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Quasi-static response, implicit scheme and incremental problem in gradient plasticity

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ABSTRACT

This paper is devoted to the study of gradient plasticity at small strains. Some time-independent dissipative processes such as brittle damage can also be considered in the same framework. Our attention is focussed on the description of the constitutive equations, on the formulation of the governing equations in terms of the energy potential and the dissipation potential of the solid. A time-discretization by the implicit scheme of the evolution equation leads to the study of the incremental problem which is different from the rate problem. The increment of the response under an increment of the loads must satisfy a variational inequality and, if the energy potential is convex, an incremental minimum principle. In particular, a local minimum of the incremental minimum principle is a stable solution to the variational inequality.

1. Introduction

Since the two last decades, gradient theories have been much discussed in elasticity, in plasticity as in damage mechanics, cf. for example [1–7]. This paper is devoted to the study of gradient plasticity at small strains. A general and consistent description including gradient plasticity and some time-independent processes such as brittle damage is presented. Our attention is focussed on the formulation of the constitutive equations and the derivation of the governing equations for the response of a solid under a loading path *in terms of the expression of the energy potential and the dissipation potential of the solid*. Such a synthetic description, still lacking in the literature, appears to be interesting for an overview on the subject. It enables us to include in the same framework all general statements that result from the basic ingredients of the theory such as the evolution equation in quasi-statics and the associated variational principles. In view of numerical applications, a time-discretization by the implicit scheme of the evolution equation is introduced. It leads to the formulation of the incremental problem which is different from the rate problem.

2. Gradient theory of plasticity and standard time-independent processes

In an isothermal transformation, the mechanical response of a solid V is described by the fields of displacement $\mathbf{u}$, of internal parameter $\boldsymbol{\Phi}$. The internal parameter is a scalar or a tensor and represents physically hidden parameters such as micro-displacements or phase proportions or an elastic strains, etc. The set of state variables $(\epsilon, \phi, \nabla\phi)$, with $\epsilon = \nabla_s u$, describes the material behavior and the constitutive equations can be given in the following way, cf. [1,8,2]:

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It is accepted that the rates $(\dot{\epsilon},\ \dot{\phi},\ \nabla\dot{\phi})$ of the state variables are associated with the generalized forces (σ, X, Y) such that a generalized virtual work equation holds

$$\begin{cases} P_i + P_j = P_e \quad \forall \quad \delta\mathbf{u}, \delta\boldsymbol{\Phi} \\ P_i = \int_V (\sigma\cdot\delta\epsilon + X\cdot\delta\phi + Y\cdot\nabla\delta\phi)\,\mathrm{d}V, \\ P_j = \int_V \rho\ddot{u}\cdot\delta u\,\mathrm{d}V, \\ P_e = \int_V (F_{uv}\cdot\delta u + F_{\phi v}\cdot\delta\phi)\,\mathrm{d}V + \int_{\partial V}(F_{us}\cdot\delta u + F_{\phi s}\cdot\delta\phi)\,\mathrm{d}a \end{cases} \tag{1}$$

where (F_{uv}, F_{us}) and $(F_{\phi v}, F_{\phi s})$ are respectively external body and surface forces associated with the displacement and the internal parameter.

Standard gradient models of plasticity also assume that there exists per unit volume an energy potential which is a smooth function $W(\epsilon, \phi, \nabla\phi)$ associated with the energy forces $\sigma, X_\mathrm{e}, Y_\mathrm{e}$:

$$\sigma = W_{,\epsilon}\,,\quad X_\mathrm{e} = W_{,\phi}\,,\quad Y_\mathrm{e} = W_{,\nabla\phi} \tag{2}$$

and a dissipation potential $D(\dot{\phi}, \nabla\dot{\phi})$ which is a convex and positively homogeneous function of degree 1

$$D(a\dot{\phi}, a\nabla\dot{\phi}) = aD(\dot{\phi}, \nabla\dot{\phi}) \ \forall \ \ a \geq 0 \tag{3}$$

associated with the dissipative forces

$$X_\mathrm{d} = \partial_{\dot{\phi}}\, D(\dot{\phi}, \nabla\dot{\phi}),\quad Y_\mathrm{d} = \partial_{\nabla\dot{\phi}}\, D(\dot{\phi}, \nabla\dot{\phi}) \tag{4}$$

such that the following equations hold:

$$X = X_\mathrm{e} + X_\mathrm{d},\quad Y = Y_\mathrm{e} + Y_\mathrm{d} \tag{5}$$

In (4), the derivatives must be understood in the sense of sub-gradients of a convex function, cf. for example [9]. The dissipation potential can be state-dependent, for example via the history of the state variable ϕ.

2.1. Standard models of gradient plasticity and brittle damage

For example, the following model has been discussed by Fleck et al., [4] with $\phi = \epsilon^\mathrm{p}$ and

$$\begin{cases} W(\epsilon, \epsilon^\mathrm{p}) = \frac{1}{2}(\epsilon - \epsilon^\mathrm{p}) : L : (\epsilon - \epsilon^\mathrm{p}), \\ D(\dot{\epsilon}^\mathrm{p}, \nabla\dot{\epsilon}^\mathrm{p}) = R(\gamma)\sqrt{\|\dot{\epsilon}^\mathrm{p}\|^2 + \ell^2\|\nabla\dot{\epsilon}^\mathrm{p}\|^2}, \\ \gamma = \int_0^t \sqrt{\|\dot{\epsilon}^\mathrm{p}\|^2 + \ell^2\|\nabla\dot{\epsilon}^\mathrm{p}\|^2})\|\mathrm{d}\tau \end{cases} \tag{6}$$

with the notation $\|\dot{\epsilon}^\mathrm{p}\| = \sqrt{\dot{\epsilon}^\mathrm{p}_{ij}\dot{\epsilon}^\mathrm{p}_{ij}}$ and $\|\nabla\dot{\epsilon}^\mathrm{p}\| = \sqrt{\dot{\epsilon}^\mathrm{p}_{ij,k}\dot{\epsilon}^\mathrm{p}_{ij,k}}$.

