C. R. Mecanique 344 (2016) 388–401

Contents lists available at ScienceDirect

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A nonlocal Fourier's law and its application to the heat conduction of one-dimensional and two-dimensional thermal lattices

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ARTICLE INFO

Article history:
Received 23 October 2015
Accepted 2 January 2016
Available online 17 March 2016

Keywords:
Heat equation
Lattice
Fourier's law
Nonlocal thermodynamics
Gradient Fourier's law
Nonlocality
Diffusion equation

ABSTRACT

This study focuses on heat conduction in unidimensional lattices also known as micro-structured rods. The lattice thermal properties can be representative of concentrated thermal interface phases in one-dimensional segmented rods. The exact solution of the linear time-dependent spatial difference equation associated with the lattice problem is presented for some given initial and boundary conditions. This exact solution is compared to the quasicontinuum approximation built by continualization of the lattice equations. A rational-based asymptotic expansion of the pseudo-differential problem leads to an equivalent nonlocal-type Fourier's law. The differential nonlocal Fourier's law is analysed with respect to thermodynamic models available in the literature, such as the Guyer–Krumhansl-type equation. The length scale of the nonlocal heat law is calibrated with respect to the lattice spacing. An error analysis is conducted for quantifying the efficiency of the nonlocal model to capture the lattice evolution problem, as compared to the local model. The propagation of error with the nonlocal model is much slower than that in its local counterpart. A two-dimensional thermal lattice is also considered and approximated by a two-dimensional nonlocal heat problem. It is shown that nonlocal and continualized heat equations both approximate efficiently the two-dimensional thermal lattice response. These extended continuous heat models are shown to be good candidates for approximating the heat transfer behaviour of microstructured rods or membranes.

1. Introduction

This paper deals with a nonlocal generalization of the heat equation that can be based on lattice arguments. Such nonlocal theories may be useful to capture the scale effects of microstructured solids, when the discreteness at a subscale may play a predominant role at a larger scale. Such scale effects have been experimentally or numerically (based on molecular dynamics simulations) observed, for small scale structures, where size-dependent thermo-mechanical behaviour is noticed. Although the paper is mainly focused on thermal diffusion, fluid infiltration in porous media or electrical conductivity may

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http://dx.doi.org/10.1016/j.crme.2016.01.001

be considered as alternative basic diffusion problems [1]. Nonlocal thermomechanics has been developed since the 1960s based on well-founded thermodynamic arguments [2]. Eringen and Kim [3] or Eringen [4] calibrated the nonlocal elasticity kernel (uncoupled mechanical problem) from lattice mechanics. Lattice mechanics is typically governed by discrete equations, whereas continuum models are known to be better suited for engineering applications, with some more mathematical available framework. There is a need to develop some continuous models which possess some information of the lattice ones. In that spirit, Collins [5] introduced the concept of quasicontinuum to representing a transition medium between the discrete lattice and the asymptotic local continuum. Collins [5] defined this quasicontinuum for a mechanical lattice, with specific application to the soliton phenomenon (see also [6,7]). Rosenau [8] obtained a nonlocal wave equation by continualization of the discrete wave equation. This nonlocal wave equation can be shown to be cast as a differential-based nonlocal model [4], also called a stress-gradient nonlocal model. More recently, the source of nonlocality has been investigated, especially with respect to the inherent microstructure, and in particular for uncoupled mechanical problems (see, recently, [9–11] for nonlocal elasticity problems). Challamel et al. [12] also showed the key role of different microstructures, namely a concentrated or some distributed microstructures. Nonlocal mechanics may be used for characterizing the behaviour of the quasicontinuum. To the authors' knowledge, this methodology has not yet been applied to the thermal analysis of the lattice, so the main aim of this paper is to fill this gap.

It has been demonstrated that the nonlocal kernel for elasticity problems may be related to the discreteness of the material at a fine scale, using a nonlocal differential model introduced by Eringen [4]:

$$\sigma - l_c^2 \sigma'' = E\varepsilon \quad \text{with } \varepsilon = u' \tag{1}$$

where σ is the uniaxial stress, ε is the uniaxial strain, u is the axial displacement, E is the Young modulus, and l_c is a characteristic length which accounts for the specific lattice effect of the equivalent quasicontinuum. For axial vibrations problems, Challamel et al. [11] showed that the length scale of the nonlocal model can be calibrated from the lattice spacing a using:

$$l_c^2 = \frac{a^2}{12} \tag{2}$$

This value is slightly different from the one calibrated by Eringen [4] by comparing the wave dispersive properties of the nonlocal model with the lattice one, also referred to as the Born–Kármán lattice model.

In this paper, we adopt the same methodology used and applied in a one-dimensional problem of thermal diffusion evolution. Nonlocal heat equations have been recently considered using space-fractional derivative operators instead of integer derivative ones [13–17]. In these approaches, the attenuation functions can be introduced by fractional derivative theory, leading to equivalent fractional power law decaying functions. Atanackovic et al. [13] considered a generalized fractional heat equation (also called fractional Cattaneo-type equation) (from the initial work of Cattaneo [18] – see also [19]) with both space- and time-fractional operators, and presented some numerical and analytical solutions. Some more general results including existence and uniqueness properties of Cattaneo-type space–time fractional heat equation (and nonlocal wave-type equations) are available in the books of Atanackovic et al. [20,21]. Michelitsch et al. [14] studied nonlocal wave propagation and nonlocal diffusion processes for self-similar harmonic interactions media using fractional derivatives. Sapora et al. [15] investigated a spatially nonlocal heat equation involving space-fractional derivative operators. Michelitsch et al. [14], Tarasov [16] or Zingales [17] built some space-fractional derivative nonlocal heat equations from a lattice model. Michelitsch et al. [14] or Tarasov [16] considered long-range lattice interactions (nearest-neighbour ones, but also interactions including some other neighbouring) for the physical justification of fractionality, whereas Zingales [17] investigated only nearest-neighbour interactions with power-law lattice non-uniformity. Deseri and Zingales [22] considered a time-fractional Darcy equation (diffusion equation), which can be also considered as a kind of generalized Cattaneo-type equation. Yu et al. [23] coupled Eringen's nonlocal elasticity (integer order spatial differential model) with time-fractional order derivative for the heat conduction. Challamel et al. [9] analytically studied wave propagation in a nonlocal fractional differential-based model, highlighting the possible link between fractional nonlocality and Eringen's differential-based model (see [4] for Eringen's differential model applied to elasticity). Peridynamic heat transfer modelling (which makes use of nonlocal type diffusion equations) has been investigated by Oterkus et al. [24]. Recently, Zhan et al. [25] numerically noticed some length-dependent thermal conductivity in a one-dimensional carbon nanomaterial – diamond nanothread (DNT) – based on non-equilibrium molecular dynamics simulations.

In this paper, we consider an Eringen-type differential model for the nonlocal one-dimensional (and later two-dimensional) generalization of Fourier's law:

$$q - l_c^2 q'' = -\lambda T' \tag{3}$$

where q is the heat flux, T is the temperature, λ is thermal conductivity, and l_c is a characteristic length which contains the microstructure information related to the discreteness of the material. The meaning of this nonlocal parameter is discussed further below. In Eq. (3), we can recognize an Eringen-type nonlocal differential model [4], where the heat flux acts as the stress and the temperature may be associated with the displacement in the analogous case of nonlocal elasticity. Eq. (3) can also be classified as a Guyer–Krumhansl-type equation [26–29], restricted to the nonlocal space contribution as recently highlighted by Sellitto et al. [30], Jou et al. [31] or Jou et al. [32]. The additional nonlocal terms may appear in the kinetic theory of gases in the so-called Burnett approximation [33]. Jou et al. [32] provided some thermodynamic arguments to justify such a family of models.

