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Elastic and piezoelectric waveguides may have infinite number of gaps in their spectra

Les guides d'ondes élastiques et piézoélectriques peuvent avoir un nombre infini de lacunes dans leur spectre

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A B S T R A C T

We consider elastic and piezoelectric waveguides composed from identical beads threaded periodically along a spoke converging at infinity. We show that the essential spectrum constitutes a non-negative monotone unbounded sequence and thus has infinitely many spectral gaps.

R É S U M É

Nous considérons des guides d'ondes élastiques et piézoélectriques réalisés à partir de perles identiques agencées de façon périodique le long d'un rayon convergeant à l'infini. Nous montrons que le spectre essentiel est une suite croissante positive non bornée. Cela prouve l'existence d'un nombre infini de trous spectraux.

1. Elastic waveguide

Let $\varpi \subset \{x = (y, z) \in \mathbb{R}^2 \times \mathbb{R} : z = x_3 \in (0, 1)\}$ be a domain with Lipschitz boundary $\partial\varpi$ and compact closure $\overline{\varpi} = \varpi \cup \partial\varpi$, where the surface $\partial\varpi$ has two planar parts $\gamma_p = \mathbb{B}^2_R \times \{p\}$ with $p = 0, 1$ and $\mathbb{B}^2_R = \{y : |y| < R\}$. We also introduce a Lipschitz domain $\omega \subset \mathbb{B}^2_{R/2}$ and an infinitesimal sequence $\{\alpha_j\}_{j\in\mathbb{Z}} \subset (0, 1)$, where $\mathbb{Z} = \{0, \pm1, \pm2, \dots\}$, as well as the sets

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$$\varpi_j = \{x : (y, z - j) \in \varpi\}, \quad \omega_j = \{x : \alpha_j^{-1} y \in \omega, z = j\} \quad \text{for } j \in \mathbb{Z} \tag{1}$$

The waveguide Π (Fig. 1a),

$$\Pi = \bigcup_{j \in \mathbb{Z}} (\varpi_j \cup \omega_j) \tag{2}$$

consists of identical beads threaded periodically along a thin spoke, which converges at infinity. In other words, the neighboring beads ϖ_{j-1} and ϖ_j are connected through the aperture ω_j.

Let waveguide (2) be filled with a homogeneous elastic material. The variational formulation of the elasticity problem on time-harmonic oscillations with a frequency $\kappa > 0$ reads as

$$(AD(\nabla)u, D(\nabla)v)_\Pi = \lambda(u, v)_\Pi + (f, v)_\Pi \quad \forall v \in H^1(\Pi) \tag{3}$$

Here, $\lambda = \rho\kappa^2$ and $\rho > 0$ are the spectral parameter and the material density, $\nabla = \text{grad}$, f denotes mass forces and the Voigt–Mandel notation is in use so that $u = (u_1, u_2, u_3)^\top \in \mathbb{R}^3$ and $\varepsilon(u) = D(\nabla)u \in \mathbb{R}^6$ are the displacement and strain columns,

$$D(\nabla)^\top = \begin{pmatrix} \partial_1 & 0 & 0 & 0 & \alpha\partial_3 & \alpha\partial_2 \\ 0 & \partial_2 & 0 & \alpha\partial_3 & 0 & \alpha\partial_1 \\ 0 & 0 & \partial_3 & \alpha\partial_2 & \alpha\partial_1 & 0 \end{pmatrix}, \; \alpha = \frac{1}{\sqrt{2}}, \; \partial_j = \frac{\partial}{\partial x_j}, \; \nabla = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}$$

and $\top$ stands for transposition. Furthermore, A denotes a symmetric and positive definite 6×6-matrix of constant elastic moduli in the Hooke's law $\sigma(u) = AD(\nabla)u$, where $\sigma_{jk}(u)$ are Cartesian components of the stress tensor composing the stress column

$$\sigma(u) = (\sigma_{11}(u), \sigma_{22}(u), \sigma_{33}(u), \sqrt{2}\sigma_{23}(u), \sqrt{2}\sigma_{31}(u), \sqrt{2}\sigma_{12}(u))^\top$$

Finally, $(\cdot, \cdot)_\Pi$ is the natural inner product in the Lebesque space $L^2(\Pi)$ and $H^1(\Pi)$ is the Sobolev space, either scalar or vector. The problem (3) realizes as the continuous mapping $B - \lambda : H^1(\Pi) \to H_0^1(\Pi)^*$, where $H_0^1(\Pi)^*$ is the dual space.