Here, the dissipation potential is state-dependent via the expression of γ. As in classical plasticity, the model leads to a plastic criterion $f(X^\mathrm{p}_\mathrm{d}, Y^\mathrm{p}_\mathrm{d}) \leq 0$, which defines the set of physically admissible forces, and to the normality law:

$$\begin{cases} f = (\|X^\mathrm{p}_\mathrm{d}\|^2 + \frac{1}{\ell^2}\|Y^\mathrm{p}_\mathrm{d}\|^2)^{1/2} - R(\gamma) \leq 0, \\ \dot{\epsilon}^\mathrm{p} = \lambda\frac{\partial f}{\partial X^\mathrm{p}_\mathrm{d}},\quad \nabla\dot{\epsilon}^\mathrm{p} = \lambda\frac{\partial f}{\partial Y^\mathrm{p}_\mathrm{d}},\quad \lambda \geq 0,\quad f\lambda = 0, \\ \dot{\gamma} = \lambda \end{cases} \tag{7}$$

The dissipation is

$$d = X^\mathrm{p}_\mathrm{d}\cdot\dot{\epsilon}^\mathrm{p} + Y^\mathrm{p}_\mathrm{d}\cdot\nabla\dot{\epsilon}^\mathrm{p} = R(\gamma)\lambda = \frac{\mathrm{d}}{\mathrm{d}t}W_\mathrm{d}(\gamma) \ \text{ with } \ R = W'_\mathrm{d}(\gamma) \tag{8}$$

$W_\mathrm{d}(\gamma)$ is the dissipated energy.

This model can be easily modified to obtain a state-independent dissipation potential. For this, the following model is introduced with $\phi = (\epsilon^\mathrm{p}, \gamma)$

$$\begin{cases} W(\epsilon, \epsilon^\mathrm{p}, \gamma) = \frac{1}{2}(\epsilon - \epsilon^\mathrm{p}) : L : (\epsilon - \epsilon^\mathrm{p}) + H(\gamma), \\ D(\dot{\epsilon}^\mathrm{p}, \dot{\gamma}, \nabla\dot{\epsilon}^\mathrm{p}) = k\sqrt{\|\dot{\epsilon}^\mathrm{p}\|^2 + \ell^2\|\nabla\dot{\epsilon}^\mathrm{p}\|^2} + \Psi_\mathrm{o}(\dot{\gamma} - \sqrt{\|\dot{\epsilon}^\mathrm{p}\|^2 + \ell^2\|\nabla\dot{\epsilon}^\mathrm{p}\|^2}) \end{cases} \tag{9}$$

where k is a constant and Ψ_o the indicator function

$$\Psi_\mathrm{o}(a) = 0 \ \text{ if } \ a = 0 \ \text{ and } \ \Psi_\mathrm{o}(a) = +\infty \ \text{ if } \ a \neq 0 \tag{10}$$

which ensures the constraint

$$\dot{\gamma} = \sqrt{\|\dot{\epsilon}^{\mathrm{p}}\|^2 + \ell^2 \|\nabla \dot{\epsilon}^{\mathrm{p}}\|^2}$$

From (9), the considered model leads to the plastic criterion

$$f(X_{\mathrm{d}}^{\mathrm{p}}, X_{\mathrm{d}}^{\gamma}, Y_{\mathrm{d}}^{\mathrm{p}}) = (\|X_{\mathrm{d}}^{\mathrm{p}}\|^2 + \frac{1}{\ell^2}\|Y_{\mathrm{d}}^{p}\|^2)^{1/2} + X_{\mathrm{d}}^{\gamma} - k \le 0 \tag{11}$$

and to the normality law

$$\begin{cases} \dot{\epsilon}^{\mathrm{p}} = \lambda \frac{\partial f}{\partial X_{\mathrm{d}}^{\mathrm{p}}}, \quad \dot{\gamma} = \lambda \frac{\partial f}{\partial X_{\mathrm{d}}^{\gamma}} = \lambda, \quad \nabla \dot{\epsilon}^{\mathrm{p}} = \lambda \frac{\partial f}{\partial Y_{\mathrm{d}}^{\mathrm{p}}}, \\ f \le 0, \quad \lambda \ge 0, \quad f\lambda = 0 \end{cases} \tag{12}$$

The dissipation is

$$d = X_{\mathrm{d}}^{\mathrm{p}} \cdot \dot{\epsilon}^{\mathrm{p}} + X_{\mathrm{d}}^{\gamma}\ \dot{\gamma} + Y_{\mathrm{d}}^{\mathrm{p}} \cdot \nabla \dot{\epsilon}^{\mathrm{p}} = k\lambda = k\dot{\gamma} = k\sqrt{\|\dot{\epsilon}^{\mathrm{p}}\|^2 + \ell^2 \|\nabla \dot{\epsilon}^{\mathrm{p}}\|^2}$$

which gives here the physical interpretation of $k\gamma$ as the dissipated energy.

Since $X_{\mathrm{d}}^{\gamma} = -X_{\mathrm{e}}^{\gamma} = H'(\gamma)$, the constitutive equation (6) is recovered with $H'(\gamma) + k = R(\gamma)$. In this model, W_{d} is the work done by plastic deformation and consists of the dissipated energy $k\gamma$ and the stored energy $H(\gamma)$.

In the same spirit, an interesting model of isotropic hardening is given by

$$\begin{cases} W(\epsilon, \epsilon^{\mathrm{p}}, \gamma, \nabla\gamma) = \frac{1}{2}(\epsilon - \epsilon^{\mathrm{p}}) : L : (\epsilon - \epsilon^{\mathrm{p}}) + H(\gamma) + \frac{g}{2}\nabla\gamma^2, \\ D(\dot{\epsilon}^{\mathrm{p}}, \dot{\gamma}, \nabla\dot{\gamma}) = k\|\dot{\epsilon}^{\mathrm{p}}\| + \kappa\|\nabla\dot{\gamma}\| + \Psi_{\mathrm{o}}(\dot{\gamma} - \|\dot{\epsilon}^{\mathrm{p}}\|) \end{cases} \tag{13}$$

