that the maximum value of the wave steepness may be limited by shear instability at the interface rather than the breaking condition at the interface. Holyer [4], on the other hand, calculated the maximum steepness for interfacial waves numerically up to 31st order. He showed that surface waves do not break at the crest if we consider the density of the air. Increasing further the

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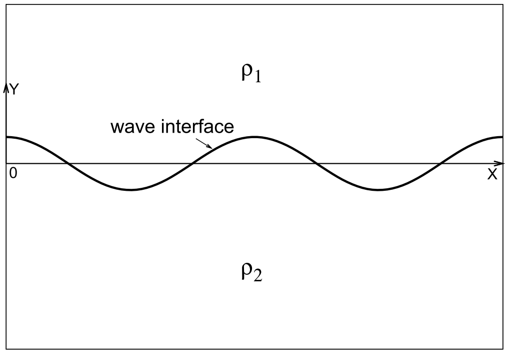

Fig. 1. The configuration of the problem.

density ratio yields an increase in both the maximum amplitude allowed by the waves and the breaking location. Boussinesq waves correspond to the limiting case of equal densities, at which the breaking location occurs at $1/4$ wavelength. Saffman & Yuen [5] considered finite-amplitude interfacial gravity waves between two unbounded fluids in relative motion and identified two different factors that limit the existence of steady gravity wave solutions. Părău & Dias [6] computed periodic waves of permanent form that propagate into a fluid system with free surface boundary conditions using Fourier series expansions. Hill [7] studied analytically the weakly nonlinear cubic interactions between surface waves and interfacial waves. He obtained a set of third-order equations describing the interactions between surface waves and interfacial waves. His study revealed the importance of cubic interactions. Yuan, Li & Cheng [8] established a diagram that demarcates the validity ranges for interfacial wave theories in a two-layer system. The proposed diagram is an extension of Le Méhauté's plot for free surface waves. Liu & Hwung [9] extended the work of Saffman & Yuen [5] to the case of two finite depths. The solutions are derived using the perturbation method. Recently, Grigor'ev, Shiryaeva & Sukhanov [10] investigated the Kelvin–Helmholtz instability of gravity-capillary waves. The lower fluid is assumed inviscid while the upper fluid is a dielectric with translational motion parallel to the interface. They showed that when the density ratio is greater than one, the resonances are absent. Liu, Hwung & Yang [11] considered the case of a two-fluid system with free surface. Second-order solutions were obtained, using the perturbation method. The aim of this study is to derive an analytical solution for interfacial progressive waves of two unbounded fluids. The fifth-order perturbation solution is presented. Moreover, the kinematic properties such as wave profile and frequency are investigated by considering the effects of capillary number and density ratio.

2. Problem formulation

Periodic gravity-capillary waves at the interface between two unbounded fluids are considered. The wave is assumed to move from left to right without change of form along an interface under the influence of gravity and surface tension. The fluids are supposed to be incompressible and inviscid, and the motion is assumed to be irrotational. The properties of the upper fluid are denoted by (1), and those of the lower fluid by (2). We present here the properties of finite-amplitude periodic waves with wavelength λ, which propagate steadily without change of shape with speed C. It is convenient to change the framework in order to reduce the wave propagation to rest by moving with the wave. The flow is then independent of time, and is sketched in Fig. 1. The two fluids are assumed to be stable and stratified, so $\rho_1 < \rho_2$. Rectangular coordinates (x, y) are chosen such that the x-axis is horizontal and the y-axis is directed vertically upwards.

Since both fluids are incompressible and the motion in each fluid is irrotational, we can define the velocity potentials that satisfy Laplace's equation

$$\triangle\phi_1 = 0, \qquad \triangle\phi_2 = 0 \tag{1}$$

subject to the following boundary conditions

$$\phi_1 = 0 \quad \text{for} \quad y \longrightarrow \infty \tag{2}$$

$$\phi_2 = 0 \quad \text{for} \quad y \longrightarrow -\infty \tag{3}$$

$$\eta_t + \phi_{ix}\eta_x - \phi_{iy} = 0 \quad \text{at} \quad y = \eta \quad i = (1, 2) \tag{4}$$

$$\rho_2\left[\phi_{2t} + \frac{1}{2}\left(\phi_{2x}^2 + \phi_{2y}^2\right)\right] - \rho_1\left[\phi_{1t} + \frac{1}{2}\left(\phi_{1x}^2 + \phi_{1y}^2\right)\right] + g\left(\rho_2 - \rho_1\right)\eta - \sigma\frac{\eta_{xx}}{\left(1+\eta_x^2\right)^{3/2}} = 0 \quad \text{at} \quad y = \eta \tag{5}$$

where g denotes the acceleration due to gravity and σ the surface tension coefficient.

For waves of permanent form, we have

$$\eta_t = -C\eta_x \quad \text{and} \quad \phi_{it} = -C\phi_{ix} \quad i = 1, 2 \tag{6}$$

Substituting (6) into (4) and (5), it follows that

$$-C\eta_x+\phi_{ix}\eta_x-\phi_{iy}=0 \quad \text{at} \quad y=\eta \quad i=(1,2) \tag{7}$$

$$\rho_2\left[-C\phi_{2x}+\frac{1}{2}\left(\phi_{2x}^2+\phi_{2y}^2\right)\right]-\rho_1\left[-C\phi_{1x}+\frac{1}{2}\left(\phi_{1x}^2+\phi_{1y}^2\right)\right]+g\left(\rho_2-\rho_1\right)\eta-\sigma\frac{\eta_{xx}}{\left(1+\eta_x^2\right)^{3/2}}=0 \tag{8}$$

It is convenient to employ the following dimensionless variables

$$\tilde{x}=kx,\ \tilde{y}=ky,\ \tilde{\eta}=k\eta,\ \tilde{\phi_i}=k/\left(gk\right)^{1/2}\phi_i,\ \tilde{\omega}=\left(Ck\right)/\left(gk\right)^{1/2} \tag{9}$$

Therefore, the Eqs. (1), (2), (3), (7) and (8) take the following dimensionless form in which the tilde will be omitted for the sake of simplicity

$$\triangle\phi_1=0, \qquad \triangle\phi_2=0 \tag{10}$$

$$-\omega\eta_x+\phi_{ix}\eta_x-\phi_{iy}=0 \quad \text{at} \quad y=\eta \quad i=(1,2) \tag{11}$$

$$\left[-\omega\phi_{2x}+\frac{1}{2}\left(\phi_{2x}^2+\phi_{2y}^2\right)\right]-\mu\left[-\omega\phi_{1x}+\frac{1}{2}\left(\phi_{1x}^2+\phi_{1y}^2\right)\right]+\left(1-\mu\right)\eta-\kappa\frac{\eta_{xx}}{\left(1+\eta_x^2\right)^{3/2}}=0 \quad \text{at } y=\eta \tag{12}$$

$$\phi_1=0 \quad \text{for} \quad y\longrightarrow\infty \tag{13}$$

$$\phi_2=0 \quad \text{for} \quad y\longrightarrow-\infty \tag{14}$$

where $\mu=\rho_1/\rho_2$ is a density ratio and $\kappa=k^2\sigma/(\rho_2 g)$ is the dimensionless capillary number. This parameter represents the reverse of the Bond number Bo, and measures the ratio between capillarity and gravity effects.

Eqs. (10)–(14) are solved for the dimensionless physical quantities ϕ_1, ϕ_2, η, and ω. The perturbation method is used for this purpose and it is presented in the next section.