The energy balance equation may be written as:

$$\rho c\dot{T} = -q' \tag{4}$$

where ρ denotes the density and c is the specific heat capacity of the rod. Coupling Eq. (3) with Eq. (4) leads to the nonlocal heat equation:

$$\dot{T} = \alpha T'' + l_c^2 \dot{T}'' \tag{5}$$

where $\alpha = \lambda/\rho c$ is the thermal diffusivity. Eq. (5) is the Eringen-type nonlocal diffusion equation, which may be also formally valid for diffusion in porous media or electrical conduction.

This equation has been considered by Barenblatt [34] for the flow of liquids in fissured rocks (see also [35]). Some mathematical solutions in reference configurations are presented by Barenblatt [34] and Ting [36] (see also [37]). Recent studies on so-called weak nonlocal thermo-mechanical problems have been carried out by Maugin [38], Berezovski et al. [39], and Filopoulos et al. [40,41].

2. Solution to the nonlocal heat equation

We are searching for a solution to this spatially nonlocal evolution equation with the following boundary and initial conditions:

$$T(0,t) = T(L,t) = 0 \quad \text{and} \quad T(x,0) = f(x) \tag{6}$$

Using the method of separation of variables, based on $T(x,t) = X(x)Z(t)$, the nonlocal evolution equation gives:

$$\frac{\dot{Z}}{Z} = \alpha \frac{X''}{X} + l_c^2 \frac{X''}{X}\frac{\dot{Z}}{Z} \tag{7}$$

We can assume that $\frac{\dot{Z}}{Z} = -\alpha\gamma$ and then $\frac{X''}{X} = \frac{-\gamma\alpha}{\alpha - \gamma\alpha l_c^2} = -\delta^2$ with the following solutions:

$$Z(t) = A\mathrm{e}^{-\alpha\gamma t} \tag{8}$$

$$X(x) = B\sin(\delta x) + C\cos(\delta x) \tag{9}$$

From the boundary conditions, $T(0,t) = T(L,t) = 0$, which is equivalent to $X(0) = X(L) = 0$, we can therefore write:

$$X(x) = B\sin\left(\frac{m\pi x}{L}\right) \quad \text{with } \delta = \frac{m\pi}{L},\ m = 1,2,3\ldots \tag{10}$$

Introducing $\delta = \frac{m\pi}{L}, m = 1,2,3\ldots$ into $\frac{X''}{X} = \frac{-\gamma\alpha}{\alpha - \gamma\alpha l_c^2} = -\delta^2$, we obtain:

$$\gamma = \frac{(\frac{m\pi}{L})^2}{1 + l_c^2(\frac{m\pi}{L})^2} \tag{11}$$

The general Fourier series solution can be expressed as:

$$T(x,t) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{L}\right)\mathrm{e}^{\frac{-\alpha m^2\pi^2 t}{l_c^2 m^2\pi^2 + L^2}} \tag{12}$$

By setting $l_c = 0$, the solution is reduced to the classical local solution (see, for instance, [1]):

$$T(x,t) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{L}\right)\mathrm{e}^{-\frac{m^2\pi^2\alpha t}{L^2}} \tag{13}$$

To satisfy the initial condition, we require for $x \in [0; L]$:

$$f(x) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{L}\right) \tag{14}$$

where the coefficient A_m is given by:

$$A_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x \tag{15}$$

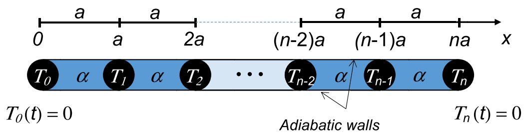

Fig. 1. Lattice thermal behaviour.

for a specific hat function such as:

$$f(x) = \begin{cases} T_0 \frac{2x}{L}, & 0 \le x \le \frac{L}{2} \\ T_0(2 - \frac{2x}{L}), & \frac{L}{2} \le x \le L \end{cases} \tag{16}$$

Based on Eq. (15), the coefficient A_m is calculated as:

$$A_m = \frac{4T_0}{L^2} \int_0^{L/2} x \sin\left(\frac{m\pi x}{L}\right) dx + \frac{4T_0}{L^2} \int_{L/2}^{L} (L - x) \sin\left(\frac{m\pi x}{L}\right) dx$$
$$= \frac{8T_0}{(m\pi)^2} \sin\left(\frac{m\pi}{2}\right) = \begin{cases} \frac{8T_0}{(m\pi)^2}(-1)^{(m-1)/2}, & m \text{ is odd} \\ 0, & m \text{ is even} \end{cases} \tag{17}$$

The general Fourier series solution can then be expressed as:

$$T(x,t) = \sum_{m=1,3,5,\dots} (-1)^{(m-1)/2} \frac{8T_0}{(m\pi)^2} \sin\left(\frac{m\pi x}{L}\right) e^{\frac{-\alpha m^2 \pi^2 t}{l_c^2 m^2 \pi^2 + L^2}} \tag{18}$$

The dimensionless parameters can be introduced as:

$$T^* = \frac{T}{T_0}, \qquad \hat{x} = \frac{x}{L}, \qquad \hat{l}_c = \frac{l_c}{L} \quad \text{and} \quad \tau = \alpha \frac{t}{L^2} \tag{19}$$

The general Fourier series solution can then be expressed in a dimensionless form:

$$T^*(\hat{x}, \tau) = \sum_{m=1,3,5,\dots} (-1)^{(m-1)/2} \frac{8}{(m\pi)^2} \sin(m\pi\hat{x}) e^{\frac{-m^2\pi^2}{1+\hat{l}_c^2 m^2 \pi^2} \tau} \tag{20}$$

3. Exact solution for the thermal lattice equation

In this study, we consider heat transfer in a lattice of n rigid elements (see Fig. 1). a is the lattice spacing, which has a physical meaning with respect to the physical discreteness of the thermal lattice. A similar lattice with additional inhomogeneities was recently studied by Zingales [17]. The discrete-based (or lattice) approach adopted here is based on the following spatially discrete Fourier's law:

$$q_i = -\lambda \frac{T_{i+1} - T_i}{a} \tag{21}$$

and the discrete energy balance equation, which may be written as:

$$\rho c \dot{T}_i = -\frac{q_i - q_{i-1}}{a} \tag{22}$$

We then investigate the discrete diffusion equation formulated with:

$$\dot{T}_i = \alpha \frac{T_{i+1} - 2T_i + T_{i-1}}{a^2} \tag{23}$$

where again, $\alpha = \lambda/\rho c$ is the thermal diffusivity and $a = L/n$ is the lattice spacing.

Exact solutions of thermal lattice equations have been published with respect to the numerical treatment of the discretized time and spatial local heat equation (see [42–44] – see more recently [45]). It is noteworthy that the thermal lattice equations in the present paper coincide exactly with the finite-difference formulation of the spatial local heat equation with a continuous time parameter.