The plastic criterion is given by two inequalities

$$f(X_{\mathrm{d}}^{\mathrm{p}}, X_{\mathrm{d}}^{\gamma}) = \|X_{\mathrm{d}}^{\mathrm{p}}\| + X_{\mathrm{d}}^{\gamma} - k \le 0, \quad \varphi(Y_{\mathrm{d}}^{\gamma}) = \|Y_{\mathrm{d}}^{\gamma}\| - \kappa \le 0 \tag{14}$$

and the normality law is

$$\begin{cases} \dot{\epsilon}^{\mathrm{p}} = \lambda \frac{\partial f}{\partial X_{\mathrm{d}}^{\mathrm{p}}}, \quad \dot{\gamma} = \lambda \frac{\partial f}{\partial X_{\mathrm{d}}^{\gamma}} \ \text{ with } \ f \le 0, \quad \lambda \ge 0, \quad \lambda f = 0, \\ \nabla\dot{\gamma} = \tau \frac{\partial \varphi}{\partial Y_{\mathrm{d}}^{\gamma}} \ \text{ with } \varphi \le 0, \quad \tau \ge 0, \quad \tau\varphi = 0 \end{cases} \tag{15}$$

The reader can also refer to [10,6,5] for interesting discussions on a model of energy $W = \frac{1}{2}(\epsilon - \epsilon^{\mathrm{p}}) : L : (\epsilon - \epsilon^{\mathrm{p}}) + \frac{1}{2}\mathrm{curl}(\epsilon^{\mathrm{p}}) : E : \mathrm{curl}(\epsilon^{\mathrm{p}})$. Here the energy potential depends on the gradient of the plastic strain via the operator curl.

The same framework also includes other standard time-independent irreversible processes. For example, the following model

$$\begin{cases} W(\epsilon, \theta, \nabla\theta) = \frac{1-\theta}{2}\epsilon : L : \epsilon + \frac{1}{2}(a\theta^2 + b\|\nabla\theta\|^2) + \Psi_{\theta \le 1-r}(\theta), \\ D(\dot{\theta}) = \bar{k}\dot{\theta} + \Psi_{\dot{\theta} \ge 0}(\dot{\theta}) \end{cases} \tag{16}$$

gives a simple modeling of brittle damage in an elastic solid. Here, $\theta \ge 0$ is the damage parameter, $\theta = 0$ for a sane state and $\theta = 1 - r$ for a total damaged state of the material. The elastic rigidity of the material varies with damage from a natural value L to a residual value rL. The indicator functions $\Psi_{\theta \le 1-r}(\theta)$ and $\Psi_{\dot{\theta} \ge 0}(\dot{\theta})$ ensure the constraints $\theta \le 1 - r$ and $\dot{\theta} \ge 0$. The interest of the terms $a\theta^2$ and $b\|\nabla\theta\|^2$ in the expression of the energy has been discussed in [1,3,11] for example. This dissipation potential leads to the damage criterion and the normality law

$$\bar{f}(X_{\mathrm{d}}^{\theta}) = X_{\mathrm{d}}^{\theta} - \bar{k} \le 0, \quad \dot{\theta} = \lambda \frac{\partial \bar{f}}{\partial X_{\mathrm{d}}^{\theta}} = \lambda \ge 0, \quad \lambda \bar{f} = 0$$

A model of brittle damage in an elastic-plastic solid can also be introduced in the same spirit:

$$\begin{cases} W(\epsilon, \epsilon^{\mathrm{p}}, \gamma, \theta, \nabla\gamma, \nabla\theta) = \frac{1-\theta}{2}(\epsilon - \epsilon^{\mathrm{p}}) : L : (\epsilon - \epsilon^{\mathrm{p}}) + \frac{1}{2}(a\theta^2 + b\|\nabla\theta\|^2) + \Psi_{\theta \le 1-r}(\theta) \\ \qquad\qquad + \frac{1}{2}(h\gamma^2 + g\|\nabla\gamma\|^2), \\ D(\dot{\theta}, \dot{\epsilon}^{\mathrm{p}}, \dot{\gamma}) = \bar{k}\dot{\theta} + \Psi_{\dot{\theta} \ge 0}(\dot{\theta}) + k\|\dot{\epsilon}^{\mathrm{p}}\| + \Psi_{\mathrm{o}}(\dot{\gamma} - \|\dot{\epsilon}^{\mathrm{p}}\|) \end{cases} \tag{17}$$

2.2. Governing equations for a solid under a loading path

For a solid submitted to a classical loading path, defined by the body forces $F_{uv}(x,t), F_{\phi,v}(x,t)$, the surface forces $F_{us}(x,t), F_{\phi,s}(x,t)$ and the imposed displacement $u_g(x,t)$, the response of the solid must satisfy the local equations

$$\begin{cases} -\ \forall t \in [0,T]: \\ \sigma = W_{,\epsilon(u)}, \quad X_{\mathrm{e}} = W_{,\phi}, \quad Y_{\mathrm{e}} = W_{,\nabla\phi}, \\ X = X_{\mathrm{e}} + X_{\mathrm{d}}, \quad Y = Y_{\mathrm{e}} + Y_{\mathrm{d}}, \quad (X_{\mathrm{d}}, Y_{\mathrm{d}}) = \partial D(\dot\phi, \nabla\dot\phi), \\ \nabla\cdot\sigma + F_{uv} = \rho\ddot{u}, \quad X + \nabla\cdot Y + F_{\phi v} = 0 \text{ in } V, \\ \sigma\cdot n = F_{us} \text{ on } \partial V_f, \quad u = u_g \text{ on } \partial V_u, \\ Y\cdot n = F_{\phi s} \text{ on } \partial V \\ -\ \text{At } t=0: \\ \mathbf{u}(\mathbf{0}) = \mathbf{u_o}, \quad \phi(\mathbf{0}) = \phi_{\mathbf{o}}, \quad \dot{\mathbf{u}}(\mathbf{0}) = \mathbf{v_o} \end{cases} \tag{18}$$

These equations describe the response of the solid from an initial position of state and velocity.

3. The quasi-static response

It is convenient to introduce as a condensed notation the general displacement $\mathbf{U} = (\mathbf{u}, \mathbf{\Phi})$ to write simply the energy and dissipation potentials of the solid as:

$$\mathbf{W}(\mathbf{U}) = \int_V W(\epsilon, \phi, \nabla\phi)\,\mathrm{d}V, \quad \mathbf{D}(\dot{\mathbf{U}}) = \int_V D(\dot\phi, \nabla\dot\phi)\,\mathrm{d}V \tag{19}$$

In the sequel, the assumption of state-independent dissipation is accepted. The conditions $F_{\phi v} = 0$ and $F_{\phi s} = 0$, although not essential, are also admitted as in (18).