3. Perturbation method

In the perturbation approach, we assume that the solution depends on the presumed small quantity ε. Therefore, the quantities are decomposed into a power series in ε:

$$\begin{cases} \phi_i(x,y)=\sum_{r=1}^{\infty}\phi_i^{(r)}(x,y)\varepsilon^r \\ \eta(x)=\sum_{r=1}^{\infty}\eta^{(r)}(x)\varepsilon^r \\ \omega=\sum_{r=0}^{\infty}\omega_r\varepsilon^r \end{cases} \tag{15}$$

We choose the expansion parameter, $\varepsilon=h$, which represents the half wave height. This choice is similar to those made by Holyer [4] for progressive interracial gravity waves, and Rottman [12] for standing interfacial gravity waves. So the semi-wave-height h is given by

$$h=\frac{1}{2}\left(\eta\left(0\right)-\eta\left(\pi\right)\right) \tag{16}$$

The dimensionless velocity potentials ϕ_1 and ϕ_2 at the interface may be expressed in terms of the Taylor expansion at $y=0$ instead of $y=\eta$

$$\phi_i\left(x,y=\eta\right)=\phi_i\left(x,0\right)+\left.\frac{\partial\phi_i}{\partial y}\right|_{y=0}\eta+\frac{1}{2}\left.\frac{\partial^2\phi_i}{\partial y^2}\right|_{y=0}\eta^2+\cdots \quad \text{for } i=1,2 \tag{17}$$

By substituting the expressions of ϕ_1, ϕ_2 and η given by (15) into (17), we obtain:

$$\phi_i\left(x,y=\eta\right)=\sum_{r=1}^{\infty}\phi_i^{(r)}(x,0)\varepsilon^r+\sum_{r=1}^{\infty}\left.\frac{\partial\phi_i^{(r)}\varepsilon^r}{\partial y}\right|_{y=0}\sum_{r=1}^{\infty}\eta^{(r)}+\frac{1}{2}\sum_{r=1}^{\infty}\left.\frac{\partial^2\phi_i^{(r)}\varepsilon^r}{\partial y^2}\right|_{y=0}\left(\sum_{r=1}^{\infty}\eta^{(r)}\right)^2+\cdots \text{ for } i=1,2 \tag{18}$$

The nonlinear term $1/\left(1+\eta_x^2\right)^{3/2}$ may be expanded as a series

$$\frac{1}{\left(1+\eta_x^2\right)^{3/2}}=1-\frac{3}{2}\eta_x^2+\frac{15}{8}\left(\eta_x^2\right)^2-\frac{35}{15}\left(\eta_x^2\right)^3+\cdots \tag{19}$$

Upon substitution the expressions (15), (18) and (19) in the system of Eqs. (10), (11), (12), (13) and (14), and equating powers of ε, we obtain a set of equations at each order of ε

$$\phi_{1xx}^{(r)} + \phi_{1yy}^{(r)} = 0 \qquad \text{for} \qquad y \geq \eta \tag{20}$$

$$\phi_{2xx}^{(r)} + \phi_{2yy}^{(r)} = 0 \qquad \text{for} \qquad y \leq \eta \tag{21}$$

$$\phi_{1y}^{(r)} + \omega_0 \eta_x^{(r)} = II_1^{(r-1)} \qquad \text{at} \qquad y = 0 \tag{22}$$

$$\phi_{2y}^{(r)} + \omega_0 \eta_x^{(r)} = II_2^{(r-1)} \qquad \text{at} \qquad y = 0 \tag{23}$$

$$-\kappa \eta_{xx}^{(r)} + (1-\mu)\,\eta^{(r)} - \omega_0 \phi_{2x}^{(r)} + \mu \omega_0 \phi_{1x}^{(r)} = I^{(r-1)} \qquad \text{at} \qquad y = 0 \tag{24}$$

where $r = 1, 2, \cdots$ is the order. The first three terms of $I^{(r-1)}, II_1^{(r-1)}$ and $II_2^{(r-1)}$ on the right-hand side are

$$I^{(0)} = II_1^{(0)} = II_2^{(0)} = 0 \tag{25}$$

$$I^{(1)} = \eta^{(1)} \omega_0 \left(\phi_{2xy}^{(1)} - \mu \phi_{1xy}^{(1)}\right) + \omega_1 \left(\phi_{2x}^{(1)} - \mu \phi_{1x}^{(1)}\right) + \frac{1}{2}\mu \left(\left(\phi_{1x}^{(1)}\right)^2 + \left(\phi_{1y}^{(1)}\right)^2\right) - \frac{1}{2}\left(\left(\phi_{2x}^{(1)}\right)^2 + \left(\phi_{2y}^{(1)}\right)^2\right) \tag{26}$$

$$II_1^{(1)} = -\omega_1 \eta_x^{(1)} + \phi_{1x}^{(1)} \eta_x^{(1)} - \phi_{1yy}^{(1)} \eta^{(1)} \tag{27}$$

$$II_2^{(1)} = -\omega_1 \eta_x^{(1)} + \phi_{2x}^{(1)} \eta_x^{(1)} - \phi_{2yy}^{(1)} \eta^{(1)} \tag{28}$$

$$\begin{aligned} I^{(2)} = & -\omega_1 \left(\mu \phi_{1x}^{(2)} - \phi_{2x}^{(2)}\right) - \phi_{2x}^{(1)} \phi_{2x}^{(2)} + \mu \phi_{1x}^{(1)} \phi_{1x}^{(2)} - \phi_{2y}^{(1)} \phi_{2y}^{(2)} + \mu \phi_{1y}^{(1)} \phi_{1y}^{(2)} - \frac{3}{2}\kappa \eta_{xx}^{(1)} \left(\eta_x^{(1)}\right)^2 \\ & - \left(-\phi_{2x}^{(1)} + \mu \phi_{1x}^{(1)}\right) \omega_2 - \frac{1}{2}\left(-\phi_{2xyy}^{(1)} + \mu \phi_{1xyy}^{(1)}\right) \omega_0 \left(\eta^{(1)}\right)^2 - \left(-\phi_{2xy}^{(1)} + \mu \phi_{1xy}^{(1)}\right) \omega_0 \eta^{(2)} \\ & - \Big[\omega_1 \left(\mu \phi_{1xy}^{(1)} - \phi_{2xy}^{(1)}\right) + \left(\phi_{2x}^{(1)} \phi_{2xy}^{(1)} - \mu \phi_{1x}^{(1)} \phi_{1xy}^{(1)}\right) \\ & \left(\phi_{2y}^{(1)} \phi_{2yy}^{(1)} - \mu \phi_{1y}^{(1)} \phi_{1xy}^{(1)}\right) + \omega_0 \left(\mu \phi_{1xy}^{(2)} - \phi_{2xy}^{(2)}\right)\Big] \eta^{(1)} \end{aligned} \tag{29}$$

$$\begin{aligned} II_1^{(2)} = & -\frac{1}{2}\phi_{1zzz}^{(1)} \left(\eta^{(1)}\right)^2 + \left(\phi_{1x}^{(2)} - \omega_2\right) \eta_x^{(1)} - \phi_{1zz}^{(1)} \eta^{(2)} - \phi_{1zz}^{(2)} \eta^{(1)} + \left(\phi_{1x}^{(1)} - \omega_1\right) \eta_x^{(2)} \\ & + \phi_{1xz}^{(1)} \eta^{(1)} \eta_x^{(1)} \end{aligned} \tag{30}$$

$$\begin{aligned} II_2^{(2)} = & -\frac{1}{2}\phi_{2zzz}^{(1)} \left(\eta^{(1)}\right)^2 + \left(\phi_{2x}^{(2)} - \omega_2\right) \eta_x^{(1)} - \phi_{2zz}^{(1)} \eta^{(2)} - \phi_{2zz}^{(2)} \eta^{(1)} + \left(\phi_{2x}^{(1)} - \omega_1\right) \eta_x^{(2)} \\ & + \phi_{2xz}^{(1)} \eta^{(1)} \eta_x^{(1)} \end{aligned} \tag{31}$$