In principle, problem (3) should be supplied with appropriate radiation conditions, see [1] for a scalar problem and [2] for elasticity. However, since we will only consider the inhomogeneous ($f \neq 0$) problem in the regularity field, it is not necessary to formulate them.

2. Motivation

The structure (2) of waveguide Π, which consists of the periodic family of identical cells (1) connected through converging apertures, comes from the previous works of the authors [3–6] and their attempts to prove or disprove the existence of infinite number of spectral gaps for periodic elastic and piezoelectric waveguides. This question is related to the classical Bethe–Sommerfeld conjecture on a finite number of spectral gaps for any periodic waveguide in $\mathbb{R}^d$, $d > 1$. The conjecture has been solved for some scalar problems only, mainly for the stationary Schrödinger equation, see [7–10] and [11] for an introduction to the topic. The (periodic) waveguide Π^ε in papers [3–6] was obtained as the union of the cells (1) and the thin cylinder $\Omega^\varepsilon = \{(y, z) : \varepsilon^{-1} y \in \omega, z \in \mathbb{R}\}$ where the domain $\omega \subset \mathbb{R}^{d-1}$ is as above, but $\varepsilon > 0$ is a small parameter. In other words, the cells are connected through small but *fixed* apertures $\omega_j^\varepsilon = \{(y, z) : \varepsilon^{-1} y \in \omega, z = j\}$. By constructing asymptotics of eigenvalues of a model problem in ϖ obtained from (3) in Π^ε by the Gelfand transform, it was proven that, for any $N \in \mathbb{N} = \{1, 2, 3, \dots\}$, there exists $\varepsilon_N > 0$ such that the spectrum of the problem in Π^ε with $\varepsilon \in (0, \varepsilon_N]$ has at least N opened spectral gaps. However, such asymptotic analysis does not seem to suffice for opening infinitely many gaps, because $\varepsilon_N \to +0$ as $N \to +\infty$. On the other hand, the waveguide (2) is not periodic because diameters $O(\alpha_j)$ of the apertures ω_j connecting the identical beads ϖ_j and ϖ_{j+1} decay as $j \to \pm\infty$. Hence, Theorem 1 does not solve the Bethe–Sommerfeld conjecture.

3. Spectrum

According to the Korn inequality in ϖ, see, e.g., [12], the left-hand side of the integral identity (3) constitutes a closed positive Hermitian form in $H^1(\Pi)$. Hence, problem (3) is associated with a positive self-adjoint unbounded operator $\mathcal{A}$ in $L^2(\Pi)$ with a domain $\mathcal{D}(\mathcal{A}) \subset H^1(\Pi)$, see [13, Ch. 10]. Its spectrum $\wp$ belongs to the closed positive real semi-axis $\overline{\mathbb{R}_+}$ and

$$\wp = \wp_{\text{di}} \cup \wp_{\text{es}}, \quad \wp_{\text{di}} \cap \wp_{\text{es}} = \varnothing \tag{4}$$

where $\wp_{di}$ and $\wp_{\text{es}}$ are the discrete and essential spectra, respectively. To describe the latter component, we mention that the model problem of the elasticity theory in the bounded Lipschitz domain ϖ

$$(AD(\nabla)U, D(\nabla)V)_\varpi = \Lambda(U, V)_\varpi \quad \forall V \in H^1(\varpi) \tag{5}$$

possesses the eigenvalue sequence

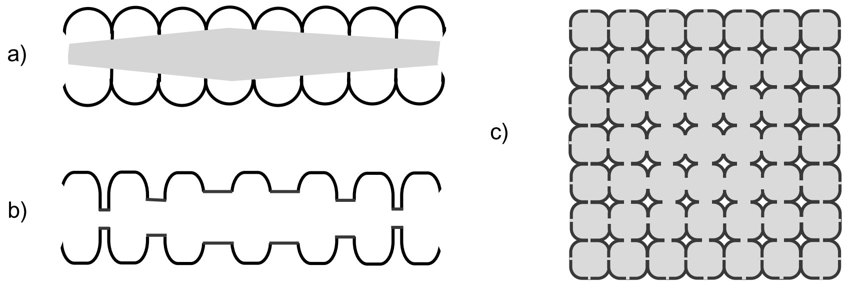

Fig. 1.

$$0 = \Lambda_1 = \cdots = \Lambda_6 < \Lambda_7 \leqslant \Lambda_8 \leqslant \cdots \leqslant \Lambda_k \leqslant \cdots \to +\infty \tag{6}$$

and the corresponding vector eigenfunctions $U^1, \ldots, U^k, \ldots \in H^1(\varpi)$ can be orthonormalized in $L^2(\varpi)$. The first six of them are the rigid motions.