3.1. Evolution equation

In quasi-static transformation, a variational and condensed form of the evolution equation for the solid can be introduced as in Classical Plasticity, cf. [9].

Evolution variational inequality For all $t \in [0,T]$, the quasi-static response $\mathbf{U}(\mathbf{t})$ of the solid submitted to a given loading path $\mathbf{F_g}(\mathbf{t}),\ \mathbf{u_g}(\mathbf{t})$ satisfies the following variational inequality[1]:

$$\mathbf{W}_{,\mathbf{U}}(\mathbf{U})\cdot(\dot{\mathbf{U}}^* - \dot{\mathbf{U}}) + \mathbf{D}(\dot{\mathbf{U}}^*) - \mathbf{D}(\dot{\mathbf{U}}) - \mathbf{F_g}\cdot(\dot{\mathbf{U}}^* - \dot{\mathbf{U}}) \geq \mathbf{0} \tag{20}$$

for all admissible response $\mathbf{U}^*(\mathbf{t})$, i.e. satisfying the imposed condition $\mathbf{U}^*(\mathbf{t}) = \mathbf{u_g}(\mathbf{t})$ on ∂V_u.

This variational inequality means explicitly that

$$\begin{cases} \int_V \sigma : (\dot\epsilon^* - \dot\epsilon)\,\mathrm{d}V - \int_V F_{vu}\cdot(\dot u^* - \dot u)\,\mathrm{d}V - \int_{\partial V_F} F_{gsu}\cdot(\dot u^* - \dot u)\,\mathrm{d}S \\ + \int_V (X_{\mathrm{e}}\cdot(\dot\phi^* - \dot\phi) + Y_{\mathrm{e}}\cdot(\nabla\dot\phi^* - \nabla\dot\phi) + D(\dot\phi^*, \nabla\dot\phi^*) - D(\dot\phi, \nabla\dot\phi))\,\mathrm{d}V \geq 0 \end{cases} \tag{21}$$

for all $(\mathbf{u}^*, \mathbf{\Phi}^*)$ admissible.

Thus, for all t, it follows from the evolution variational inequality that the equilibrium equation holds

$$\mathbf{W}_{,\mathbf{u}}(\mathbf{U})\cdot\delta\mathbf{u} - \mathbf{F_{gu}}\cdot\delta\mathbf{u} = 0 \ \ \forall\ \delta\mathbf{u} = 0 \ \text{ on } \partial V_u$$

and that $\dot{\mathbf{\Phi}}$ satisfies the following minimum principle:

$$\mathbf{I}(\dot{\mathbf{\Phi}}) = \min_{\delta\mathbf{\Phi}} \mathbf{I}(\delta\mathbf{\Phi}), \quad \mathbf{I}(\delta\mathbf{\Phi}) = \mathbf{W}_{,\mathbf{\Phi}}\cdot\delta\mathbf{\Phi} + \mathrm{D}(\delta\mathbf{\Phi}) \tag{22}$$

which is the minimum principle I in Fleck & Willis [4]. Such a minimum must satisfy $\mathbf{I}(\dot{\mathbf{\Phi}}) = 0$, since $\mathbf{I}(\delta\mathbf{\Phi})$ is a sum of a linear and a homogeneous functional. The stationary condition at the minimum shows that $\dot{\mathbf{\Phi}}$ must satisfy the condition:

$$\begin{cases} \exists\ (\mathbf{X}_{\mathrm{d}}, \mathbf{Y}_{\mathrm{d}}) \text{ associated with } \dot{\mathbf{\Phi}} \text{ by the normality law and the plastic criterion such that:} \\ X_{\mathrm{e}} + X_{\mathrm{d}} - \nabla\cdot(Y_{\mathrm{e}} + Y_{\mathrm{d}}) = 0, \quad X_{\mathrm{e}} = W_{,\phi}, \quad Y_{\mathrm{e}} = W_{,\nabla\phi} \text{ in } V, \\ (Y_{\mathrm{e}} + Y_{\mathrm{d}})\cdot n = 0 \text{ on } \partial V \end{cases} \tag{23}$$

thus the force–flux relationship follows and vice-versa.

[1] The theory of evolution variational inequality is well developed in mathematics, cf. [12].

The question of the existence of the response has been much discussed in classical plasticity. In gradient plasticity, many discussions have been recently proposed for the existence, regularity and the numerical analysis of a solution, cf. [13,7,5,14,6].

The stationary condition at the minimum of $\mathbf{I}(\delta\boldsymbol{\Phi})$ shows that the following equation holds for the solution to the minimum principle

$$-\mathbf{W},_{\boldsymbol{\Phi}} \in \partial\mathbf{D}(\dot{\boldsymbol{\Phi}}) \tag{24}$$

which is an extended form of the well-known Biot equation in classical visco-elasticity, cf. for example [9].

Finally, the evolution equation (20) can be also schematically condensed as

$$\begin{cases} \mathbf{W},_{\mathbf{u}} = \mathbf{F_u} \\ -\mathbf{W},_{\boldsymbol{\Phi}} \in \partial\mathbf{D}(\dot{\boldsymbol{\Phi}}) \end{cases} \tag{25}$$

This discussion shows in particular that higher gradients can also be included in the same framework. The force–flux relation is still given by the Biot equation for the solid (24) and the response of the solid in a quasi-static transformation is governed by the evolution variational inequality (20).

3.2. Uniqueness

Let $(\mathbf{U}_i,\ i = 1, 2)$ be two possible solutions to (20). If the dissipation potential is state-independent, then the combination of the governing equations associated with these solutions gives in quasi-statics

$$(\mathbf{W},_{\mathbf{U}}(\mathbf{U_2}) - \mathbf{W},_{\mathbf{U}}(\mathbf{U_1})) \cdot (\dot{\mathbf{U}}_2 - \dot{\mathbf{U}}_1) \le 0$$

Under the assumption of a quadratic energy potential $\mathbf{W}(\mathbf{U})$, since

$$\frac{\mathrm{d}}{\mathrm{d}t}((\mathbf{W}(\mathbf{U_2} - \mathbf{U_1}) = (\mathbf{W},_{\mathbf{U}}(\mathbf{U_2}) - \mathbf{W},_{\mathbf{U}}(\mathbf{U_1})) \cdot (\dot{\mathbf{U}}_2 - \dot{\mathbf{U}}_1) \le 0$$

there is *a contraction of the energy distance between two responses*. It follows that

$$\mathbf{W}(\mathbf{U_2}(\mathbf{t}) - \mathbf{U_1}(\mathbf{t})) \le \mathbf{W}(\mathbf{U_2}(\mathbf{0}) - \mathbf{U_1}(\mathbf{0}) = 0$$

Thus $\mathbf{U_2}(\mathbf{t}) = \mathbf{U_1}(\mathbf{t})$ for all $t \ge 0$ if the energy potential of the solid is positive-definite.