The general solutions for dimensionless interface elevation and dimensionless velocity potentials at each order ε^r are as following:

$$\begin{cases} \eta^{(r)}(x) = \sum_{m=1}^{r} a_m^{(r)} \cos(mx) \\ \phi_1^{(r)}(x, y) = \sum_{m=1}^{r} b_m^{(r)} \sin(mx)\, \mathrm{e}^{-my} \\ \phi_2^{(r)}(x, y) = \sum_{m=1}^{r} c_m^{(r)} \sin(mx)\, \mathrm{e}^{my} \end{cases} \tag{32}$$

The kinematic and dynamic boundary conditions on the right-hand side are trigonometric products resulting from previous orders, so it may be expressed as follows:

$$\begin{cases} I^{(r-1)} = \sum_{m=1}^{r-1} \chi_m^{(r-1)} \cos(mx) \\ II_1^{(r-1)} = \sum_{m=1}^{r-1} \alpha_m^{(r-1)} \sin(mx) \\ II_2^{(r-1)} = \sum_{m=1}^{r-1} \beta_m^{(r-1)} \sin(mx) \end{cases} \tag{33}$$

Substitution of (32) and (33) into (22)–(24) gives

$$a_m^{(r)} \left(\kappa m^2 - 1 + \mu\right) + m\,\mu\,\omega_0\, b_m^{(r)} - \omega_0\, m\, c_m^{(r)} = \chi_m^{(r-1)} \tag{34}$$

$$-m\omega_0 a_m^{(r)} - m\, b_m^{(r)} = \alpha_m^{(r-1)} \tag{35}$$

$$-m\omega_0 a_m^{(r)} + m\, c_m^{(r)} = \beta_m^{(r-1)} \tag{36}$$

And the solutions are

$$a_m^{(r)} = \frac{\omega_0 \left(\beta_m^{(r-1)} + \mu \alpha_m^{(r-1)}\right) + \chi_m^{(r-1)}}{(m-1)(m\kappa - 1 + \mu)} \tag{37}$$

$$b_m^{(r)} = -\frac{\alpha_m^{(r-1)} + m\omega_0 a_m^{(r)}}{m} \tag{38}$$

$$c_m^{(r)} = \frac{\beta_m^{(r-1)} + m\omega_0 a_m^{(r)}}{m} \tag{39}$$

For $m = 1$, a secular term occurs in (37), which is then replaced by

$$\omega_0 \left(\beta_1^{(r-1)} + \mu \alpha_1^{(r-1)}\right) + \chi_1^{(r-1)} = 0 \tag{40}$$

The above equation allows us to derive the solution for the wave frequency ω_{r-1} at each order ε^r. The coefficients $a_1^{(r)}$ for an odd r value, are obtained by using the Eq. (16). The values of $b_1^{(r)}$ and $c_1^{(r)}$ for an odd r value are then determined by (38) and (39).

The calculation procedure can be summarized in the following steps

(i) Specify the order, r, of the perturbation expansion.
(ii) Calculate the right-hand side $I^{(r-1)}, II_1^{(r-1)}$ and $II_2^{(r-1)}$ (Eqs. (22), (23) and (24)).
(iii) Seek these relations in the forms given by Eq. (33).
(iv) Calculate, for any value of $m \neq 1$, the coefficients $a_m^{(r)}$, $b_m^{(r)}$ and $c_m^{(r)}$ using Eqs. (37), (38) and (39).
(v) Calculate ω_{r-1} using Eq. (40).
(vi) Finally, calculate, for $m = 1$, the coefficients $a_1^{(r)}$, $b_1^{(r)}$ and $c_1^{(r)}$ using Eqs. (16), (38) and (39).

The solutions for third order are given by

$$a_1^{(1)} = 1 \tag{41}$$

$$b_1^{(1)} = -\omega_0 \tag{42}$$

$$c_1^{(1)} = \omega_0 \tag{43}$$

$$\omega_0^2 = \frac{1 - \mu + \kappa}{1 + \mu} \tag{44}$$

$$a_2^{(2)} = -\frac{1}{2}\frac{(\mu - 1)(-1 + \mu - \kappa)}{(1 + \mu)(\mu - 1 + 2\kappa)} \tag{45}$$

$$b_2^{(2)} = -\frac{1}{2}\frac{\omega_0(-2 + 2\mu + 3\kappa\mu + \kappa)}{(1 + \mu)(\mu - 1 + 2\kappa)} \tag{46}$$

$$c_2^{(2)} = -\frac{1}{2}\frac{\omega_0\left(-2\mu + 2\mu^2 + \kappa\mu + 3\kappa\right)}{(1 + \mu)(\mu - 1 + 2\kappa)} \tag{47}$$

$$a_1^{(2)} = b_1^{(2)} = c_1^{(2)} = \omega_1 = 0 \tag{48}$$

Table 1
Coefficients $a_m^{(r)}$ and ω_r for the special case $\mu=0$ and $\kappa=0$ up to the fifth order.

$a_1^{(1)}=1$
$\omega_0^2=1$
$a_2^{(2)}=1/2$
$a_3^{(3)}=3/8$
$a_1^{(3)}=-3/8$
$\omega_2=1/2$
$a_2^{(4)}=1/3$
$a_4^{(4)}=1/3$
$a_3^{(5)}=99/128$
$a_5^{(5)}=125/384$
$a_5^{(5)}-211/192$
$\omega_4=1/8$

$$\omega_2=\frac{\left(12\,\mu-2\,\mu^2-2\right)\kappa^2+(\mu-1)\left(\mu^2-30\,\mu+1\right)\kappa-8\,\left(\mu^2+1\right)(\mu-1)^2}{16\,(1+\mu)^3\,(\mu-1+2\,\kappa)\,\omega_0} \tag{49}$$

$$a_3^{(3)}=\frac{2\,(3\,\mu-1)\,(\mu-3)\,\kappa^2-(\mu-1)\left(21\,\mu^2-22\,\mu+21\right)\kappa+2\,(3\,\mu-1)\,(\mu-3)\,(\mu-1)^2}{16\,(1+\mu)^2\,(\mu-1+2\,\kappa)\,(\mu-1+3\,\kappa)} \tag{50}$$

$$b_3^{(3)}=\frac{-2\,\omega_0\left(39\,\mu^2+26\,\mu+3\right)\kappa^2+\omega_0\,(\mu-1)\left(15\,\mu^2-82\,\mu-33\right)\kappa+8\,\omega_0\,(\mu-3)\,(\mu-1)^2}{16\,(1+\mu)^2\,(\mu-1+2\,\kappa)\,(\mu-1+3\,\kappa)} \tag{51}$$