Theorem 1. *There holds* $\wp_{\rm es} = \{\Lambda_k\}_{k \in \mathbb{N}}$, *where* $\mathbb{N} = \{1, 2, \ldots\}$.

Since the sequence (5) is monotone and unbounded, Theorem 1 demonstrates that the operator $\mathcal{A}$ gets infinitely many spectral gaps, namely finite open intervals in $\overline{\mathbb{R}_+}$, which are free of the essential spectrum $\wp_{\rm es}$ but have both endpoints in $\wp_{\rm es}$. It looks quite probable that each point $\Lambda_k \in \wp_{\rm es}$ is an accumulation point of the discrete spectrum $\wp_{\rm di}$; however, at the moment the authors are not able to confirm this hypothesis. Instead, first, we have proven that the total multiplicity of the point spectrum in the segment $[\Lambda_k - \delta, \Lambda_k + \delta]$ cannot be finite for any $\delta > 0$ and, second, we construct the asymptotics of the eigenvalues $\lambda^\varepsilon_{jk} \to \Lambda_k$, $j \to +\infty$ in the case $\alpha_j = \varepsilon \alpha^0_j$, for $\varepsilon \in (0, \varepsilon_k)$ and sufficiently small $\varepsilon_k > 0$.

4. Abstract spectral equation

To prove Theorem 1, it is convenient to reformulate the elasticity problem as the abstract equation

$$\mathcal{B}u = \mu u \text{ in } \mathcal{H} \tag{7}$$

where $\mathcal{H}$ is the Sobolev space $H^1(\Pi)$ with the inner product (recall the Korn inequality mentioned above)

$$\langle u, v \rangle = (AD(\nabla)u, D(\nabla)v)_\Pi + (u, v)_\Pi \tag{8}$$

and $\mathcal{B}$ is a self-adjoint positive continuous operator in $\mathcal{H}$ determined by the formula

$$\langle \mathcal{B}u, v \rangle = (u, v)_\Pi \quad \forall u, v \in \mathcal{H} \tag{9}$$

Comparing (7)–(9) with (3) yields the relationship of the spectral parameters

$$\mu = (1 + \lambda)^{-1} \tag{10}$$

Notice that $\mathcal{B}$ is related to the inverse operator $(B + 1)^{-1}$ for the isomorphism $B + 1 : \mathcal{H} \to \mathcal{H}^*$.

5. Weyl singular sequence

For any eigenvalue $\Lambda_k = \mu_k^{-1} - 1$ in (6), we will construct a sequence $\{u^{kj}\}_{j \in \mathbb{N}} \subset \mathcal{H}$ such that
1°. $\|u^{kj}; \mathcal{H}\| \leqslant c_k$;
2°. $u^{kj} \rightharpoonup 0$ weakly in $\mathcal{H}$;
3°. $\mathcal{B}u^{kj} - \mu_k u^{kj} \to 0$ strongly in $\mathcal{H}$ when $j \to +\infty$.

Then, the Weyl criterion, cf. [13, §9.1], implies that μ_k belongs to the essential spectrum of the operator $\mathcal{B}$, and we thus obtain the inclusion $\{\Lambda_j\}_{j \in \mathbb{N}} \subset \wp_{\rm es}$ using (10). Let χ be a standard smooth cut-off function, $\chi(x) = 1$ for $|x| < R/2$ and $\chi(x) = 0$ for $|x| > R$. Set

$$u^{jk}(x) = X_j(x) U^k(y, z - j) \tag{11}$$

$$X_j(x) = 1 - \chi(\alpha^{-1}_{j+1}(y, z - j - 1)) - \chi(\alpha^{-1}_j(y, z - j)), \quad x \in \varpi_j \tag{12}$$