Following the methodology already described for the nonlocal heat equation, we use the method of separation of variables, based on $T_i(t) = X_i Z(t)$, thus leading to:

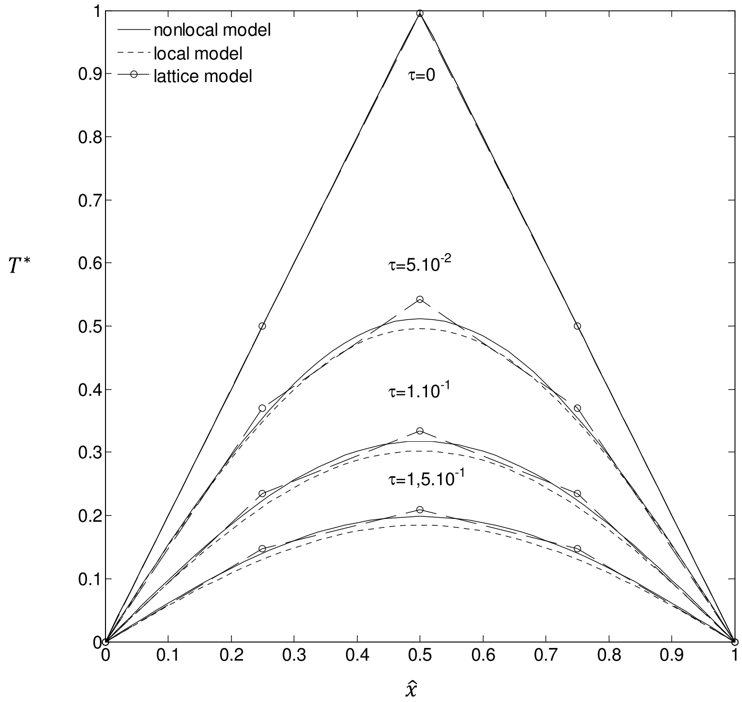

Fig. 2. Evolution of heat transfer using the lattice model, the local model and the nonlocal model based on $l_c^2 = a^2/12$ and $n = 4$.

$$\frac{\dot{Z}}{Z} = \alpha \frac{X_{i+1} - 2X_i + X_{i-1}}{a^2 X_i} \tag{24}$$

We can assume that $\dot{Z} = -\alpha\gamma Z$, which is easily integrated:

$$Z(t) = A\mathrm{e}^{-\alpha\gamma t} \tag{25}$$

and then a linear second-order difference equation in space needs to be solved:

$$X_{i+1} + (\beta - 2)X_i + X_{i-1} = 0 \quad \text{with } \beta = \gamma a^2 \tag{26}$$

whose solution can be expressed by (see, for example, [11]):

$$X_i = B\sin(\phi i) + C\cos(\phi i) \quad \text{with } \phi = \arccos\left(1 - \frac{\beta}{2}\right) \tag{27}$$

From the boundary conditions, $T_0(t) = T_n(t) = 0$, which is equivalent to $X_0 = X_n = 0$, we can therefore write:

$$X_i = B\sin\left(\frac{m\pi i}{n}\right) \quad \text{with } \phi = \frac{m\pi}{n},\ m = 1, 2, 3\ldots \tag{28}$$

Introducing $\phi = \frac{m\pi}{n}, m = 1, 2, 3\ldots$ into $\phi = \arccos(1 - \frac{\gamma a^2}{2})$, we obtain:

$$\gamma = \frac{4n^2\sin^2(\frac{m\pi}{2n})}{L^2} \tag{29}$$

The general Fourier series solution can be expressed as:

$$T_i(t) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi i}{n}\right)\mathrm{e}^{\frac{-\alpha 4n^2\sin^2(\frac{m\pi}{2n})}{L^2}t} \tag{30}$$

The local solution is again found when n tends to infinity. Using the Fourier decomposition given by Eq. (17), the exact solution of the thermal lattice equation is then written as:

$$T_i^*(\tau) = \sum_{m=1,3,5,\ldots} (-1)^{(m-1)/2}\frac{8}{(m\pi)^2}\sin\left(\frac{m\pi i}{n}\right)\mathrm{e}^{-[4n^2\sin^2(\frac{m\pi}{2n})]\tau} \tag{31}$$

The temperature evolution of the lattice is represented in Figs. 2 and 3 for a lattice with $n = 4$ elements. It is clearly shown that the temperatures of the lattice model are higher than the values obtained with the local model.