$$c_3^{(3)}=\frac{2\,\omega_0\left(26\,\mu+3\,\mu^2+39\right)\kappa^2+\omega_0\,(\mu-1)\left(33\,\mu^2+82\,\mu-15\right)\kappa+8\,\omega_0\mu\,(3\,\mu-1)\,(\mu-1)^2}{16\,(1+\mu)^2\,(\mu-1+2\,\kappa)\,(\mu-1+3\,\kappa)} \tag{52}$$

$$a_1^{(3)}=-\frac{2\,(3\,\mu-1)\,(\mu-3)\,\kappa^2-(\mu-1)\left(21\,\mu^2-22\,\mu+21\right)\kappa+2\,(3\,\mu-1)\,(\mu-3)\,(\mu-1)^2}{16\,(1+\mu)^2\,(\mu-1+2\,\kappa)\,(\mu-1+3\,\kappa)} \tag{53}$$

$$\begin{aligned}b_1^{(3)}=\frac{1}{D_{13}}\Big[&-2\left(9\,\mu^2+20\,\mu-9\right)\kappa^3+(\mu-1)\left(17\,\mu^2+62\,\mu-43\right)\kappa^2+\left(23\,\mu^2+2\,\mu+35\right)(\mu-1)^2\,\kappa\\&-4\left(\mu^2-3\,\mu-2\right)(\mu-1)^3\Big]\end{aligned} \tag{54}$$

$$\begin{aligned}c_1^{(3)}=\frac{1}{D_{13}}\Big[&-2\left(9\,\mu^2-20\,\mu-9\right)\kappa^3+(\mu-1)\left(43\,\mu^2-62\,\mu-17\right)\kappa^2-\left(35\,\mu^2+2\,\mu+23\right)(\mu-1)^2\,\kappa\\&-4\left(2\,\mu^2+3\,\mu-1\right)(\mu-1)^3\Big]\end{aligned} \tag{55}$$

$$D_{13}=8\,(1+\mu)^3\,(\mu+2\,\kappa-1)\,(\mu-1+3\,\kappa)\,\omega_0 \tag{56}$$

The fourth- and fifth-order coefficients are given in Appendix A.

4. Results and discussion

4.1. Validation of present coefficients

4.1.1. Comparison with Stokes surface gravity wave theory

For $\mu=0$ and $\kappa=0$, the coefficients must coincide with a Stokes surface gravity wave theory. The comparison is made with Fenton's solution [13]. Our coefficients in this case are given in Table 1.

This solution coincides perfectly with Fenton's solution up to the fifth order.

4.1.2. Comparison with Stokes surface gravity-capillary waves

Two-dimensional progressive surface gravity-capillary waves correspond to a particular case $\mu=0$. In this case $a_m^{(r)}$ and ω_r up to the third order are:

$$a_2^{(2)}=\frac{1}{2}\,\frac{-1-\kappa}{-1+2\,\kappa} \tag{57}$$

Table 2
Comparison of the frequency ω of the present results with those of Debiane & Kharif [15] for $\kappa = 0.000075$ and $\mu = 0$.

h	Present method	Debiane & Kharif [15] method
0.0655	1.002 185 2	1.002 185 193 4
0.1309	1.008 642 7	1.008 642 968 4
0.1964	1.019 512 5	1.019 515 210 5
0.2619	1.034 926 2	1.034 934 921 0
0.3274	1.055 077 0	1.055 049 554 5

Table 3
Comparison of ω/ω_0 of the present results with those of Saffman & Yuen [5] for $\kappa = 0$ and $\mu = 0.1$.

h	Present method	Saffman & Yuen [5] method
0.05	1.001 043 3	1.001 043 3
0.10	1.004 172 6	1.004 172 5
0.15	1.009 385 6	1.009 385 1
0.20	1.016 678 7	1.016 676 3
0.25	1.026 046 8	1.026 038 1

$$a_3^{(3)} = -\frac{1}{8}\,\frac{6 + 6\kappa^2 + 21\kappa}{(-1 + 2\kappa)(2 - 6\kappa)} \tag{58}$$

$$\omega_2 = -\frac{1}{16}\,\frac{8 + \kappa + 2\kappa^2}{-1 + 2\kappa} \tag{59}$$

The present solution for gravity-capillary waves coincides with solution of Nayfeh [14].

We compare also in Table 2 the frequency ω, calculated up to the fifth order, with the numerical results of Debiane & Kharif [15]. For small value of the wave steepness h, the accuracy reaches six digits. When h is large, the agreement reaches only four digits.

4.1.3. Comparison with interfacial gravity waves

When $\kappa = 0$, the problem reduces to progressive interfacial gravity waves. The expressions of $a_m^{(r)}$ and ω_r up to the third order are

$$a_2^{(2)} = -\frac{1}{2}\,\frac{\mu - 1}{1 + \mu} \tag{60}$$

$$a_3^{(3)} = \frac{1}{8}\,\frac{3\mu^2 - 10\mu + 3}{(1 + \mu)^2} \tag{61}$$

$$\omega_2 = \frac{\omega_0}{2}\,\frac{1 + \mu^2}{(1 + \mu)^2} \tag{62}$$

which is identical to those of Thorpe [16] and Tsuji & Nagata [3].

In Table 3, the present results of ω/ω_0 is compared with those obtained by Saffman & Yuen [5] for $\mu = 0.1$. We note a good agreement to six decimal places between the two results for low values of h.

4.2. Some properties of interfacial gravity-capillary waves

4.2.1. Influence of the order

A comparison of profiles of wave theory for different orders, and over two wavelengths, is depicted in Fig. 2. The profile of the linear theory is characterized by a sinusoidal form. The fifth order modified the structure of the wave profile with flatten crest and narrow trough. This property is more pronounced when wave steepness is large.

4.2.2. Influence of capillary number

Fig. 3 shows two wave profiles calculated for $\kappa = 0$ and $\kappa = 0.7$ with $\mu = 0.1$ and $h = 0.2$. Fig. 3a, which corresponds to interface gravity waves, is marked by a narrow crest and a broader trough. The case of interfacial gravity-capillary interfacial waves is depicted in Fig. 3b. This figure shows the impact of the capillary coefficient κ on the structure of wave profile. In this case, the crest is more flattened than the trough.

4.2.3. Influence of the density ratio

Fig. 4 illustrates the influence of the density ratio μ for fixed values of κ et h. Fig. 4a is a characteristic of gravity-capillary waves with round crests and narrow trough. When the density ratio increases, the asymmetry is observed to be less important (Fig. 4(b)). If μ further increases, the wave interface seems to be symmetrical (Fig. 4c).

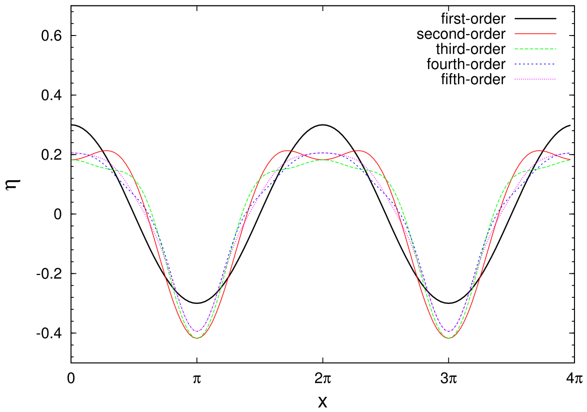

Fig. 2. Wave profiles of gravity-capillary waves as a function of the wave theory with $\mu = 0.1$, $\kappa = 0.7$, $h = 0.3$.