4. Time-discretization by implicit scheme

4.1. Implicit scheme and incremental problem

The numerical analysis of the response of a solid to a given loading path is considered in this section. In a time-like discretization, the present value $\mathbf{U}$ is assumed at a current step. The incremental problem consists in determining the incremental response $\Delta\mathbf{U}$ to an increment of load $(\Delta\mathbf{F_g},\ \ \Delta\mathbf{u_g})$.

A time discretization of the evolution variational inequality (20) following the implicit scheme consists in replacing $\dot{\mathbf{U}},\ \dot{\mathbf{U}}^*,\ \dot{\mathbf{F}}$ respectively by $\frac{\Delta\mathbf{U}}{\Delta t},\ \frac{\Delta\mathbf{U}^*}{\Delta t},\ \frac{\Delta\mathbf{F}}{\Delta t}$ and $\mathbf{U}$ by $\mathbf{U}_+ = \mathbf{U} + \Delta\mathbf{U}$, $\mathbf{F}$ by $\mathbf{F}_+ = \mathbf{F} + \Delta\mathbf{F}$ in the expression (20).

Since the dissipation potential is positively homogeneous of degree 1, it follows that the incremental response $\Delta\mathbf{U}$ must be a solution to the incremental problem, i.e. must satisfy the following variational inequality:

$$\begin{cases} \mathbf{W},_{\mathbf{U}}(\mathbf{U} + \Delta\mathbf{U}) \cdot (\Delta\mathbf{U}^* - \Delta\mathbf{U}) + \mathbf{D}(\Delta\mathbf{U}^*) - \mathbf{D}(\Delta\mathbf{U}) \\ -(\mathbf{F_g} + \Delta\mathbf{F_g}) \cdot (\Delta\mathbf{U}^* - \Delta\mathbf{U}) \ge \mathbf{0} \quad \forall \quad \Delta\mathbf{U}^* \text{ admissible} \end{cases} \tag{26}$$

The implicit scheme ensures that the equilibrium equation and the normality law are satisfied by the increments of the displacement and the internal parameter at the next step.

At the limit, when $\Delta t \to 0$, then $\frac{\Delta\mathbf{F}}{\Delta t} \to \dot{\mathbf{F}}$, $\frac{\Delta\mathbf{U}}{\Delta t} \to \dot{\mathbf{U}}$ and $\frac{\Delta\mathbf{U}^*}{\Delta t} \to \dot{\mathbf{U}}^*$ and the evolution equation (20) is recovered.

Since $\Delta\mathbf{U}$ must be small, it is interesting to introduce the condition $\Delta\mathbf{U} \in \mathcal{N}(\mathbf{U})$ where $\mathcal{N}(\mathbf{U})$ denotes a neighborhood of $\mathbf{U}$ with respect to a suitable distance. The variational inequality (26) can be considered with the additional condition

$$\Delta\mathbf{U} \in \mathcal{N}(\mathbf{U}) \text{ and } \Delta\mathbf{U}^* \in \mathcal{N}(\mathbf{U}) \tag{27}$$

When $\Delta t \to 0$, it is clear that (26) + (27) leads also to the evolution equation (20).

4.2. Incremental minimum principle and stable response

If the energy potential is a convex function (as in the models (9) and (13)), a solution $\Delta\mathbf{U}$ of the variational inequality (26) is also a solution to the minimization problem:

Incremental minimum principle The increment $\Delta\mathbf{U}$ minimizes the functional

$$\mathbf{K}(\Delta\mathbf{U}^*) = \mathbf{W}(\mathbf{U}+\Delta\mathbf{U}^*) + \mathbf{D}(\Delta\mathbf{U}^*) - (\mathbf{F_g}+\Delta\mathbf{F_g})\cdot\Delta\mathbf{U}^* \tag{28}$$

among the set of admissible increments $\Delta\mathbf{U}^*$.

Indeed, the minimum principle (28) results from the variational inequality (26), since the convexity of the energy potential ensures that $\mathbf{W}(\mathbf{U}+\Delta\mathbf{U}^*) - \mathbf{W}(\mathbf{U}+\Delta\mathbf{U}) \geq \mathbf{W},_{\mathbf{U}}(\mathbf{U}_+)\cdot(\Delta\mathbf{U}^* - \Delta\mathbf{U})$.

The same conclusion also holds if the energy potential is only convex in $\mathcal{N}(\mathbf{U})$. In this case, the solution $\Delta\mathbf{U}$ of (26) + (27) is a local minimum of the functional $\mathbf{K}(\Delta\mathbf{U}^*)$.

Conversely, a local minimum $\Delta\mathbf{U}\in\mathcal{N}(U)$ of the functional $\mathbf{K}(\Delta\mathbf{U}^*)$ is necessarily a solution to the variational inequality (26) + (27) for any smooth energy potential (not necessarily convex as in the model (16)). Indeed, for any $\Delta\mathbf{U}^*\in\mathcal{N}(\mathbf{U})$

$$\begin{cases} \mathbf{K}(\Delta\mathbf{U}) \leq \mathbf{K}((\mathbf{1}-\alpha)\Delta\mathbf{U}+\alpha\Delta\mathbf{U}^*) \leq \mathbf{W}(\mathbf{U}_+ + \alpha(\Delta\mathbf{U}^*-\Delta\mathbf{U})) + \\ (\mathbf{1}-\alpha)\mathbf{D}(\Delta\mathbf{U}) + \alpha\mathbf{D}(\Delta\mathbf{U}^*) - \mathbf{F_{g+}}\cdot(\Delta\mathbf{U}+\alpha(\Delta\mathbf{U}^*-\Delta\mathbf{U})) \quad \forall\ \alpha\in[0,1] \end{cases}$$

since $\mathbf{D}$ is a convex function. It follows that

$$\frac{1}{\alpha}(\mathbf{W}(\mathbf{U}_+ + \alpha(\Delta\mathbf{U}^*-\Delta\mathbf{U})) - \mathbf{W}(\mathbf{U}_+)) - \mathbf{F_{g+}}\cdot(\Delta\mathbf{U}^*-\Delta\mathbf{U}) + \mathbf{D}(\Delta\mathbf{U}^*) - \mathbf{D}(\Delta\mathbf{U}) \geq 0$$

thus (26) results for vanishing α.