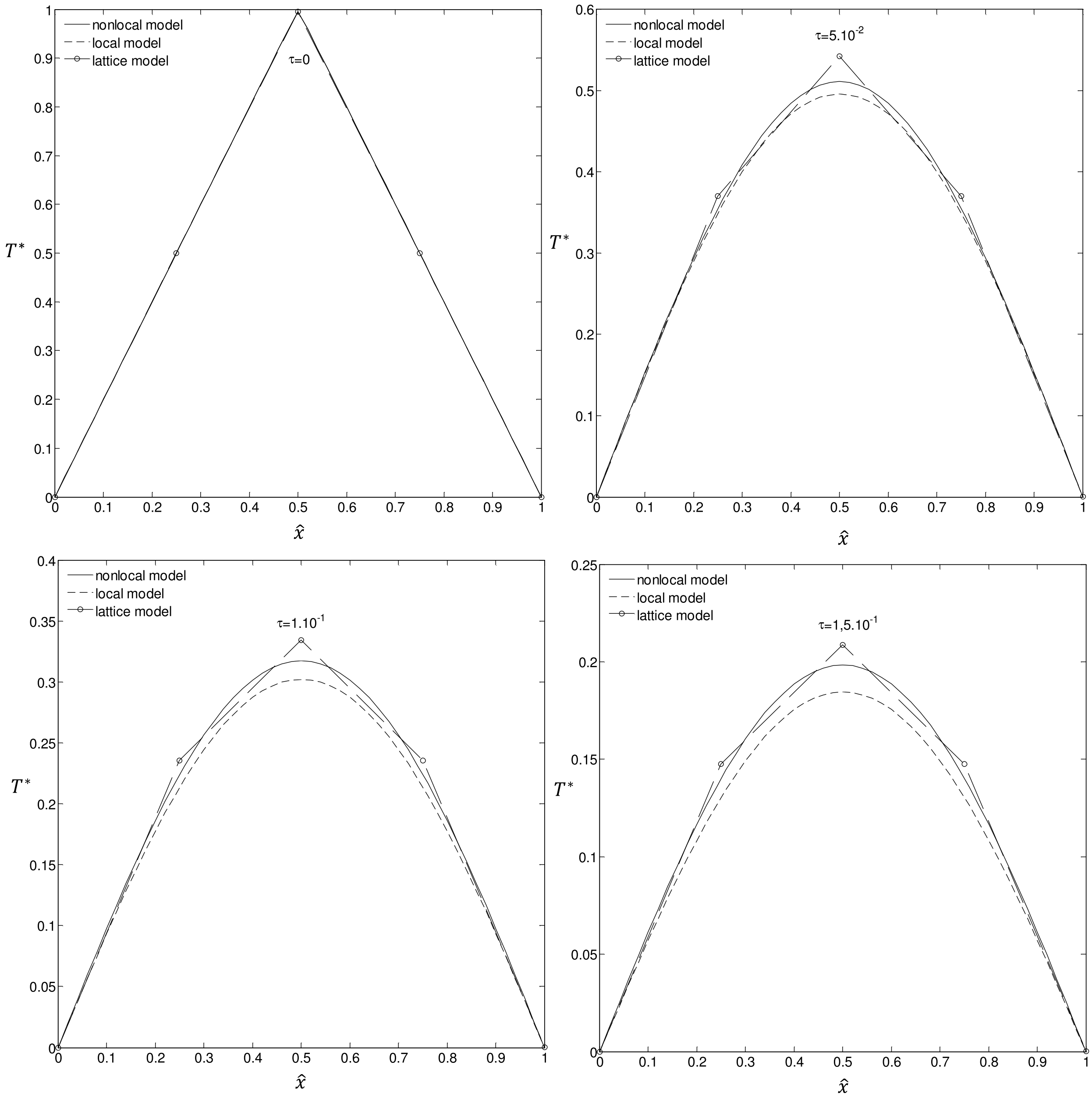

Fig. 3. Comparison of heat transfer evolution for the lattice model, the local model and the nonlocal model based on $l_c^2 = a^2/12$ and $n = 4$.

4. Identification of the length scale through microstructure analysis

Using a continualization procedure, it is possible to expand the spatial difference operators in Eq. (23) with a Taylor expansion based on $T_i = T(x = ia)$ for a sufficiently smooth temperature function:

$$T(x+a) = \sum_{k=0}^{\infty} \frac{a^k \partial_x^k}{k!} T(x) = \mathrm{e}^{a\partial_x} T(x) \quad \text{with } \partial_x = \frac{\partial}{\partial x} \tag{32}$$

This continualization method has been widely used for mechanical problems (see [5,6,8,46–48], but very few results are available for the diffusion equation. It is worth mentioning that Nielsen and Teakle [49] used a similar continualization approach for a Fick diffusion process, based on a Taylor-based asymptotic expansion of the field variable defined at discrete points.

The second-order finite difference operator can then be formulated using the pseudo-differential operator:

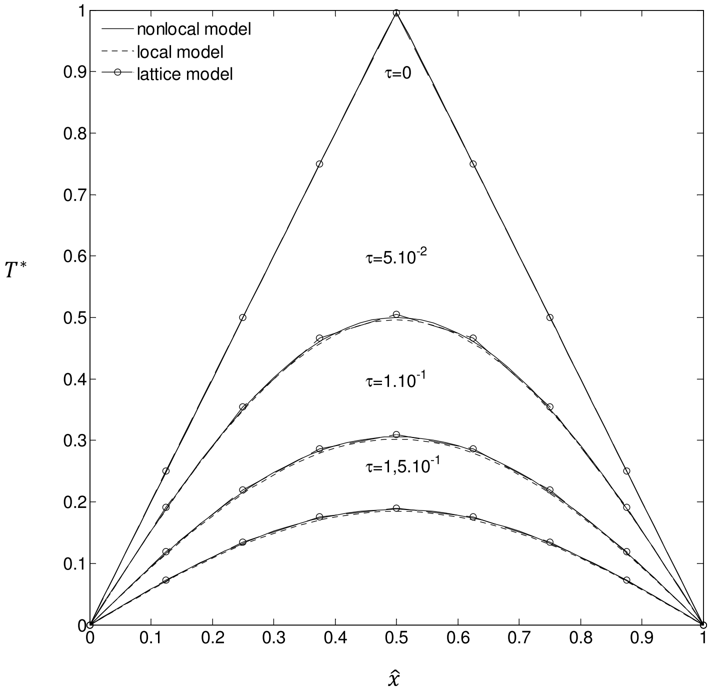

Fig. 4. Evolution of heat transfer using the lattice model, the local model and the non-local model based on $l_c^2 = a^2/12$ and $n = 8$.

$$T_{i-1} + T_{i+1} - 2T_i = \left[e^{a\partial_x} + e^{-a\partial_x} - 2\right]T(x) = 4\sinh^2\left(\frac{a}{2}\partial_x\right)T(x) \tag{33}$$

The diffusion problem is then governed by the following system of pseudo-differential equations, obtained from Eq. (33):

$$\dot{T} = 4\frac{\alpha}{a^2}\sinh^2\left(\frac{a}{2}\partial_x\right)T \tag{34}$$

Since the pseudo-differential operator can be efficiently approximated by a Padé approximant (for axial wave applications, see [8,46,48,49]):

$$\frac{4}{a^2}\sinh^2\left(\frac{a}{2}\partial_x\right) = \frac{\partial_x^2}{1 - l_c^2\partial_x^2} + \cdots \quad \text{with } l_c^2 = \frac{a^2}{12} \tag{35}$$

By inserting the rational approximation Eq. (35) in Eq. (34) and by multiplying by $1 - l_c^2\partial_x^2$, Eq. (34) can then be efficiently approximated by the second-order differential equation:

$$\dot{T} = \alpha T'' + \frac{a^2}{12}\dot{T}'' \tag{36}$$

which is strictly equivalent to Eq. (3) with the length scale correspondence $l_c^2 = \frac{a^2}{12}$.

Hence, the nonlocal diffusion equation may be physically supported by a lattice heat model. Figs. 2 and 3 show the efficiency of the quasicontinuous model (or nonlocal heat model) with respect to the lattice model for $n = 4$. Figs. 4 and 5 also compare both the nonlocal and the local models with respect to the reference lattice model for $n = 8$. The efficiency of the nonlocal model with respect to the local model is higher for small n values, i.e. in presence of strong microstructured effects. In fact, it is possible to quantify the relative error of the nonlocal model with respect to the reference lattice model from the following time-dependent function:

$$\mathrm{Err}(i,\tau) = \frac{T_i^*(\tau) - T^*(\hat{x} = i/n, \tau)}{T_i^*(\tau)} = \frac{\sum_{m=1,3,5,\ldots}(-1)^{(m-1)/2}\frac{8}{(m\pi)^2}\sin(\frac{m\pi i}{n})[e^{-[4n^2\sin^2(\frac{m\pi}{2n})]\tau} - e^{\frac{-m^2\pi^2}{1+\hat{l}_c^2 m^2\pi^2}\tau}]}{\sum_{m=1,3,5,\ldots}(-1)^{(m-1)/2}\frac{8}{(m\pi)^2}\sin(\frac{m\pi i}{n})e^{-[4n^2\sin^2(\frac{m\pi}{2n})]\tau}} \tag{37}$$

The error is maximum at the middle of the lattice bar:

$$\mathrm{Err}(n/2,\tau) = \frac{T_{n/2}^*(\tau) - T^*(\hat{x} = 1/2, \tau)}{T_{n/2}^*(\tau)} = \frac{\sum_{m=1,3,5,\ldots}\frac{8}{(m\pi)^2}[e^{-[4n^2\sin^2(\frac{m\pi}{2n})]\tau} - e^{\frac{-m^2\pi^2}{1+\hat{l}_c^2 m^2\pi^2}\tau}]}{\sum_{m=1,3,5,\ldots}\frac{8}{(m\pi)^2}e^{-[4n^2\sin^2(\frac{m\pi}{2n})]\tau}} \tag{38}$$