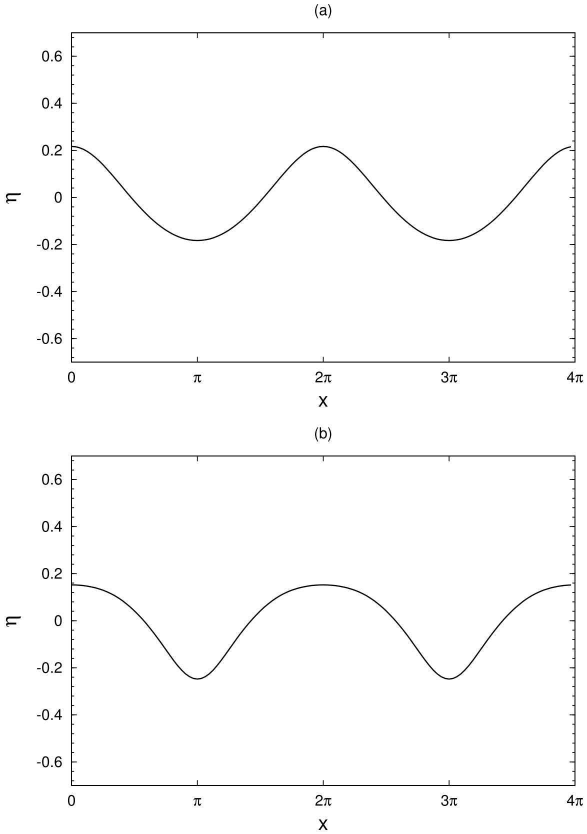

Fig. 3. Wave profiles for various values of κ at fixed values of $\mu = 0.1$ and $h = 0.2$. (a) $\kappa = 0$. (b) $\kappa = 0.7$.

4.2.4. Evolution of frequency-harmonic resonance

We consider the evolution of the frequency ω with the amplitude h up to the third order. This can be found by substituting the relations (44) and (49) into the following equation:

$$\omega = \omega_0 + h^2 \omega_2 \tag{63}$$

The difference between the frequency ω and the linear frequency ω_0 is due to the term $h^2\omega_2$, which can be positive or negative depending on the values of κ and μ. The regions of positive and negative ω_2 are shown in Fig. 5. To the right of the curve named (A) and to the left of the curve named (B), ω_2 is positive, and between these two curves it is negative. Curve (A) corresponds to a sign change in the numerator of (49), so that for all values of κ and μ fitting on this curve, ω_2 is zero. In this case, the dispersion relation becomes amplitude independent ($\omega_2 = 0$).

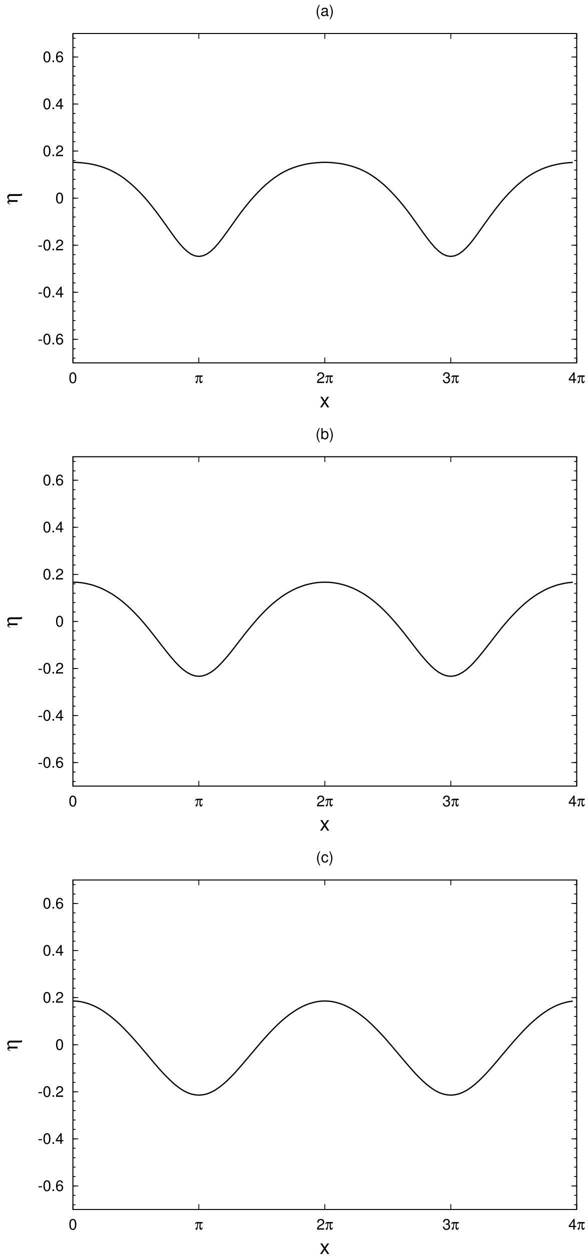

Fig. 4. Wave profiles for various values of μ at fixed values of $\kappa = 0.7$ and $h = 0.3$. (a) $\mu = 0.1$, (b) $\mu = 0.2$, (c) $\mu = 0.4$.

Curve (B) corresponds to the sign change of the term $(\mu - 1 + 2\kappa)$ in the denominator of (49). For values of κ and μ lying on this curve, the denominator in the relation (49) vanished. This case represents a resonance condition for the second harmonic. In the vicinity of curve (B), the coefficients of second harmonic in the expressions of η, ϕ_1 and ϕ_2 become very large.

Curve (C) represents the sign change of the term $(\mu - 1 + 3\kappa)$ in the denominator of third-order harmonic coefficients (expressions (50), (51), (52) and (53)) and represents a resonance condition for the third-order harmonic.

If the solution is calculated to the mth order, a division by zero occurs in the expression of the coefficients (37), which may cause a zero radius of convergence. This phenomenon corresponds to the occurrence of harmonic resonance. The equation relating the parameters is given by

$$\mu + 1 - m\kappa = 0 \tag{64}$$

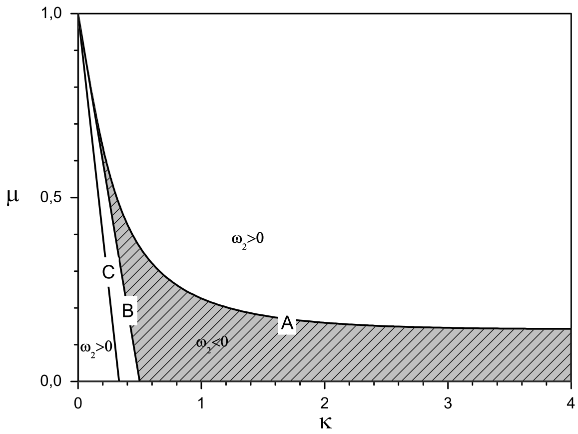

Fig. 5. The location of zeros and poles of ω_2 and the poles of $\eta^{(3)}, \phi_1^{(3)}$ and $\phi_2^{(3)}$. ω_2 is zero along (A). ω_2 has poles along (B). $\eta^{(3)}, \phi_1^{(3)}$ and $\phi_2^{(3)}$ have poles along (C).