Since the vector eigenfunction U^k is smooth near the flat surfaces γ_p, we have $\|u^{kj}; \mathcal{H}\|^2 = \Lambda_k + 1 + O(\alpha_{j-1})$. Hence, condition 1° is satisfied, and also condition 2° becomes evident, because the supports of u^{kj} and u^{km} are disjoint for $j \neq m$. Finally, (8)–(10) yield

$$\|\mathcal{B}u^{kj}-\mu_k u^{kj};\mathcal{H}\| = \sup\left|\langle \mathcal{B}u^{kj}-\mu_k u^{kj}, v\rangle\right| = (1+\Lambda_k)^{-1}\sup\left|(AD(\nabla)u^{kj}, D(\nabla)v)_\Pi - \Lambda_k(u^{kj}, v)_\Pi\right| \tag{13}$$

where the supremum is computed over the unit ball of $\mathcal{H}$. Commuting the differential operator $D(\nabla)$ with cut-off functions in (11) and applying the Hardy inequality to the test function v, we can evaluate the expression (13) as $O(\alpha_{j-1})$. Thus, 3° is true, too.

6. Parametrix

The most complicated part of our proof of Theorem 1 is to verify that $\mathbb{C}\setminus\{\Lambda_k\}_{k\in\mathbb{N}}$ is the regularity field of the operator $\mathcal{A}$. To this end, we employ general results [14, Ch. 4], which provide "almost inverse" operators for elliptic boundary-value problems in domains with singularly perturbed boundaries. Indeed, the beads ϖ_j and ϖ_{j-1} in (1) are connected in (2) through the aperture ω_j, the diameter $O(\alpha_j)$ of which decays when $j\to\pm\infty$ so that ω_j is nothing but a singular nucleation in the boundary $\partial\varpi_j\cup\partial\varpi_{j+1}$. Using these results, we construct for any $\lambda\notin\{\Lambda_k\}_{k\in\mathbb{N}}$ a right parametrix $\mathcal{R}(\lambda)$ for the problem (3), that is, a continuous operator

$$\mathcal{R}(\lambda): H^1(\Pi)^*\to H^1(\Pi)$$

such that $(B-\lambda)\mathcal{R}(\lambda)-1$ is a compact mapping in $H^1(\Pi)^*$, the space of (anti)linear functionals in $H^1(\Pi)$.

Assuming that $f(v)=(f,v)_\Pi$ of (3) belongs to $H^1(\Pi)^*$, we search for $\mathcal{R}(\lambda)f$ in the form

$$\mathcal{R}(\lambda)f(x) = \sum_{j\in\mathbb{Z}} X_j(x)\mathcal{U}^j(y,z+j) + \sum_{j\in\mathbb{Z}}\chi_j(x)\mathcal{W}^j(\xi^j) \tag{14}$$

where $\xi^j=(\alpha_j^{-1}y,\alpha_j^{-1}(z-j))$ are stretched coordinates, X_j is defined in (12) and the smooth cut-off function χ_j satisfies $\chi_j(x)=1$ for $r_j:=(|y|^2+|z-j|^2)^{-1/2}<R/2$, $\chi_j(x)=0$ for $r_j>R$. Furthermore, $\mathcal{U}^j\in H^1(\varpi)^3$ is a solution to the following problem on the etalon bead ϖ,

$$(AD(\nabla)\mathcal{U}^j, D(\nabla)\mathcal{V})_\varpi - \lambda(\mathcal{U}^j,\mathcal{V})_\varpi = \mathcal{F}^j(\mathcal{V}) \quad \forall\mathcal{V}\in H^1(\varpi)\,,\ \text{where} \tag{15}$$

$$\mathcal{F}^j(\mathcal{V}) = f(X_j\mathcal{V}^j),\quad \mathcal{V}^j(y,z)=\mathcal{V}(y,z-j),\quad (y,z)\in\varpi_j \tag{16}$$

Since λ differs from the eigenvalues in (6), the variational problem (15) has a unique solution subject to the estimate $\|\mathcal{U}^j;H^1(\varpi)\|\leqslant c_\lambda\|\mathcal{F}^j;H^1(\varpi)^*\|$.