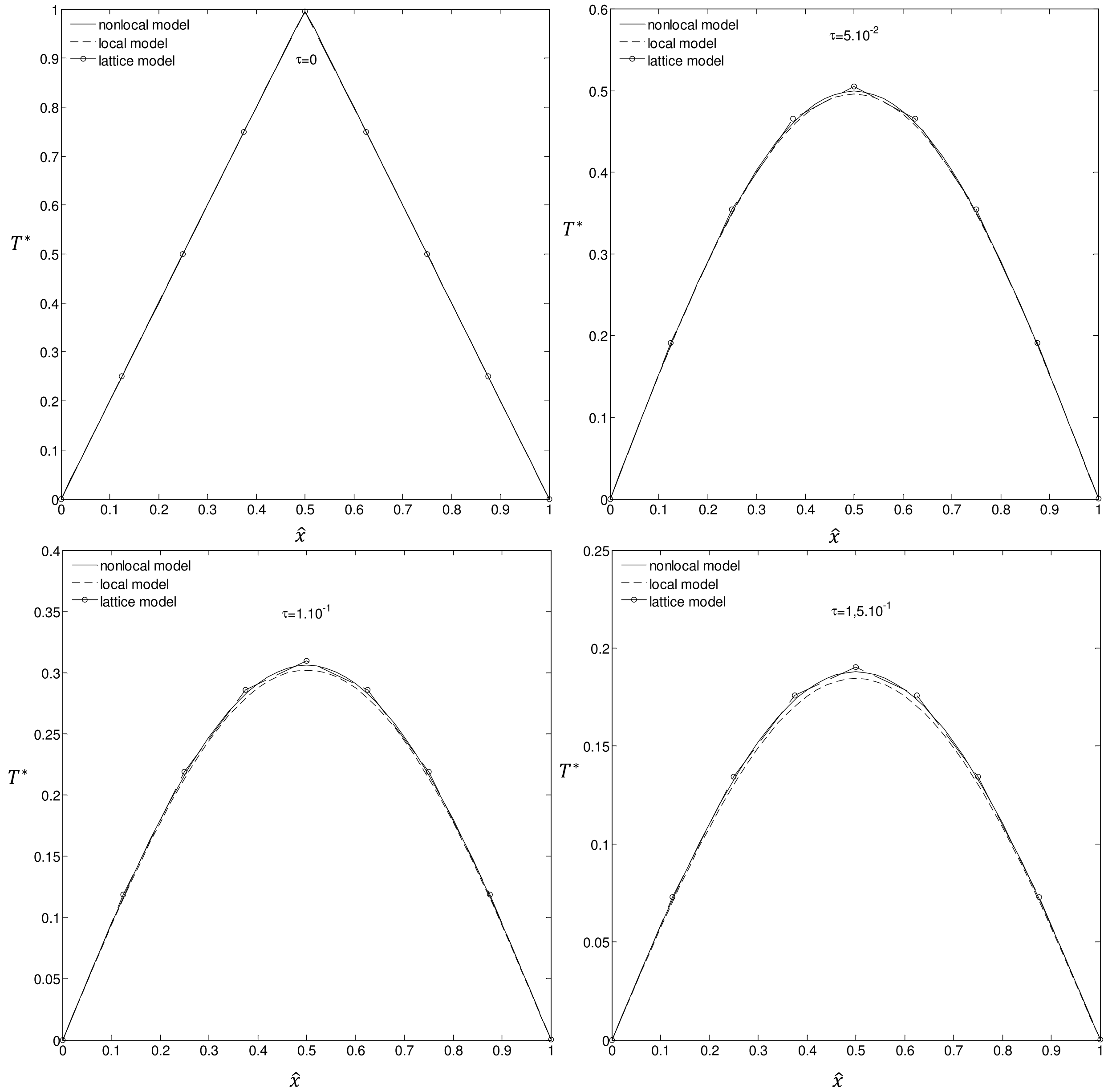

Fig. 5. Comparison of heat transfer evolution for the lattice model, the local model and the nonlocal model based on $l_c^2 = a^2/12$ and $n = 8$.

Error propagation during the diffusion process is shown in Fig. 6, and again, the superiority of the nonlocal model as compared to the local model clearly appears. Moreover, it is possible to calibrate the length scale of the nonlocal model to fit exactly the lattice model, from the fitting coefficient:

$$\beta_{\text{opt}}(n/2,\tau) = \frac{1}{n^2 \hat{l}_{\text{c,opt}}^2} \quad \text{or} \quad l_{\text{c,opt}}^2 = \frac{a^2}{\beta_{\text{opt}}(n/2,\tau)} \tag{39}$$

The best nonlocal parameter is numerically computed from:

$$\text{Err}(n/2,\tau,\beta) = \frac{T_{n/2}^*(\tau) - T^*(\hat{x}=1/2,\tau,\beta)}{T_{n/2}^*(\tau)} = \frac{\sum_{m=1,3,5,\ldots} \frac{8}{(m\pi)^2}\left[\mathrm{e}^{-[4n^2 \sin^2(\frac{m\pi}{2n})]\tau} - \mathrm{e}^{\frac{-m^2\pi^2}{1+\frac{m^2\pi^2}{\beta n^2}}\tau}\right]}{\sum_{m=1,3,5,\ldots} \frac{8}{(m\pi)^2}\mathrm{e}^{-[4n^2 \sin^2(\frac{m\pi}{2n})]\tau}} = 0 \tag{40}$$

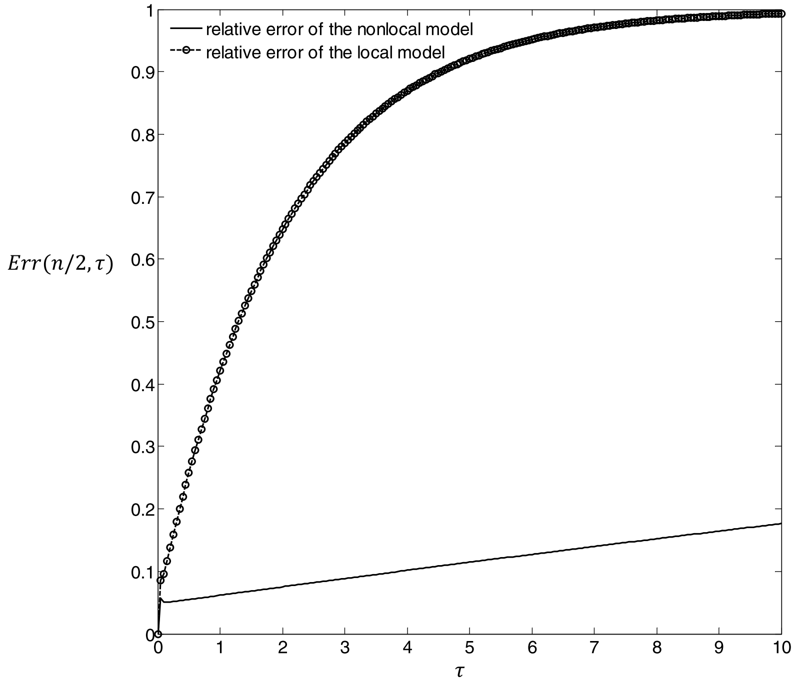

Fig. 6. Time evolution of the relative temperature error at the middle of the bar of the nonlocal model and the local model with respect to the reference lattice model for $n=4$; $\mathrm{Err}(n/2,\tau)=\frac{T^*_{n/2}(\tau)-T^*(\hat{x}=1/2,\tau)}{T^*_{n/2}(\tau)}$; $l_c^2=a^2/12$ is chosen for the lattice model.