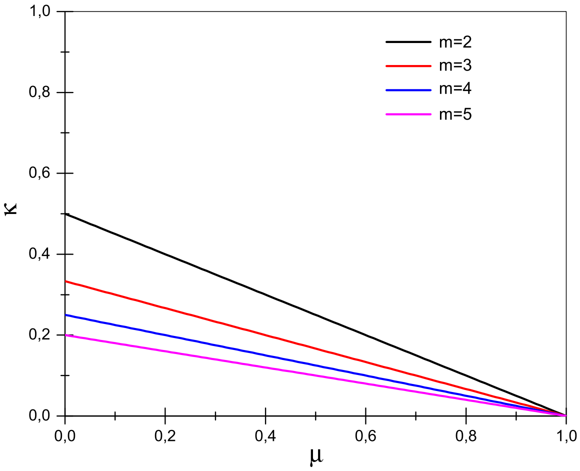

Fig. 6. Variation of κ with μ for various values of m.

This relation is a generalization of the Wilton singularity to interfacial waves, in which the amount of the corresponding harmonic present in η, ϕ_1 and ϕ_2 becomes infinite and the perturbation scheme breaks down. The latter singularities are the internal wave analogue of the Wiltons ripple due to the interplay of gravity and surface tension.

Some comments on Eq. (64) may be emphasized as following.

(i) When $\mu = 0$, this relation reduces to the well-known resonance condition of surface gravity-capillary waves given by

$$\kappa = \frac{1}{m} \tag{65}$$

(ii) The solution to Eq. (6) for various values of an integer m is presented in Fig. 6. Note that κ varies linearly with μ. Moreover, near these curves, the solution is not unique and the perturbation method used in the present work breaks down. In order to study the behavior of the solutions near the singularity, we must use a modified expansion such as the wave number perturbation technique (see Pierson and Fife [17]).

(iii) There exist regions where the solutions are unique. For a fixed density ratio μ, these region correspond to the following expression

$$\kappa > \frac{1+\mu}{m} \tag{66}$$

(iv) Notice that neither of these singularities occur if $\kappa = 0$ (pure gravity waves).

In this work, we consider values of parameters for which the denominators are non-singular. The computation of internal wave ripples is planned as an avenue of future study.

5. Conclusion

In this paper, a fifth-order solution to the problem of two-dimensional gravity-capillary progressive interfacial waves in infinite depths has been presented. The method of perturbation is used to calculate the properties of waves such as frequency and wave profile. The advantage of this technique is that it highlights the harmonic resonance. Furthermore, it is always of interest to develop analytical solutions to allow the validation of further highly nonlinear numerical solutions. The main results reveal two competing aspects:

(a) increasing the capillary coefficient tends to flatten the crest of the wave,
(b) increasing the density ratio makes the wave more symmetrical.

Appendix A. Fourth and fifth orders

$$a_2^{(4)} = \frac{1}{D_{24}}\Big[6\,(\mu-1)\left(\mu^2+34\,\mu+1\right)\kappa^4+\left(5\,\mu^2-302\,\mu+5\right)(\mu-1)^2\,\kappa^3 - 18\left(7\,\mu^2+10\,\mu+7\right)(\mu-1)^3\,\kappa^2-3\left(13\,\mu^2+18\,\mu+13\right)(\mu-1)^4\,\kappa - 8\left(\mu^2-\mu+1\right)(\mu-1)^5\Big] \tag{67}$$

$$b_2^{(4)} = \frac{1}{D'_{24}}\Big[\left(300-1404\,\mu^3-3372\,\mu^2+444\,\mu\right)\kappa^5+8\,(\mu-1)\left(90\,\mu^3+317\,\mu^2-482\,\mu-49\right)\kappa^4 + \left(2109\,\mu^3+2249\,\mu^2+2539\,\mu-1057\right)(\mu-1)^2\,\kappa^3-3\left(45\,\mu^3-493\,\mu^2-945\,\mu-343\right)(\mu-1)^3\,\kappa^2 - 2\left(135\,\mu^3-4\,\mu^2-485\,\mu-442\right)(\mu-1)^4\,\kappa-16\left(3\,\mu^3-\mu^2+\mu-13\right)(\mu-1)^5\Big] \tag{68}$$

$$c_2^{(4)} = \frac{1}{D'_{24}}\Big[\left(-1404+300\,\mu^3+444\,\mu^2-3372\,\mu\right)\kappa^5-8\,(\mu-1)\left(49\,\mu^3+482\,\mu^2-317\,\mu-90\right)\kappa^4 - \left(1057\,\mu^3-2539\,\mu^2-2249\,\mu-2109\right)(\mu-1)^2\,\kappa^3+3\left(343\,\mu^3+945\,\mu^2+493\,\mu-45\right)(\mu-1)^3\,\kappa^2 + 2\left(442\,\mu^3+485\,\mu^2+4\,\mu-135\right)(\mu-1)^4\,\kappa+16\left(13\,\mu^3-\mu^2+\mu-3\right)(\mu-1)^5\Big] \tag{69}$$

$$a_4^{(4)} = \frac{1}{D_{44}}\Big[12\,(\mu-1)\left(\mu^2-22\,\mu+1\right)\kappa^4-4\left(65\,\mu^2-54\,\mu+65\right)(\mu-1)^2\,\kappa^3 + 3\left(57\,\mu^2-46\,\mu+57\right)(\mu-1)^3\,\kappa^2+3\left(31\,\mu^2+30\,\mu+31\right)(\mu-1)^4\,\kappa - 16\,(\mu-1)^5\left(\mu^2-6\,\mu+1\right)\Big] \tag{70}$$

$$b_4^{(4)} = \frac{1}{D_{44}}\Big[-12\,\omega_0\left(3+197\,\mu^3+197\,\mu^2+51\,\mu\right)\kappa^4-8\,\omega_0\,(\mu-1)\left(7\,\mu^3+496\,\mu^2+353\,\mu+48\right)\kappa^3 + \omega_0\left(385\,\mu^3-279\,\mu^2-2529\,\mu-905\right)(\mu-1)^2\,\kappa^2-\omega_0\left(35\,\mu^3-417\,\mu^2+357\,\mu+617\right)(\mu-1)^3\,\kappa - 16\,\omega_0\left(\mu^2-7\,\mu+8\right)(\mu-1)^4\Big] \tag{71}$$

$$c_4^{(4)} = \frac{1}{D_{44}}\Big[-12\,\omega_0\left(197+3\,\mu^3+51\,\mu^2+197\,\mu\right)\kappa^4-8\,\omega_0\,(\mu-1)\left(48\,\mu^3+353\,\mu^2+496\,\mu+7\right)\kappa^3 - \omega_0\left(905\,\mu^3+2529\,\mu^2+279\,\mu-385\right)(\mu-1)^2\,\kappa^2-\omega_0\left(617\,\mu^3+357\,\mu^2-417\,\mu+35\right)(\mu-1)^3\,\kappa - 16\,\omega_0\mu\left(8\,\mu^2-7\,\mu+1\right)(\mu-1)^4\Big] \tag{72}$$

$$a_3^{(5)} = \frac{1}{D_{35}}\Big[\left(6048+51072\,\mu^3-88128\,\mu^2+51072\,\mu+6048\,\mu^4\right)\kappa^6 - 8\,(\mu-1)\left(855\,\mu^4+13148\,\mu^3-26102\,\mu^2+13148\,\mu+855\right)\kappa^5 - 12\left(3055\,\mu^4-2284\,\mu^3+5834\,\mu^2-2284\,\mu+3055\right)(\mu-1)^2\,\kappa^4 + 2\left(4887\,\mu^4-1564\,\mu^3-46694\,\mu^2-1564\,\mu+4887\right)(\mu-1)^3\,\kappa^3$$