The minimum principle (28) + (27) deals with stable solutions to the variational inequality (26). The stability is understood here in the sense of a positive external work in any perturbation of the equilibrium $\mathbf{U}_+$, cf. [15,9]:

An equilibrium $\mathbf{U}_+ = \mathbf{U}+\Delta\mathbf{U}$ under the applied force $\mathbf{F}_+$ and imposed displacement $\mathbf{u_{g+}}$ is stable if in any perturbation of this equilibrium, defined by a perturbed path in function of a kinematic time τ

$$\mathbf{U}[\tau],\ \tau\in[0,1],\ \mathbf{U}[0]=\mathbf{U}_+,\ \mathbf{U}[1]=\mathbf{U}_+^*\in\mathcal{N}(\mathbf{U})$$

under the action of some perturbation forces, the work provided by these forces is non-negative.

Indeed, in such a perturbation, the energy balance, which results from the constitutive equations (1)–(5) of the solid, shows that the amount of work provided by the perturbed forces is

$$W_{\text{per}} = \mathbf{W}(\mathbf{U}_+^*) - \mathbf{W}(\mathbf{U}_+) + \int_{\mathbf{0}}^{\mathbf{1}} \mathbf{D}(\frac{\mathrm{d}\Phi}{\mathrm{d}\tau}[\tau])\,\mathrm{d}\tau - \mathbf{F_{g+}}\cdot(\mathbf{U}_+^* - \mathbf{U}_+) \tag{29}$$

From the fact that the dissipation potential is a kind of norm

$$\int_0^1 \mathbf{D}(\frac{\mathrm{d}\boldsymbol{\Phi}}{\mathrm{d}\tau}[\tau])\,\mathrm{d}\tau \geq \mathbf{D}(\Delta\mathbf{U}^*-\Delta\mathbf{U}) \geq \mathbf{D}(\Delta\mathbf{U}^*) - \mathbf{D}(\Delta\mathbf{U}) \tag{30}$$

it follows that

$$W_{\text{per}} \geq \mathbf{W}(\mathbf{U}_+^*) - \mathbf{W}(\mathbf{U}_+) + \mathbf{D}(\Delta\mathbf{U}^*) - \mathbf{D}(\Delta\mathbf{U}) - \mathbf{F_{g+}}\cdot(\Delta\mathbf{U}^*-\Delta\mathbf{U}) \geq 0 \tag{31}$$

The incremental minimum principle can also written as the following minimum principle concerning the response at the next step $\mathbf{U}_+ = \mathbf{U}+\Delta\mathbf{U}$.

Displacement minimum principle At time $t+\Delta t$, the generalized displacement $\mathbf{U}_+$ minimizes the functional

$$\bar{\mathbf{K}}(\mathbf{U}_+^*) = \mathbf{W}(\mathbf{U}_+^*) + \mathbf{D}(\mathbf{U}_+^* - \mathbf{U}) - \mathbf{F_{g+}}\cdot\mathbf{U}_+^* \tag{32}$$

among the set of admissible displacements $\mathbf{U}_+^*\in\mathcal{N}(\mathbf{U})$.

The reader can refer to [13,7,5] for an original mathematical formulation of stable responses. In their approach, the starting point is the displacement minimum principle (32) instead of the evolution equation (20) and the implicit scheme. Their results show in particular that the convergence of the implicit scheme is ensured under the assumption of convexity of the energy potential.

For a model of potentials $W = W(\epsilon, \phi, \nabla\phi)$ and $D = D(\dot{\phi}, \nabla\dot{\phi})$, the minimum principle (28) leads to the following variational equations

$$\begin{cases} \int_V (\sigma + \Delta\sigma)\delta\epsilon \, \mathrm{d}V \; - \; \mathbf{F_{gu+}} \cdot \delta\mathbf{u} = 0, \\ \int_V \{X_{e+} \cdot \delta\phi + Y_{e+} \cdot \nabla\delta\phi + X^{\phi}_{d+}\delta\phi + Y^{\phi}_{d+} \cdot \nabla\delta\phi)\} \, \mathrm{d}V = 0 \\ \text{with } (X^{\phi}_{d+}, \; Y^{\phi}_{d+}) \in \partial D(\Delta\phi, \nabla\Delta\phi) \end{cases} \tag{33}$$

in which the first line is the equilibrium equation. The second line is not identical to (23), since the forces X_{e+}, Y_{e+} depend on the increments $\nabla\Delta u, \Delta\phi, \nabla\Delta\phi$.

In particular, if the current state is the natural state and if the load increment is the final load, the implicit scheme gives the response of the associated deformation model under the final load.

It is also interesting to remark that for a time-dependent system, i.e. when the dissipation potential is a smooth function, Biot equation holds under the classical form

$$(\mathbf{W},_{\mathbf{U}} + \mathbf{D},_{\dot{\mathbf{U}}}) \cdot (\dot{\mathbf{U}}^* - \dot{\mathbf{U}}) - \mathbf{F_g} \cdot (\dot{\mathbf{U}}^* - \dot{\mathbf{U}}) = 0 \quad \forall \, \mathbf{U}^* \text{ admissible} \tag{34}$$

and leads to the following minimum principle after discretization by the implicit scheme, cf. [16]:

$$\Delta\mathbf{U} = \text{Arg.} \min_{\Delta\mathbf{U}^*\text{adm.}} \mathbf{W}(\mathbf{U} + \Delta\mathbf{U}^*) + \Delta t \mathbf{D}(\frac{\Delta\mathbf{U}^*}{\Delta \mathbf{t}}) - \mathbf{F_{g+}} \cdot \Delta\mathbf{U}^* \tag{35}$$

when the energy and dissipation potentials are convex functions.