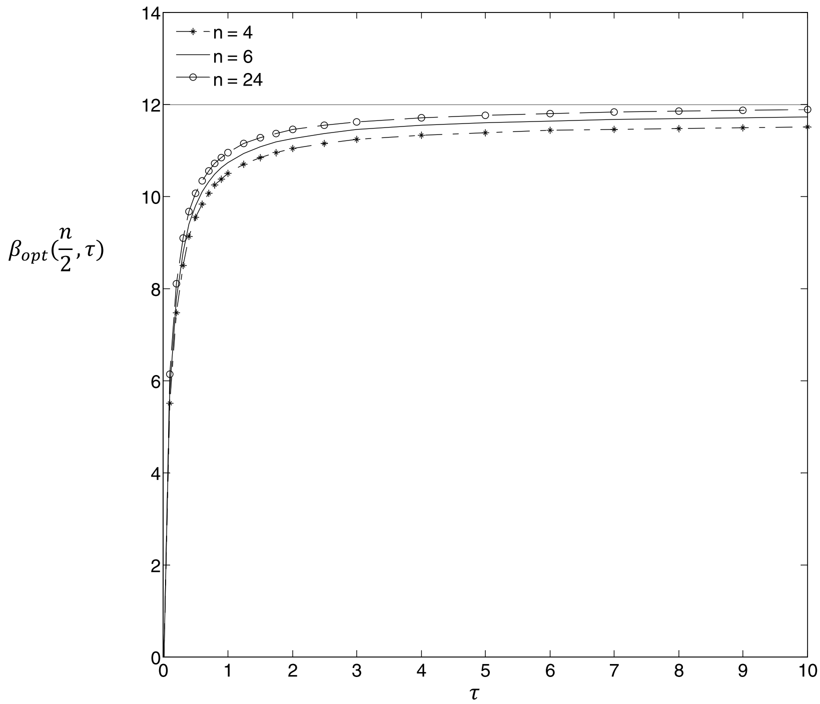

Fig. 7. Numerical evaluation of $\beta_{\mathrm{opt}}(n/2,\tau)=\frac{1}{n^2\hat{l}^2_{\mathrm{c,opt}}}$ as a function of time τ for $n=4$, $n=6$, and $n=24$ in order to have $\mathrm{Err}(n/2,\tau)=\frac{T^*_{n/2}(\tau)-T^*(\hat{x}=1/2,\tau)}{T^*_{n/2}(\tau)}=0$.

Fig. 7 shows that β_{opt} is a monotonic increasing function of time which grows up to 12, especially for larger values of n, which means that the time independent length scale $l_c^2=\frac{a^2}{12}$ is reached after a transitory time for a sufficiently large number of lattice elements.

5. Two-dimensional nonlocal heat problem

We are now considering the heat diffusion in a rectangular microstructured membrane of size $L_1 \times L_2$. The two-dimensional nonlocal heat equation is expressed from the two-dimensional generalization of Eq. (5) as:

$$\dot{T} = \alpha \Delta T + l_c^2 \Delta \dot{T} \tag{41}$$

where $\alpha = \lambda/\rho c$ is the thermal diffusivity. We are searching for a solution of this spatially nonlocal evolution equation with the following boundary and initial conditions:

$$T(x=0, y, t) = T(x=L_1, y, t) = T(x, y=0, t) = T(x, y=L_2, t) = 0 \quad \text{and} \quad T(x, y, 0) = f(x, y) \tag{42}$$

Using the method of separation of variables, based on $T(x, y, t) = X(x)Y(y)Z(t)$, the nonlocal evolution equation gives:

$$\left(1 - l_c^2 \Delta\right) XY\dot{Z} = \alpha Z \Delta(XY) \tag{43}$$

We can assume that $\frac{\dot{Z}}{Z} = -\alpha\gamma$ and then

$$-\gamma XY = \left(1 - \gamma l_c^2\right)\Delta(XY) \tag{44}$$

For the boundary conditions considered in Eq. (42), the following solution may be considered:

$$Z(t) = A\mathrm{e}^{-\alpha\gamma t} \tag{45}$$

$$X(x)Y(y) = B \sin\left[\frac{m\pi x}{L_1}\right] \sin\left[\frac{p\pi y}{L_2}\right] \tag{46}$$

Introducing the solution of Eq. (46) into the partial differential equation in Eq. (44), we obtain:

$$\gamma = \frac{(\frac{m\pi}{L_1})^2 + (\frac{p\pi}{L_2})^2}{1 + l_c^2[(\frac{m\pi}{L_1})^2 + (\frac{p\pi}{L_2})^2]} \tag{47}$$

The general Fourier series solution can be expressed as

$$T(x, y, t) = \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} A_{m,p} \sin\left(\frac{m\pi x}{L_1}\right) \sin\left(\frac{p\pi y}{L_2}\right) \mathrm{e}^{-\alpha \frac{(\frac{m\pi}{L_1})^2 + (\frac{p\pi}{L_2})^2}{1 + l_c^2[(\frac{m\pi}{L_1})^2 + (\frac{p\pi}{L_2})^2]} t} \tag{48}$$

To satisfy the initial condition, we require for $(x, y) \in [0; L_1] \times [0; L_2]$:

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} A_{m,p} \sin\left(\frac{m\pi x}{L_1}\right) \sin\left(\frac{p\pi y}{L_2}\right) \tag{49}$$

where the coefficient $A_{m,p}$ is given by (see [1]):

$$A_{m,p} = \frac{4}{L_1 L_2} \int_0^{L_1}\int_0^{L_2} f(x, y) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{p\pi y}{L_2}\right) \mathrm{d}x\mathrm{d}y \tag{50}$$

We study in this paper a two-dimensional hat-type temperature distribution characterized by:

$$f(x, y) = \begin{cases} T_0 \frac{2x}{L_1} \times \frac{2y}{L_2}, & 0 \le x \le \frac{L_1}{2}, \quad 0 \le y \le \frac{L_2}{2} \\ T_0(2 - \frac{2x}{L_1}) \times \frac{2y}{L_2}, & \frac{L_1}{2} \le x \le L_1, \quad 0 \le y \le \frac{L_2}{2} \\ T_0 \frac{2x}{L_1} \times (2 - \frac{2y}{L_2}), & 0 \le x \le \frac{L_1}{2}, \quad \frac{L_2}{2} \le y \le L_2 \\ T_0(2 - \frac{2x}{L_1}) \times (2 - \frac{2y}{L_2}), & \frac{L_1}{2} \le x \le L_1, \quad \frac{L_2}{2} \le y \le L_2 \end{cases} \tag{51}$$

It is easy to check that:

$$A_{m,p} = \frac{64 T_0}{(m\pi)^2 (p\pi)^2} \sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{p\pi}{2}\right) \tag{52}$$

and the nonlocal temperature solution is finally written as:

$$T(x, y, t) = \sum_{m=1,3,5,\ldots}\sum_{p=1,3,5,\ldots} (-1)^{(m-1)/2} (-1)^{(p-1)/2} \frac{64 T_0}{(m\pi)^2 (p\pi)^2} \sin\left(\frac{m\pi x}{L_1}\right) \sin\left(\frac{p\pi y}{L_2}\right) \mathrm{e}^{-\alpha \frac{(\frac{m\pi}{L_1})^2 + (\frac{p\pi}{L_2})^2}{1 + l_c^2[(\frac{m\pi}{L_1})^2 + (\frac{p\pi}{L_2})^2]} t} \tag{53}$$

6. Two-dimensional thermal lattice – exact solution and continualization approach

The heat diffusion in the rectangular microstructured membrane of size $L_1 \times L_2$ with $L_1 = n_1 a$ and $L_2 = n_2 a$ is studied, where a is the lattice spacing in each direction. Following Rosenau's reasoning for mechanical two-dimensional lattices [49], the thermal lattice difference equations are written by:

$$\dot{T}_{i,j} = \alpha \frac{T_{i+1,j} + T_{i,j+1} + T_{i-1,j} + T_{i,j-1} - 4T_{i,j}}{a^2} \tag{54}$$

where a is the lattice spacing in each direction of the microstructured membrane. One recognizes the spatial finite difference equations of the local heat equation.