$$+\left(12141\,\mu^4-10028\,\mu^3-35122\,\mu^2-10028\,\mu+12141\right)(\mu-1)^4\,\kappa^2$$
$$+3\left(931\,\mu^4-1988\,\mu^3+306\,\mu^2-1988\,\mu+931\right)(\mu-1)^5\,\kappa$$
$$+2\left(297\,\mu^4-1388\,\mu^3+2006\,\mu^2-1388\,\mu+297\right)(\mu-1)^6\Big] \tag{73}$$

$$b_3^{(5)}=\frac{1}{D'_{35}}\Big[\left(9216-608640\,\mu^3-97920\,\mu^2+80256\,\mu-224640\,\mu^4\right)\kappa^7$$
$$+32\,(\mu-1)\left(3375\,\mu^4+9392\,\mu^3-24074\,\mu^2-8128\,\mu+363\right)\kappa^6$$
$$+8\left(43227\,\mu^4+64808\,\mu^3+70478\,\mu^2-49696\,\mu-13729\right)(\mu-1)^2\,\kappa^5$$
$$-4\left(8615\,\mu^4-82772\,\mu^3-178102\,\mu^2-96500\,\mu+13255\right)(\mu-1)^3\,\kappa^4$$
$$-2\left(39791\,\mu^4+29500\,\mu^3-117702\,\mu^2-225284\,\mu-56433\right)(\mu-1)^4\,\kappa^3$$
$$-\left(12333\,\mu^4+65812\,\mu^3+18478\,\mu^2-147500\,\mu-97139\right)(\mu-1)^5\,\kappa^2$$
$$-\left(805\,\mu^4+7220\,\mu^3+14782\,\mu^2-6412\,\mu-30139\right)(\mu-1)^6\,\kappa$$
$$+32\left(2\,\mu^4-37\,\mu^3+71\,\mu^2-151\,\mu+123\right)(\mu-1)^7\Big] \tag{74}$$

$$c_3^{(5)}=\frac{1}{D'_{35}}\Big[\left(224640-80256\,\mu^3+97920\,\mu^2+608640\,\mu-9216\,\mu^4\right)\kappa^7$$
$$-32\,(\mu-1)\left(363\,\mu^4-8128\,\mu^3-24074\,\mu^2+9392\,\mu+3375\right)\kappa^6$$
$$+8\left(13729\,\mu^4+49696\,\mu^3-70478\,\mu^2-64808\,\mu-43227\right)(\mu-1)^2\,\kappa^5$$
$$+4\left(13255\,\mu^4-96500\,\mu^3-178102\,\mu^2-82772\,\mu+8615\right)(\mu-1)^3\,\kappa^4$$
$$-2\left(56433\,\mu^4+225284\,\mu^3+117702\,\mu^2-29500\,\mu-39791\right)(\mu-1)^4\,\kappa^3$$
$$-\left(97139\,\mu^4+147500\,\mu^3-18478\,\mu^2-65812\,\mu-12333\right)(\mu-1)^5\,\kappa^2$$
$$-\left(30139\,\mu^4+6412\,\mu^3-14782\,\mu^2-7220\,\mu-805\right)(\mu-1)^6\,\kappa$$
$$-32\left(123\,\mu^4-151\,\mu^3+71\,\mu^2-37\,\mu+2\right)(\mu-1)^7\Big] \tag{75}$$

$$a_5^{(5)}=\frac{1}{D_{55}}\Big[\left(-3360-17280\,\mu^3+45888\,\mu^2-17280\,\mu-3360\,\mu^4\right)\kappa^5$$
$$-8\,(\mu-1)\left(1595\,\mu^4-9580\,\mu^3+11314\,\mu^2-9580\,\mu+1595\right)\kappa^4$$
$$+4\left(9715\,\mu^4-17300\,\mu^3+30962\,\mu^2-17300\,\mu+9715\right)(\mu-1)^2\,\kappa^3$$
$$-6\left(1145\,\mu^4-7676\,\mu^3+5910\,\mu^2-7676\,\mu+1145\right)(\mu-1)^3\,\kappa^2$$
$$-\left(6245\,\mu^4-10348\,\mu^3-4514\,\mu^2-10348\,\mu+6245\right)(\mu-1)^4\,\kappa$$
$$+4\left(125\,\mu^4-1516\,\mu^3+3118\,\mu^2-1516\,\mu+125\right)(\mu-1)^5\Big] \tag{76}$$

$$b_5^{(5)}=\frac{1}{D_{55}}\Big[-96\,\omega_0\left(15+7220\,\mu^3+2978\,\mu^2+420\,\mu+5415\,\mu^4\right)\kappa^5$$
$$+8\,\omega_0\,(\mu-1)\left(20415\,\mu^4-110240\,\mu^3-134186\,\mu^2-40120\,\mu-2925\right)\kappa^4$$
$$+4\,\omega_0\left(17295\,\mu^4+63840\,\mu^3-164762\,\mu^2-150040\,\mu-23725\right)(\mu-1)^2\,\kappa^3$$
$$-6\,\omega_0\left(3455\,\mu^4-18484\,\mu^3-19510\,\mu^2+46636\,\mu+20655\right)(\mu-1)^3\,\kappa^2$$
$$+\omega_0\left(945\,\mu^4-20108\,\mu^3+63286\,\mu^2+212\,\mu-55455\right)(\mu-1)^4\,\kappa$$

$$+\,64\,\omega_0\left(6\,\mu^3-73\,\mu^2+196\,\mu-125\right)(\mu-1)^5\Big] \tag{77}$$

$$\begin{aligned}
c_5^{(5)}=\frac{1}{D_{55}}\Big[&96\,\omega_0\left(5415+420\,\mu^3+2978\,\mu^2+7220\,\mu+15\,\mu^4\right)\kappa^5\\
&+8\,\omega_0\,(\mu-1)\left(2925\,\mu^4+40120\,\mu^3+134186\,\mu^2+110240\,\mu-20415\right)\kappa^4\\
&+4\,\omega_0\left(23725\,\mu^4+150040\,\mu^3+164762\,\mu^2-63840\,\mu-17295\right)(\mu-1)^2\,\kappa^3\\
&+6\,\omega_0\left(20655\,\mu^4+46636\,\mu^3-19510\,\mu^2-18484\,\mu+3455\right)(\mu-1)^3\,\kappa^2\\
&+\omega_0\left(55455\,\mu^4-212\,\mu^3-63286\,\mu^2+20108\,\mu-945\right)(\mu-1)^4\,\kappa\\
&+64\,\omega_0\mu\,\left(125\,\mu^3-196\,\mu^2+73\,\mu-6\right)(\mu-1)^5\Big]
\end{aligned} \tag{78}$$