In plasticity, for the model (13) with linear isotropic elasticity and linear hardening $H(\gamma) = \frac{h}{2}\gamma^2$, the increment $\Delta\mathbf{u}$, $\Delta\epsilon^{\mathrm{p}}$, $\Delta\gamma$ must satisfy the minimum of the functional

$$\begin{cases} \mathbf{K}(\Delta\mathbf{u}, \Delta\epsilon^{\mathrm{p}}, \; \Delta\gamma) = \int_V \; \{ \frac{1}{2}(\epsilon_+ - \epsilon^{\mathrm{p}}_+) : L : (\epsilon_+ - \epsilon^{\mathrm{p}}_+) + \; \frac{h}{2}\gamma^2_+ \\ + \frac{g}{2}\|\nabla\gamma_+\|^2 + k\|\Delta\epsilon^p\| + \Psi_o(\Delta\gamma - \|\Delta\epsilon^p\|) \; + \kappa\|\nabla\Delta\gamma\|\} \, \mathrm{d}V - \Delta\mathbf{F_{gu+}} \cdot \Delta\mathbf{u} \end{cases} \tag{36}$$

For the model (16), stable solutions $\mathbf{U}_+ = (\mathbf{u}_+, \mathbf{\Theta}_+)$ are the local minima of the functional

$$\mathbf{K}(\Delta\mathbf{u}, \; \Delta\mathbf{\Theta}) = \int_V \{\frac{1}{2}((1-\theta_+)\epsilon_+ : L : \epsilon_+ + a\,\theta^2_+ + b\|\nabla\theta_+\|^2) + \; \bar{k}\Delta\Theta\} \, \mathrm{d}V - \mathbf{F_{gu+}} \cdot \mathbf{u}_+ \tag{37}$$

under the constraints $\Delta\theta \geq 0$ and $\theta + \Delta\theta \leq 1 - r$.

5. Regularization and iterative methods

The minimization problem (28) can be solved numerically by usual methods of minimization of a functional. Since the energy potential is a smooth function by assumption, the principal difficulty in the minimization problem (28) concerns the dissipation potential.

The regularization method consists in replacing the dissipation potential D by a smooth function D_s. For example, the dissipation potential $\kappa\|\nabla\dot{\gamma}\|$ can be regularized as

$$\begin{cases} D_s = \kappa\|\nabla\dot{\gamma}\| \qquad \text{if} \quad \|\nabla\dot{\gamma}\| \geq \frac{\kappa}{s}, \\ D_s = \frac{1}{2}s\|\nabla\dot{\gamma}\|^2 + \frac{\kappa}{2s} \quad \text{if} \quad \|\nabla\dot{\gamma}\| \leq \frac{k}{s}, \\ D_s \to D \;\text{ when }\; s \to +\infty \end{cases} \tag{38}$$

After regularization, a problem of minimization of a smooth functional must be solved. The search for the minimum can be performed following some classical descent methods in optimization.

In a complementary direction, some direct iterative methods can also be explored. These methods consist in solving the incremental problem by successive iterations in two steps:

i.– Starting from a given $\Delta\mathbf{\Phi}$, compute $\Delta\mathbf{u}$.

ii.– For a given $\Delta\mathbf{u}$, compute $\Delta\mathbf{\Phi}$ and return to i.

For the model (13), the minimization with respect to $\Delta\epsilon^{\mathrm{p}}$ shows that

$$-\sigma' - 2\mu\Delta\epsilon' + 2\mu\Delta\epsilon^{\mathrm{p}} + X^{\mathrm{p}}_{\mathrm{d}} - m\frac{\Delta\epsilon^{\mathrm{p}}}{\|\Delta\epsilon^{\mathrm{p}}\|} = 0 \quad \forall \quad x \in V$$

It is concluded that

$$\Delta\epsilon^{\mathrm{p}} = \lambda Z \;\text{ with }\; Z = \frac{\sigma' + 2\mu\Delta\epsilon'}{\|\sigma' + 2\mu\Delta\epsilon'\|} \tag{39}$$

Finally, $\lambda = \Delta\gamma = \|\Delta\epsilon^{\mathrm{p}}\|$ and, after a regularization of the function $\kappa\|\nabla\Delta\gamma\|$, the field λ minimizes the functional

$$\int_V \{(h\gamma + k - \|\sigma' + 2\mu\Delta\epsilon'\|)\lambda + \frac{h+2\mu}{2}\lambda^2 + \frac{g}{2}\|\nabla\lambda\|^2 + g\nabla\gamma\cdot\nabla\lambda + D_s(\nabla\lambda)\}\,\mathrm{d}V \tag{40}$$

under the constraint $\lambda \geq 0$ in V when the displacement field is assumed to be known.

A different approach consists in replacing the model (13) by a regularized model defined as

$$\begin{cases} W(\epsilon, \epsilon^{\mathrm{p}}, \gamma, \nabla\gamma, \beta) = \frac{1}{2}(\epsilon - \epsilon^{\mathrm{p}}) : L : (\epsilon - \epsilon^{\mathrm{p}}) + H(\gamma) + \frac{g}{2}\nabla\gamma^2 + \frac{e}{2}\|\nabla\gamma - \beta\|^2 \\ D(\dot\epsilon^{\mathrm{p}}, \dot\gamma, \dot\beta) = k\|\dot\epsilon^{\mathrm{p}}\| + \kappa\|\dot\beta\| + \Psi_{\mathrm{o}}(\dot\gamma - \|\dot\epsilon^{\mathrm{p}}\|) \end{cases} \tag{41}$$

Thus, an additional variable β is included in order to approximate $\nabla\gamma$ when e is high enough and to avoid the difficulty due to the gradient term in the dissipation potential. In this case, the minimization with respect to β gives

$$e(\nabla\gamma - \beta + \nabla\lambda - \Delta\beta) + X_{\mathrm{d}}^{\beta} = 0 \text{ in } V$$

It is concluded again that

$$\Delta\beta = \eta\zeta \text{ with } \zeta = \frac{\nabla(\gamma+\lambda) - \beta}{\|\nabla(\gamma+\lambda) - \beta\|} \text{ and } \eta = \frac{1}{e} < e\|\nabla(\gamma+\lambda) - \beta\| - \kappa > \tag{42}$$

Finally, the field λ minimizes the functional

$$\int_V \{(h\gamma + k - \|\sigma' + 2\mu\Delta\epsilon'\|)\lambda + \frac{h+2\mu}{2}\lambda^2 + \frac{g+e}{2}\|\nabla\lambda\|^2 + (e\beta_+ + (g+e)\nabla\gamma)\cdot\nabla\lambda\}\,\mathrm{d}V \tag{43}$$

under the constraint $\lambda \geq 0$ when ϵ_+ and β_+ are given.