We are searching for a solution of these two-dimensional time-dependent lattice equations with the following boundary and initial conditions:

$$T_{0,j}(t) = T_{n_1,j}(t) = T_{i,0}(t) = T_{i,n_2}(t) = 0 \quad \text{and} \quad T_{i,j}(t=0) = f_{i,j} \tag{55}$$

The initial conditions are the same as the ones of the nonlocal two-dimensional problem, i.e. $f_{i,j} = f(ai, aj)$ in Eq. (51).

Following the methodology already described for the nonlocal heat equation, we use the method of separation of variables, based on $T_{i,j}(t) = X_i Y_j Z(t)$, thus leading to:

$$X_i Y_j \frac{\dot{Z}}{Z} = \alpha \left[\frac{X_{i+1}Y_j + X_i Y_{j+1} + X_{i-1}Y_j + X_i Y_{j-1} - 4X_i Y_j}{a^2} \right] \tag{56}$$

We can assume that $\dot{Z} = -\alpha\gamma Z$, which is easily integrated:

$$Z(t) = A\mathrm{e}^{-\alpha\gamma t} \tag{57}$$

and then, the spatial difference equation in space to be solved is written by:

$$X_{i+1}Y_j + X_i Y_{j+1} + X_{i-1}Y_j + X_i Y_{j-1} + (\gamma a^2 - 4) X_i Y_j = 0 \tag{58}$$

the solution to which, for the boundary conditions given in Eq. (55), can be expressed by:

$$X_i Y_j = B \sin\left[\frac{m\pi a i}{L_1}\right] \sin\left[\frac{p\pi a j}{L_2}\right] \tag{59}$$

Introducing the solution of Eq. (59) in Eq. (58), we obtain:

$$\gamma = \frac{4}{a^2}\left[\sin^2\left(\frac{m\pi a}{2L_1}\right) + \sin^2\left(\frac{p\pi a}{2L_2}\right)\right] \tag{60}$$

The general Fourier series solution can be expressed for the two-dimensional lattice as:

$$T_{i,j}(t) = \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} A_{m,p} \sin\left(\frac{m\pi i}{n_1}\right) \sin\left(\frac{p\pi j}{n_2}\right) \mathrm{e}^{-\alpha \frac{4}{a^2}[\sin^2(\frac{m\pi}{2n_1}) + \sin^2(\frac{p\pi}{2n_2})]t} \tag{61}$$

The local solution is again found when n_1 and n_2 tend to infinity. For a square lattice, we have $L_1 = L_2$ and $n_1 = n_2$. The maximum temperature at the centre of the square lattice can then be expressed in a dimensionless form as:

$$T^*_{n/2,n/2}(\tau) = \sum_{m=1,3,5,\ldots}\ \sum_{p=1,3,5,\ldots} \frac{64}{(m\pi)^2(p\pi)^2} \mathrm{e}^{-4n^2[\sin^2(\frac{m\pi}{2n}) + \sin^2(\frac{p\pi}{2n})]\tau} \tag{62}$$

We will now show that the nonlocal heat model can be derived from continualization of the lattice equations, following the continualization reasoning of Rosenau [50], Andrianov and Awrejcewicz [51] or Lombardo and Askes [52] for mechanical lattices. The thermal lattice equations Eq. (54) can be continualized from the pseudopartial-differential operator:

$$\dot{T} = 4\frac{\alpha}{a^2}\left[\sinh^2\left(\frac{a}{2}\partial_x\right) + \sinh^2\left(\frac{a}{2}\partial_y\right)\right] T \tag{63}$$

A rational-based asymptotic expansion leads to:

$$\dot{T} = \alpha \left[\frac{\partial_x^2}{1 - \frac{a^2}{12}\partial_x^2} + \frac{\partial_y^2}{1 - \frac{a^2}{12}\partial_y^2} \right] T \tag{64}$$

which can be approximated by the following form:

$$\left(1 - \frac{a^2}{12}\Delta\right)\dot{T} = \alpha \left[\Delta T - \frac{a^2}{6}\frac{\partial^4 T}{\partial x^2 \partial y^2}\right] \tag{65}$$

Some similar differential operators have been obtained by Rosenau [50], Andrianov and Awrejcewicz [51] or Lombardo and Askes [52] for the mechanical lattice. If the last coupling term is omitted (as discussed by Rosenau [50] for the mechanical lattice), this continualized model reduces to the simplified nonlocal heat equation considered in Eq. (41) with $l_c^2 = a^2/12$:

$$\left(1 - \frac{a^2}{12}\Delta\right)\dot{T} = \alpha\Delta T \tag{66}$$

which can be obtained from the nonlocal two-dimensional formulation:

$$\underline{q} - l_c^2 \underline{\underline{\nabla^2}}\underline{q} = -\lambda\underline{\nabla}T \quad \text{and} \quad \rho c\dot{T} = -\underline{\nabla}.\underline{q} \quad \text{with } \underline{q} = \begin{pmatrix} q_x \\ q_y \end{pmatrix} \text{ and } \underline{\nabla} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \tag{67}$$

with $\underline{\underline{\nabla^2}} = \Delta\underline{\underline{1}}$ and $\underline{\nabla}$ is the gradient operator. This is an Eringen's type nonlocal Fourier law. Eq. (66) can be also obtained from the following alternative nonlocal law:

$$\underline{q} - l_c^2 \underline{\nabla}(\underline{\nabla}\cdot\underline{q}) = -\lambda\underline{\nabla}T \quad \text{and} \quad \rho c\dot{T} = -\underline{\nabla}\cdot\underline{q} \tag{68}$$

Note that the two nonlocal Fourier laws differ, even if the partial differential equation for the temperature coincides, due to the fact that:

$$\underline{\underline{\nabla^2}} = \Delta\underline{\underline{1}} = \begin{pmatrix} \partial_x^2 + \partial_y^2 & 0 \\ 0 & \partial_x^2 + \partial_y^2 \end{pmatrix} \neq \begin{pmatrix} \partial_x^2 & \partial_x\partial_y \\ \partial_x\partial_y & \partial_y^2 \end{pmatrix} \tag{69}$$