Due to the cut-off functions X_j in (16) and (14), the first sum in (14) leaves discrepancies in the original problem (3), which are located in the $c\alpha_j$-neighborhoods of the apertures ω_j and belong to $L^2(\overline{\varpi_j}\cup\overline{\varpi_{j+1}})$, but they nevertheless give rise to a non-compact mapping in $H^1(\varpi)^*$ because the domain Π is unbounded. To compensate for these discrepancies, we follow [14, Ch. 4] and construct boundary layers $\mathcal{W}^j$: they must be written in the stretched coordinates and, according to (1), they are solutions to the elasticity problem in the union Ξ of the half-spaces $\mathbb{R}^3_\pm=\{\xi\in\mathbb{R}^3:\pm\xi_3>0\}$ connected through the aperture ω in the wall $\partial\mathbb{R}^3_\pm=\{\xi:\xi_3=0\}$. Data of these problems have compact supports and they thus have solutions $\mathcal{W}^j$ with finite Dirichlet integrals

$$\|\nabla_\xi\mathcal{W}^j;L^2(\Xi)\|^2+\|(1+|\xi|)^{-1}\mathcal{W}^j;L^2(\Xi)\|^2<+\infty$$

due to the Hardy inequality, cf. [12] and [15, Ch. 6]. Owing to the decay properties of the boundary layers $\mathcal{W}^j(\xi)=O((1+|\xi|)^{-1})$, the whole discrepancy $h=f-(B-\lambda)\mathcal{R}(\lambda)f$ of the function (14) in problem (3) falls into the weighted Lebesgue space with the norm

$$\Big(\sum_{j\in\mathbb{Z}}\alpha_j^{-2}\|h;L^2(\varpi_j)\|^2\Big)^{1/2}$$

which is compactly embedded into $H^1(\Pi)^*$. This concludes the proof of Theorem 1.

7. Some remarks

1°. Theorem 1 remains valid in two-dimensional waveguides of similar shapes, although solutions to the model elasticity problem in the planar aperture domain $\Xi\subset\mathbb{R}^2$ do not decay at infinity. However, a modification of the procedure in Section 5 according to [14, Ch. 2] yields the result.

2°. The above consideration can be adapted to waveguides of different shapes; for example, beads can be connected by thin and short ligaments, see Fig. 1b, and [5,4,16], or can constitute a double-periodic family, see Fig. 1c.

3°. Both steps in our proof of Theorem 1 can be readily generalized for higher-order elliptic differential operators and systems. In Section 7, we also consider a non-self-adjoint elliptic system, the spectrum of which has the same pathological structure.

8. Piezoelectric waveguide

In the following, we use the notation of Section 1 for mechanical components, but supply them with the superscript M, and moreover we introduce the electric potential u^{E}, the electric field strength column $-\nabla u^{\mathsf{E}}$ and the matrices of sizes 9×9 and 9×4

$$A = \begin{pmatrix} A^{\mathsf{MM}} & A^{\mathsf{ME}} \\ -A^{\mathsf{EM}} & A^{\mathsf{EE}} \end{pmatrix}, \quad D(\nabla) = \begin{pmatrix} D^{\mathsf{M}}(\nabla) & \mathbb{O}_{6\times 1} \\ \mathbb{O}_{3\times 3} & \nabla \end{pmatrix} \tag{17}$$

Here, $\mathbb{O}_{m\times n}$ is the null $m \times n$-matrix, A^{EE} is a symmetric positive definite 3×3-matrix of dielectric moduli and the block $A^{\mathsf{ME}} = (A^{\mathsf{EM}})^\top \neq \mathbb{O}_{6\times 3}$ is comprised of piezoelectric moduli. We consider the low- and middle-frequency ranges of the spectrum, where the influence of electro-magnetic waves is negligible, cf. [17], and we formulate the spectral problem for the piezoelectric waveguide (2), the surface of which is traction-free and in contact with an absolute insulator:

$$(AD(\nabla)u, D(\nabla)v)_\Pi = \Lambda(u^{\mathsf{M}}, v^{\mathsf{M}})_\Pi \quad \forall v = (v^{\mathsf{M}}, v^{\mathsf{E}}) \in H^1(\Pi) \tag{18}$$

Although the matrix A in (17) is not symmetric and the form on the left of (18) is not Hermitian, the absence of the electric component u^{E} on the right-hand side of (18) allows us to reduce the problem to a positive self-adjoint operator with the spectrum (4), see [6], [18]. Also the model problem in ϖ, derived from (18), is similar to (5) and has an eigenvalue sequence of type (6), although with seven null eigenvalues (a zero electric potential is added).

Theorem 2. *Theorem 1 remains valid for the piezoelectricity problem (18).*

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