For the model (16), the increment $\Delta\boldsymbol{\Theta}$ minimizes the functional:

$$\mathcal{K}(\Delta\boldsymbol{\Theta}^*) = \int_V \{(-\frac{1}{2}\epsilon_+ : L : \epsilon_+ + \bar{k})\Delta\theta^* + \frac{a}{2}(\theta + \Delta\theta^*)^2 + \frac{b}{2}\|\nabla(\theta + \Delta\theta^*)\|^2\}\mathrm{d}V \tag{44}$$

under the constraints $\Delta\theta^* \geq 0$ and $\theta + \Delta\theta^* \leq 1 - r$ when $\mathbf{u}_+$ is assumed to be known.

After a spatial discretization by the finite element method, the numerical determination of $\Delta\theta$ can be obtained by the projected gradient method in the following way.

Let $N_i(x)$ be the interpolation function at node i, $i = 1, N$ and θ_i, Δ_i the nodal values of θ and $\Delta\theta$. The following matrices are introduced

$$A_{ij} = \int_V (aN_iN_j + b\nabla N_i\cdot\nabla N_j)\,\mathrm{d}V, \quad C_i = \int_V (-\frac{1}{2}\epsilon_+ : L : \epsilon_+ + \bar{k})N_i\,\mathrm{d}V + A_{ij}\theta_j, \quad L_i = 1 - r - \theta_i \tag{45}$$

From (44), it follows that $[\Delta]$ must satisfy the stationary condition of the associated Lagrangian

$$[A][\Delta] + [C] - [m] + [\bar{m}] = [0] \tag{46}$$

where m_i and $\bar{m}_i$ are Lagrange multipliers with Kuhn–Tucker conditions:

$$\Delta_i \geq 0,\ m_i \geq 0,\ m_i\Delta_i = 0, \quad \text{and } L_i - \Delta_i \geq 0,\ \bar{m}_i \geq 0,\ \bar{m}_i(L_i - \Delta_i) = 0 \tag{47}$$

Starting from the initial value $[\Delta^0] = [0]$, the projected gradient method consists in computing $[\Delta^n]$ from $[\Delta^{n-1}], n = 1, 2, \ldots$ until convergence from the following equation

$$[\Delta]^n = [\Delta]^{n-1} - \rho([A][\Delta]^{n-1} + [C] - [m]^n + [\bar{m}]^n) \tag{48}$$

where the multipliers m_i^n and $\bar{m}_i^n$ are associated with Δ_i^n following (47), and the coefficient $\rho > 0$ will be chosen such that the sequence $[\Delta]^n$ is a minimizing sequence of the function

$$\mathcal{K}([\Delta]) = \frac{1}{2}[\Delta]^{\mathrm{T}}[A][\Delta] + [\Delta]^{\mathrm{T}}[C]$$

i.e. of the functional (44).

Let $[T]^n = [\Delta]^{n-1} - \rho([A][\Delta]^{n-1} + [C])$. From (48), the expression of $[\Delta]^n$ is straightforward:

$$\begin{cases} \text{If } T_i^n \leq 0 \text{ then } \Delta_i^n = 0,\ m_i^n = -T_i,\ \bar{m}_i^n = 0, \\ \text{If } 0 < T_i^n < L_i \text{ then } \Delta_i^n = T_i^n,\ m_i^n = 0,\ \bar{m}_i^n = 0, \\ \text{If } T_i^n \geq L_i \text{ then } \Delta_i^n = L_i,\ m_i^n = 0,\ \bar{m}_i^n = T_i^n - L_i \end{cases} \tag{49}$$

The sequence $[\Delta]^n$ is a minimizing sequence if $\mathcal{K}([\Delta]^{n-1}) - \mathcal{K}([\Delta]^n) \geq 0$ for all n. From (48), the following expression holds

$$\begin{cases} \mathcal{K}([\Delta]^{n-1}) - \mathcal{K}([\Delta]^n) = \frac{1}{\rho}([\Delta]^n - [\Delta]^{n-1})^{\mathrm{T}}([\Delta]^n - [\Delta]^{n-1}) - \\ - \frac{1}{2}([\Delta]^n - [\Delta]^{n-1})^{\mathrm{T}}[A]([\Delta]^n - [\Delta]^{n-1}) + [m]^{\mathrm{T}}[\Delta]^{n-1} - [\bar{m}]^{\mathrm{T}}([\Delta]^{n-1} - [L]) \end{cases} \tag{50}$$

Since the two last terms are non-negative, the sequence $[\Delta]^n$ is a minimizing sequence if

$$\rho < \min_{[\Delta]} \frac{2[\Delta]^{\mathrm{T}}[\Delta]}{[\Delta]^{\mathrm{T}}[A][\Delta]} = \frac{2}{\Lambda}, \quad \Lambda \text{ denotes the highest eigenvalue of the matrix } [A] \tag{51}$$

It is then clear that, if $\mathcal{K}([\Delta]^{n-1}) - \mathcal{K}([\Delta]^n) = 0$, then $[\Delta]^n = [\Delta]^{n-1} = [\Delta]$ is the solution to (46).

The same method can be applied to compute the solution to (43).

6. Conclusion

Within the framework of standard plasticity, the theory of gradient plasticity and of time-independent processes such as brittle damage is discussed. The governing equations of the response of a solid under a loading path are written in terms of the energy and the dissipation potentials. It is shown that the quasi-static response of the solid is a solution to a variational inequality as in classical plasticity and that higher gradients can also be included in the same spirit. A time-discretization by the implicit scheme of the evolution equation leads to the study of the incremental problem, which is different from the rate problem. The increment of the response under an increment of the loads must satisfy a variational inequality and, if the energy potential is convex, an incremental minimum principle. In particular, a local minimum of the incremental minimum principle is a stable solution to the variational inequality.

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