For the square lattice, the nonlocal approximation of the dimensionless temperature in the centre, based on the continualization of Eq. (66) is then given by:

$$T^*_{n/2,n/2}(\tau) = \sum_{m=1,3,5,\ldots}\sum_{p=1,3,5,\ldots} \frac{64}{(m\pi)^2(p\pi)^2} \mathrm{e}^{-\frac{(m\pi)^2+(p\pi)^2}{1+\frac{1}{12n^2}[(m\pi)^2+(p\pi)^2]}\tau} \tag{70}$$

Considering now the continualization model given by Eq. (65), and following the reasoning presented for the truncated nonlocal model, the temperature field for the rectangular membrane would be calculated as:

$$T(x,y,t) = \sum_{m=1,3,5,\ldots}\sum_{p=1,3,5,\ldots} (-1)^{(m-1)/2}(-1)^{(p-1)/2} \frac{64T_0}{(m\pi)^2(p\pi)^2} \times \sin\left(\frac{m\pi x}{L_1}\right)\sin\left(\frac{p\pi y}{L_2}\right) \mathrm{e}^{-\alpha\frac{(\frac{m\pi}{L_1})^2+(\frac{p\pi}{L_2})^2+\frac{a^2}{6}(\frac{m\pi}{L_1})^2(\frac{p\pi}{L_2})^2}{1+\frac{a^2}{12}[(\frac{m\pi}{L_1})^2+(\frac{p\pi}{L_2})^2]}t} \tag{71}$$

For the square lattice, the nonlocal approximation of the dimensionless temperature in the centre, based on the continualization of Eq. (65) is then given by:

$$T^*_{n/2,n/2}(\tau) = \sum_{m=1,3,5,\ldots}\sum_{p=1,3,5,\ldots} \frac{64}{(m\pi)^2(p\pi)^2} \mathrm{e}^{-\frac{(m\pi)^2+(p\pi)^2+\frac{(m\pi)^2(p\pi)^2}{6n^2}}{1+\frac{1}{12n^2}[(m\pi)^2+(p\pi)^2]}\tau} \tag{72}$$

The thermal evolution at the centre of the squared membrane is plotted in Fig. 8, and compared to the nonlocal heat model, the continualized heat model and the local one for $n = 4$. The continualized model and the nonlocal one lead to very close responses, and approximated very efficiently the response of the lattice. In Fig. 9, the relative error is plotted with respect to the time. It is shown that the continualized nonlocal model – Eq. (65) – gets better result than the truncated nonlocal model – Eq. (66) –, especially for a sufficiently large time.

7. Conclusions

In this paper, we show that a microstructured lattice during heat transfer behaves as a quasicontinuum governed by a nonlocal Fourier's law. The nonlocal Fourier's law is similar to the differential model of Eringen [4] for mechanical elastic interactions. The nonlocal length scale is calibrated from the lattice spacing using a continualization procedure. It is shown that the nonlocality is similar for thermal and mechanical behaviour, with the same length scale for both phenomena. The results are valid for one-dimensional thermal lattice and two-dimensional thermal lattices, even if the two-dimensional continualized model may slightly differ from a phenomenological differential-based nonlocal thermal model. Definitely, nonlocal heat models are comforted by lattice arguments and inherently possess some length scale that can be decisive in presence of strong microstructured effects.

We have not explored the possibility for the time variable to belong to a discrete space (see Lee [53]), as we implicitly assumed a steady flow with respect to time. However, it is formally also possible to relax this assumption for coupled time–space discrete problems associated with both time and space nonlocality.

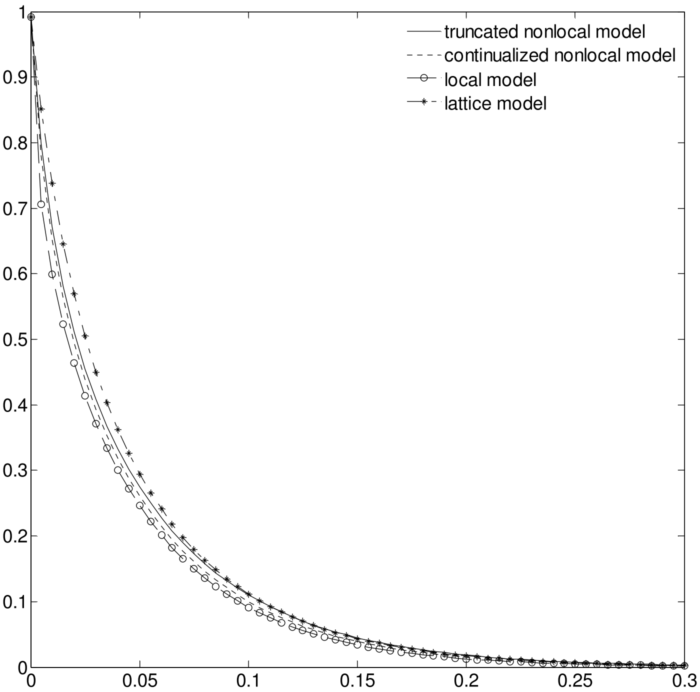

Fig. 8. Evolution of $T^*_{n/2,n/2}(\tau)$ for $n = 4$ in the centre of the square membrane.

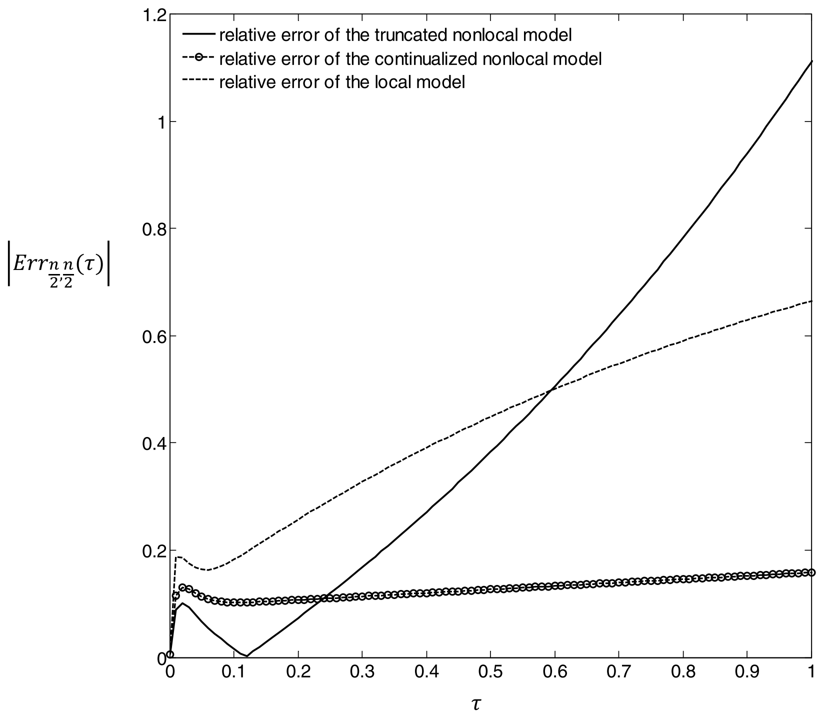

Fig. 9. Evolution of the relative error $|\,\mathrm{Err}(n/2, n/2, \tau)| = \frac{|T^*_{n/2,n/2}(\tau) - T^*(\hat{x}=1/2, \hat{y}=1/2, \tau)|}{T^*_{n/2,n/2}(\tau)}$ for $n = 4$ in the centre of the square membrane.

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