$$\begin{aligned}
a_1^{(5)}=\frac{1}{D_{15}}\Big[&\left(-407040\,\mu^3+605952\,\mu^2-407040\,\mu-40320\,\mu^4-40320\right)\kappa^7\\
&+32\,(\mu-1)\left(4677\,\mu^4+18008\,\mu^3-49946\,\mu^2+18008\,\mu+4677\right)\kappa^6\\
&+8\left(26785\,\mu^4-1804\,\mu^3-6746\,\mu^2-1804\,\mu+26785\right)(\mu-1)^2\,\kappa^5\\
&-4\left(41185\,\mu^4+7628\,\mu^3-189482\,\mu^2+7628\,\mu+41185\right)(\mu-1)^3\,\kappa^4\\
&-2\left(53999\,\mu^4+58316\,\mu^3-282182\,\mu^2+58316\,\mu+53999\right)(\mu-1)^4\,\kappa^3\\
&-\left(17117\,\mu^4-18284\,\mu^3+878\,\mu^2-18284\,\mu+17117\right)(\mu-1)^5\,\kappa^2\\
&-\left(7781\,\mu^4-59660\,\mu^3+108830\,\mu^2-59660\,\mu+7781\right)(\mu-1)^6\,\kappa\\
&-8\left(211\,\mu^4-1452\,\mu^3+2562\,\mu^2-1452\,\mu+211\right)(\mu-1)^7\Big]
\end{aligned} \tag{79}$$

$$\begin{aligned}
\omega_4=\frac{1}{D_4}\Big[&\left(120-3488\,\mu^3+1488\,\mu^2-3488\,\mu+120\,\mu^4\right)\kappa^6\\
&+4\,(\mu-1)\left(179\,\mu^4+1836\,\mu^3-3150\,\mu^2+1836\,\mu+179\right)\kappa^5\\
&-2\left(713\,\mu^4+2484\,\mu^3-6570\,\mu^2+2484\,\mu+713\right)(\mu-1)^2\,\kappa^4\\
&-\left(1147\,\mu^4+3212\,\mu^3+546\,\mu^2+3212\,\mu+1147\right)(\mu-1)^3\,\kappa^3\\
&+\left(985\,\mu^4-252\,\mu^3-6570\,\mu^2-252\,\mu+985\right)(\mu-1)^4\,\kappa^2\\
&-4\left(71\,\mu^4-396\,\mu^3+666\,\mu^2-396\,\mu+71\right)(\mu-1)^5\,\kappa\\
&+64\left(\mu^4-14\,\mu^3+30\,\mu^2-14\,\mu+1\right)(\mu-1)^6\Big]
\end{aligned} \tag{80}$$

$$\begin{aligned}
b_1^{(5)}=\frac{1}{D'_{15}}\Big[&\left(-381600-370560\,\mu^3-1167552\,\mu^2+1910400\,\mu-185760\,\mu^4\right)\kappa^9\\
&+8\,(\mu-1)\left(35145\,\mu^4+150988\,\mu^3+599582\,\mu^2-733508\,\mu+23649\right)\kappa^8\\
&+4\left(211551\,\mu^4+578924\,\mu^3-820342\,\mu^2+887436\,\mu-48049\right)(\mu-1)^2\,\kappa^7\\
&-2\left(79777\,\mu^4+1023388\,\mu^3-391074\,\mu^2-1813780\,\mu-569079\right)(\mu-1)^3\,\kappa^6\\
&-\left(814725\,\mu^4+434692\,\mu^3-1200562\,\mu^2-436316\,\mu-109611\right)(\mu-1)^4\,\kappa^5\\
&+4\left(21393\,\mu^4+57068\,\mu^3+143582\,\mu^2-442884\,\mu-37063\right)(\mu-1)^5\,\kappa^4\\
&+4\left(56275\,\mu^4+99172\,\mu^3-133878\,\mu^2-139276\,\mu-50373\right)(\mu-1)^6\,\kappa^3\\
&+2\left(27837\,\mu^4+74428\,\mu^3-159370\,\mu^2+42028\,\mu-68811\right)(\mu-1)^7\,\kappa^2
\end{aligned}$$

$$+\left(13881\,\mu^4-33868\,\mu^3+31718\,\mu^2-3948\,\mu-25303\right)(\mu-1)^8\,\kappa$$
$$+64\left(37\,\mu^4-185\,\mu^3+270\,\mu^2-121\,\mu-9\right)(\mu-1)^9\Big] \tag{81}$$

$$c_1^{(5)}=\frac{1}{D'_{15}}\Big[\left(185760-1910400\,\mu^3+1167552\,\mu^2+370560\,\mu+381600\,\mu^4\right)\kappa^9$$
$$-8\,(\mu-1)\left(23649\,\mu^4-733508\,\mu^3+599582\,\mu^2+150988\,\mu+35145\right)\kappa^8$$
$$+4\left(48049\,\mu^4-887436\,\mu^3+820342\,\mu^2-578924\,\mu-211551\right)(\mu-1)^2\,\kappa^7$$
$$-2\left(569079\,\mu^4+1813780\,\mu^3+391074\,\mu^2-1023388\,\mu-79777\right)(\mu-1)^3\,\kappa^6$$
$$-\left(109611\,\mu^4+436316\,\mu^3+1200562\,\mu^2-434692\,\mu-814725\right)(\mu-1)^4\,\kappa^5$$
$$+4\left(37063\,\mu^4+442884\,\mu^3-143582\,\mu^2-57068\,\mu-21393\right)(\mu-1)^5\,\kappa^4$$
$$+4\left(50373\,\mu^4+139276\,\mu^3+133878\,\mu^2-99172\,\mu-56275\right)(\mu-1)^6\,\kappa^3$$
$$+2\left(68811\,\mu^4-42028\,\mu^3+159370\,\mu^2-74428\,\mu-27837\right)(\mu-1)^7\,\kappa^2$$
$$+\left(25303\,\mu^4+3948\,\mu^3-31718\,\mu^2+33868\,\mu-13881\right)(\mu-1)^8\,\kappa$$
$$+64\left(9\,\mu^4+121\,\mu^3-270\,\mu^2+185\,\mu-37\right)(\mu-1)^9\Big] \tag{82}$$

$$D_{24}=24\,(\mu-1+3\kappa)(1+\mu)^3(\mu+2\kappa-1)^3 \tag{83}$$
$$D'_{24}=96\,(1+\mu)^4(\mu+2\kappa-1)^3(\mu-1+3\kappa)\,\omega_0 \tag{84}$$
$$D_{44}=48\,(1+\mu)^3(\mu+2\kappa-1)^2(\mu-1+3\kappa)(\mu+4\kappa-1) \tag{85}$$
$$D_{35}=768\,(\mu+4\kappa-1)(1+\mu)^4(\mu+2\kappa-1)^3(\mu-1+3\kappa)^2 \tag{86}$$
$$D'_{35}=768\,(1+\mu)^5(\mu-1+2\kappa)^3(\mu-1+3\kappa)^2(\mu+4\kappa-1)\,\omega_0 \tag{87}$$
$$D_{55}=1536\,(1+\mu)^4(\mu-1+2\kappa)^2(\mu-1+3\kappa)(\mu+4\kappa-1)(\mu-1+5\kappa) \tag{88}$$
$$D_{15}=1536\,(\mu-1+5\kappa)(\mu+4\kappa-1)(1+\mu)^4(\mu-1+2\kappa)^3(\mu-1+3\kappa)^2 \tag{89}$$
$$D_4=512\,(1+\mu)^6(\mu-1+2\kappa)^3(\mu-1+3\kappa)\,{\omega_0}^3 \tag{90}$$
$$D'_{15}=1536\,(\mu-1+5\kappa)(\mu+4\kappa-1)(1+\mu)^6(\mu-1+2\kappa)^3(\mu-1+3\kappa)^2\,{\omega_0}^3 \tag{91}$$

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